How quantum mechanics might be when measuring commuting observables

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(Dated: July 21, 2023)

Abstract

We consider how quantum mechanics might be when measuring commuting observables if we accept the Kronecker delta. Quantum mechanics is reduced to classical theory when we consider only commuting observables. Using this fact, we discuss an inconsistency within quantum mechanics when accepting the Kronecker delta without extra assumptions about the reality of observables. One of the objectives of this paper is for us to remain wondering the extension of quantum mechanical axiom to concrete commuting observables themselves.

PACS numbers: 03.65.Ta, 03.65.Ca
Keywords: Quantum measurement theory, Formalism

I. INTRODUCTION

Quantum mechanics (cf. [1–7]) is an important physical theory in order to explain the microscopic behaviors of the nature. The Kochen-Specker theorem based on the Kronecker delta is discussed by Nagata, Patro, and Nakamura [8]. The Kronecker delta is explained as follows: The two-variable function $\delta_{ll'}$ that takes the value 1 when $l = l'$ and the value 0 otherwise. If the elements of a square matrix are defined by the delta function, the matrix produced will be the identity matrix [9].

Nagata *et al.* derive [10–12] some inconsistency in quantum mechanics. Barros claims in [13] that the inconsistencies do not come from quantum mechanics, but from extra assumptions about the reality of observables. We suppose the inconsistency comes from quantum mechanics, without extra assumptions about the reality of observables. We show here the inconsistency in a unitary space when measuring commuting observables.

We discuss some inconsistency comes from quantum mechanics using commuting observables σ_z^1 and σ_z^2 if we introduce the Kronecker delta. We do not introduce an assumption about the reality of observables because we consider only commuting observables. And we show here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables.

In this paper, we discuss there is some inconsistency within quantum mechanics when we limit ourselves to commuting observables and we introduce the Kronecker delta. One of the objectives of this paper is for us to remain wondering the extension of quantum mechanical axiom to concrete commuting observables themselves.

We define an inconsistency as follows, when considering only two commuting Hermitian matrices:

1. Define two commuting Hermitian matrices A_1, A_2.
2. Define a two-variable function $f(X, Y)$, where f is an appropriate function and X, Y are two variables.
3. Derive a value of $f(A_1, A_2) = a$ by substituting A_1, A_2 into X, Y, respectively, without the property of the Kronecker delta.
4. Introduce the Kronecker delta.
5. Derive another value of $f(A_1, A_2) = b(\neq a)$ under the supposition that we use the Kronecker delta.
6. We cannot assign simultaneously the truth value "1" for the two suppositions $f(A_1, A_2) = a$ and $f(A_1, A_2) = b$.
7. Confirm the inconsistency derived only by the two commuting Hermitian matrices A_1, A_2.

The paradox cannot be avoidable by Matrix theory because of the Kronecker delta. In what follows, we apply such an inconsistency into quantum mechanics based on the nature of Matrix theory.

II. HOW QUANTUM MECHANICS MIGHT BE WHEN MEASURING COMMUTING OBSERVABLES

In this section, we want to discuss our argumentations in the qubit system. Let σ_z^1, σ_z^2 be two z-component Pauli operators, where they are also supposed to be commutative. They could be defined respectively as follows:

$$\sigma_z^1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \sigma_z^2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be eigenstates of σ_z such that $\sigma_z|\uparrow\rangle = +1|\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -1|\downarrow\rangle$. The measured results of trials are either $+1$ or -1.

Let us consider a simultaneous eigenstate of σ_z^1, σ_z^2, that is, $|\uparrow\downarrow\rangle$. In Physics, practically quantum mechanics uses $+1$ and -1. So we mainly consider $+1$ and -1. Let us consider a simultaneous eigenstate of σ_z^1 and σ_z^2. We might be in an inconsistency when the first result is $+1$

by the measured observable σ_z^1, the second result is -1 by the measured observable σ_z^2, and then $[\sigma_z^1, \sigma_z^2] = 0$.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either $+1$ or -1. We suppose the number of trials of obtaining the result -1 is equal to the number of trials of obtaining the result $+1$. We can depict experimental data r_1, r_2 as follows: $r_1 = +1$ and $r_2 = -1$. Let us write V as follows:

$$V = \sum_{l=1}^{2} r_l. \tag{2}$$

We are very interested in the following value:

$$V \times V = \left(\sum_{l=1}^{2} r_l\right)^2 = \left(\sum_{l=1}^{2} r_l\right) \times \left(\sum_{l'=1}^{2} r_{l'}\right). \tag{3}$$

Surprisingly, we cannot define $V \times V$ as zero as shown below.

Without the property of the Kronecker delta, we have

$$\begin{aligned} V \times V \times \delta_{ll'} &= \left(\sum_{l=1}^{2} r_l\right)^2 \delta_{ll'} \\ &= ((+1) + (-1))^2 \delta_{ll'} = 0 \times \delta_{ll'} = 0. \end{aligned} \tag{4}$$

We derive a necessary condition of the product $V \times V \times \delta_{ll'}$ of the value V without the property of the Kronecker delta. In this case, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 0. \tag{5}$$

This is the necessary condition without the property of the Kronecker delta.

In the following, we evaluate another value of $(V \times V \times \delta_{ll'})$ and derive another necessary condition within the property of the Kronecker delta.

We introduce the Kronecker delta then we have

$$\begin{aligned} & V \times V \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2} r_l\right)^2 \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2} r_l\right) \times \left(\sum_{l'=1}^{2} r_{l'}\right) \times \delta_{ll'} \\ &= \sum_{l=1}^{2} (r_l)^2 = 2. \end{aligned} \tag{6}$$

Clearly, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 2. \tag{7}$$

These argumentations are possible for the case that we utilize the property of the Kronecker delta. We cannot assign simultaneously the truth value "1" for the two suppositions (5) and (7). We derive the inconsistency when we utilize the property of the Kronecker delta.

In summary, we have been in the inconsistency when the first result is $+1$, the second result is -1, and then $[\sigma_z^1, \sigma_z^2] = 0$, where the quantum state is a simultaneous eigenstate of σ_z^1, σ_z^2, that is, $|\uparrow\downarrow\rangle$.

III. HIGH DIMENSIONAL SPACE

Let us move ourselves to the general case. Especially, we show here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables. Let A_1, A_2 be two commuting Hermitian operators. Let us consider a simultaneous eigenstate of A_1 and A_2. We might be in an inconsistency when the first result is x by the measured observable A_1, the second result is $y (\neq x)$ by the measured observable A_2, and then $[A_1, A_2] = 0$.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either x or y. We suppose the number of trials of obtaining the result x is equal to the number of trials of obtaining the result y. If the number of trials is two, then we have

$$V = x + y. \tag{8}$$

We derive a necessary condition of the product $V \times V \times \delta_{ll'}$ of the value V without the property of the Kronecker delta. In this case, we have the following calculation result:

$$(V \times V \times \delta_{ll'}) = (x + y)^2 \delta_{ll'}. \tag{9}$$

This is the necessary condition without the property of the Kronecker delta.

We can depict experimental data r_1, r_2 as follows: $r_1 = x$ and $r_2 = y$. Let us write V as follows:

$$V = \sum_{l=1}^{2} r_l. \tag{10}$$

In the following, we evaluate another value $(V \times V \times \delta_{ll'})$ and derive another necessary condition within the property of the Kronecker delta.

We introduce the Kronecker delta then we have

$$\begin{aligned} & V \times V \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2} r_l\right)^2 \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2} r_l\right) \times \left(\sum_{l'=1}^{2} r_{l'}\right) \times \delta_{ll'} \\ &= x^2 + y^2. \end{aligned} \tag{11}$$

Clearly, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = x^2 + y^2. \tag{12}$$

This is possible within the property of the Kronecker delta. We cannot assign simultaneously the truth value "1" for the two suppositions (9) and (12). We derive the inconsistency when we utilize the property of the Kronecker delta. The specific case is that $x = +1$ and $y = -1$.

In summary, we have been in the inconsistency when the first result is x, the second result is y, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2.

IV. CONCLUSIONS AND DISCUSSIONS

In conclusions, if we accept the Kronecker delta, we have discussed there is some inconsistency within quantum mechanics, even we limit ourselves to commuting observables. One of the objectives of this paper has been for us to remain wondering the extension of quantum mechanical axiom to concrete commuting observables themselves.

The inconsistency can be derived only by commuting observables. Thus, non-commutativeness of Matrix theory is needless for the derivation of the inconsistency based on the Kronecker delta. The most important is only commutativeness of Matrix theory for our purpose as shown here.

If the problem were simply an inconsistency, there are multiple logical systems that can cope with such a problem with robustness (see [14]).

ACKNOWLEDGMENTS

We thank Soliman Abdalla, Jaewook Ahn, Josep Batle, Do Ngoc Diep, Mark Behzad Doost, Ahmed Farouk, Han Geurdes, Preston Guynn, Shahrokh Heidari, Wenliang Jin, Hamed Daei Kasmaei, Janusz Milek, Mosayeb Naseri, Santanu Kumar Patro, Germano Resconi, and Renata Wong for their valuable support.

DECLARATIONS

Ethical approval

The authors are in an applicable thought to ethical approval.

Competing interests

The authors state that there is no conflict of interest.

Author contributions

Koji Nagata and Tadao Nakamura wrote and read the manuscript.

Funding

Not applicable.

Data availability

No data associated in the manuscript.

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