The Determination of Integer Coordinates of Elliptic Curves

Abdelmajid Ben Hadj Salem[1,2*]

[1*]Residence Bousten 8, Bloc B, Av. Mosquée Raoudha, Soukra, 1181 Soukra-Raoudha, Tunisia.

Corresponding author(s). E-mail(s): abenhadjsalem@gmail.com;

Abstract

In this paper, we give an elliptic curve $\boldsymbol{(E)}$ given by the equation:

$$\boldsymbol{y^2 = \varphi(x) = x^3 + px + q}$$

with $\boldsymbol{p, q} \in \mathbb{Z}$ not null simultaneous. We study the conditions verified by $\boldsymbol{(p, q)}$ so that $\exists\, \boldsymbol{(x, y)} \in \mathbb{Z}^2$ the coordinates of a point of the elliptic curve $\boldsymbol{(E)}$ given by the equation above.

Keywords: elliptic curves, integer points, solutions of polynomial equations of degree three, solutions of Diophantine equations.

MSC Classification: 11AXX , 11M26.

This paper is dedicated to the memory of my Father who taught me arithmetic,
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

1 Introduction

Elliptic curves are related to number theory, geometry, cryptography, string theory, data transmission,... We consider an elliptic curve (E) given by the equation:

$$y^2 = \varphi(x) = x^3 + px + q \tag{1}$$

where p and q are two integers and we assume in this article that p, q are not simultaneous equal to zero. For our proof, we consider the equation :

$$\varphi(x) - y^2 = x^3 + px + q - y^2 = 0 \tag{2}$$

of the unknown the parameter x, and p, q, y given with the condition that $y \in \mathbb{Z}^+$. We resolve the equation (2) and we discuss so that x is an integer.

2 Proof

Proof. We suppose that $y > 0$ is an integer, to resolve (2), let:

$$x = u + v \tag{3}$$

where u, v are two complexes numbers. Equation (2) becomes:

$$u^3 + v^3 + q - y^2 + (u+v)(3uv + p) = 0 \tag{4}$$

With the choose of:

$$3uv + p = 0 \Longrightarrow uv = -\frac{p}{3} \tag{5}$$

then, we obtain the two conditions:

$$uv = -\frac{p}{3} \tag{6}$$

$$u^3 + v^3 = y^2 - q \tag{7}$$

Hence, u^3, v^3 are solutions of the equation of second order:

$$X^2 - (y^2 - q)X - \frac{p^3}{27} = 0 \tag{8}$$

Let Δ the discriminant of the above equation, it is given by:

$$\Delta = (y^2 - q)^2 + \frac{4p^3}{27} \tag{9}$$

2.1 Case $\Delta = 0$

In this case, the equation (8) has one double root :

$$X_1 = X_2 = \frac{y^2 - q}{2} \tag{10}$$

As $\Delta = 0 \Longrightarrow \dfrac{4p^3}{27} = -(y^2 - q)^2 \Longrightarrow p < 0$. As y, q are integers then $3|p \Longrightarrow p = 3p_1$, $p_1 < 0$ and $4p_1^3 = -(y^2 - q)^2 \Longrightarrow p_1 = -p_2^2 \Longrightarrow y^2 - q = \pm 2p_2^3$ and $p = -3p_2^2$. As

$y^2 = q \pm 2p_2^3$, it exists solutions if:

$$\boxed{q \pm 2p_2^3 \text{ is a square}} \tag{11}$$

We suppose that $q \pm 2p_2^3$ is a square. The solution $X = X_1 = X_2 = \dfrac{y^2 - q}{2} = \pm p_2^3$. Using the unknowns u, v, we have two cases:

1 - $u^3 = v^3 = p_2^3$,
2 - $u^3 = v^3 = -p_2^3$.

2.1.1 Case: $u^3 = v^3 = p_2^3$

The solutions of $u^3 = p_2^3$ are :

a - $u_1 = p_2$,
b - $u_2 = j.p_2$ with $j = \dfrac{-1 + i\sqrt{3}}{2}$ is the unitary cubic complex root,
c - $u_3 = j^2.p_2 = \bar{j}.p_2$.

Case a: $u_1 = v_1 = p_2 \Longrightarrow x = u_1 + v_1 = 2p_2 \Rightarrow u_1.v_1 = p_2^2 = -p/3$. Then the condition (6) $uv = u_1.v_1 = -p/3$ is verified. The integers coordinates of the elliptic curve (E) are :

$$(2p_2, +\alpha), \quad (2p_2, -\alpha)\, and\, \alpha = \sqrt{\varphi(2p_2)} \tag{12}$$

Case b: $u_2 = j.p_2, v_2 = j^2.p_2 = \bar{j}.p_2 \Longrightarrow x = u_2 + v_2 = p_2(j + \bar{j}) = -p_2$ and the condition (6) is verified. In this case, the integers coordinates of the elliptic curve (E) are :

$$(-p_2, +\alpha), \quad (-p_2, -\alpha)\, and\, \alpha = +\sqrt{\varphi(-p_2)} \tag{13}$$

Case c: $u_3 = j^2.p_2 = \bar{j}.p_2, v_3 = j.p_2$, then $x = u_3 + v_3 = -p_2$ and $u_3.v_3 = -p/3$. It is the same as case b above.

2.1.2 Case: $u^3 = v^3 = -p_2^3$

The solutions of $u^3 = -p_2^3$ are :

d - $u_1 = -p_2$;
e - $u_2 = -j.p_2$;
f - $u_3 = -j^2.p_2 = -\bar{j}.p_2$.

Case d: $u_1 = v_1 = -p_2 \Longrightarrow x = -2p_2$. The condition $u_1.v_1 = -p/3$ is verified. The integers coordinates of the elliptic curve (E) are :

$$(-2p_2, +\alpha), \quad (-2p_2, -\alpha)\, and\, \alpha = \varphi(-2p_2) \tag{14}$$

Case e: $u_2 = -j.p_2, v_2 = -j^2.p_2 = -\bar{j}.p_2 \Longrightarrow x = u_2 + v_2 = -p_2(j + \bar{j}) = +p_2$ and the condition (6) is verified. In this case, the integers coordinates of the elliptic curve (E) are :

$$(p_2, +\alpha), \quad (p_2, -\alpha)\, and\, \alpha = \varphi(p_2) \tag{15}$$

Case f: $u_2 = -j^2.p_2, v_2 = -j.p_2$. It gives the same of case e above.

2.2 Case $\Delta > 0$

We suppose that $\Delta > 0$ and $\Delta = m^2$ where $m \in \mathbb{R}^*$ is a positive real number.

$$\Delta = (y^2 - q)^2 + \frac{4p^3}{27} = \frac{27(y^2-q)^2 + 4p^3}{27} = m^2 \tag{16}$$

$$27(y^2-q)^2 + 4p^3 = 27m^2 \Longrightarrow 27(m^2 - (y^2-q)^2) = 4p^3 \tag{17}$$

2.2.1 We suppose that $3|p$

We suppose that $3|p \Longrightarrow p = 3p_1$. We consider firstly that $|p_1| = 1$.

Case $p_1 = 1 \Rightarrow p = 3$. The equation (17) is written as:

$$m^2 - (y^2-q)^2 = 4 \Longrightarrow m^2 = 4 + (y^2-q)^2 \Rightarrow m^2 \text{ is an integer} \tag{18}$$

We consider the case m is a positive integer: $m > 0$. From the last equation above, we obtain :

$$(m + y^2 - q)(m - y^2 + q) = 2 \times 2 \tag{19}$$

That gives 3 systems of equations (with $m > 0$) :

$$\begin{cases} m + y^2 - q = 1 \\ m - y^2 + q = 4 \end{cases} \Longrightarrow m = 5/2 \text{ not an integer} \tag{20}$$

$$\begin{cases} m + y^2 - q = 2 \\ m - y^2 + q = 2 \end{cases} \Longrightarrow m = 2 \text{ and } y^2 - q = 0 \tag{21}$$

$$\begin{cases} m + y^2 - q = 4 \\ m - y^2 + q = 1 \end{cases} \Longrightarrow m = 5/2 \text{ not an integer} \tag{22}$$

As $y^2 - q = 0$ from the case (21), if $q = q'^2$ with q' a positive integer, we obtain the integer coordinates of the elliptic curve (E):

$$y^2 = x^3 + 3x + q'^2 \tag{23}$$

$$(0, q'); (0, -q') \tag{24}$$

If q is not a square, then m can not be an integer.

Case $p_1 = -1 \Rightarrow p = -3$. Using the same method as above, we arrive to the acceptable value $m = 0$, then it is a particular case of $\Delta = 0$ studied above.

Now, we consider that $|p_1| > 1$.

We suppose that $p_1 > 1$

The equation (17) is written as:

$$m^2 - (y^2-q)^2 = 4p_1^3 \Longrightarrow m^2 - (y^2-q)^2 = 4p_1^3 \tag{25}$$

We consider that $m > 0$ is an integer. From the last equation (25), $(m, y^2 - q)$ (respectively in the case $y^2 - q \leq 0, (m, q - y^2)$) are solutions of the Diophantine equation :

$$X^2 - Y^2 = N \quad X > 0, Y > 0 \tag{26}$$

where N is a positive integer equal to $4p_1^3$.

For the general solutions of the equation (26), let $Q(N)$ the number of solutions of (26) and $\tau(N)$ the number of factorization of N, then we give the following result concerning the solutions of (26) (see theorem 27.3 of [1]):

- if $N \equiv 2 (\mathrm{mod}\, 4)$, then $Q(N) = 0$;
- if $N \equiv 1$ or $N \equiv 3 (\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N)/2]$;
- if $N \equiv 0 (\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the largest integer less or equal to x.

As $N = 4p_1^3 \Longrightarrow N \equiv 0 (\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N/4)/2] = [\tau(p_1^3)/2] > 1$. A solution (X', Y') of (26) is used if $Y' = y^2 - q \Longrightarrow q + Y'$ is a square (respectively if $Y' = q - y^2 \Longrightarrow q - Y'$ is a square), then $X' = m > 0$ and $\pm y = \pm\sqrt{q + Y'}$ (respectively $\pm y = \pm\sqrt{q - Y'}$). The roots of (8) are :

$$X_1 = \frac{y^2 - q + m}{2} = \frac{Y' + m}{2} > 0 \tag{27}$$

$$X_2 = \frac{y^2 + q - m}{2} = \frac{Y' - m}{2} < 0 \tag{28}$$

(Respectively, the roots of (8) are :

$$X_1 = \frac{y^2 - q + m}{2} = \frac{-Y' + m}{2} > 0 \tag{29}$$

$$X_2 = \frac{y^2 + q - m}{2} = \frac{-Y' - m}{2} < 0 \tag{30}$$

). From $X'^2 - Y'^2 = 4p_1^3 = N$, $2|(Y' - m)$ and $2|(Y' - m + 2m) \Longrightarrow 2|(Y' + m) \Longrightarrow X_1, X_2 \in \mathbb{Z}$, and we obtain the equations:

$$u^3 = X_1 \Longrightarrow u_1 = \sqrt[3]{X_1}; u_2 = j\sqrt[3]{X_1}; u_3 = j^2\sqrt[3]{X_1} \tag{31}$$

$$v^3 = X_2 \Longrightarrow v_1 = \sqrt[3]{X_2}; v_2 = j\sqrt[3]{X_2}; v_3 = j^2\sqrt[3]{X_2} \tag{32}$$

A real x is obtained if $x = u_1 + v_1 = \sqrt[3]{X_1} + \sqrt[3]{X_2}$. If X_1, X_2 are cubic integers : $X_1 = t_1^3, X_2 = t_2^3$, then we obtain an integer solution :

$$x = t_1 + t_2, \quad \pm y = \pm\sqrt{Y' + q} \quad \text{respectively} \quad \pm y = \pm\sqrt{q - Y'} \tag{33}$$

If not, there are no integer coordinates of the elliptic curve (E).

We suppose that $p < 0 \Longrightarrow p_1 < -1$ ***:***
in this case, $(y^2 - q, m)$ (respectively $(q - y^2, m)$) is a solution of the Diophantine equation :

$$X^2 - Y^2 = N' \quad X > 0, Y > 0 \tag{34}$$

and N' is a positive integer equal to $-4p_1^3 > 0$. As seen above, a solution (X', Y') of (34) is used if $X' = y^2 - q \Longrightarrow q + X'$ is a square (respectively $X' = q - y^2 \Rightarrow q - X'$ is a square), then $\pm y' = \pm\sqrt{q + X'}$ (respectively $\pm y' = \pm\sqrt{q - X'}$) and $Y' = m > 0$. The roots of (8) are :

$$X_1' = \frac{y^2 - q + m}{2} = \frac{X' + m}{2} > 0 \tag{35}$$

$$X_2' = \frac{y^2 + q - m}{2} = \frac{X' - m}{2} > 0 \tag{36}$$

(Respectively the roots of (8) are :

$$X_1' = \frac{y^2 - q + m}{2} = \frac{-X' + m}{2} > 0 \tag{37}$$

$$X_2' = \frac{y^2 + q - m}{2} = \frac{-X' - m}{2} < 0 \tag{38}$$

) From $X'^2 - Y'^2 = -4p_1^3 = N'$, $2|(X' - m)$ and $2|(X' + m) \Longrightarrow X_1', X_2' \in \mathbb{Z}$, and we obtain the equations:

$$u'^3 = X_1' \Longrightarrow u_1' = \sqrt[3]{X_1'}; u_2' = j\sqrt[3]{X_1'}; u_3' = j^2\sqrt[3]{X_1'} \tag{39}$$

$$v'^3 = X_2' \Longrightarrow v_1' = \sqrt[3]{X_2'}; v_2' = j\sqrt[3]{X_2'}; v_3' = j^2\sqrt[3]{X_2'} \tag{40}$$

A real x' is obtained if $x' = u_1' + v_1' = \sqrt[3]{X_1'} + \sqrt[3]{X_2'}$. If X_1', X_2' are cubic integers : $X_1' = t_1'^3, X_2' = t_2'^3$ then we obtain an integer solution :

$$x' = t_1' + t_2', \quad \pm y' = \pm\sqrt{X' + q} \quad (\text{respectively} \quad \pm y' = \pm\sqrt{q - X'}) \tag{41}$$

If not, there are no integer coordinates of the elliptic curve (E).

2.2.2 We suppose that $3 \nmid p$

We rewrite the equations (8) and (17):

$$X^2 - (y^2 - q)X - \frac{p^3}{27} = 0$$

$$\Delta = (y^2 - q)^2 + \frac{4p^3}{27} = \frac{27(y^2 - q)^2 + 4p^3}{27} = m^2$$

with $m^2 > 0$ is a rational number, then m is not an integer. It follows there are no integer coordinates of the elliptic curve (E).

2.3 Case $\Delta < 0$

The expression of Δ is given by (71) :

$$\Delta = (y^2 - q)^2 + \frac{4p^3}{27}$$

We suppose that $\Delta < 0 \Longrightarrow (y^2 - q)^2 + \frac{4p^3}{27} < 0 \Longrightarrow (y^2 - q)^2 < -\frac{4p^3}{27}$, then $p < 0$. Let $p' = -p > 0 \Longrightarrow \Delta = (y^2 - q)^2 - \frac{4p'^3}{27}$.

2.3.1 We suppose $3|p'$:

We suppose that $3|p' \Longrightarrow p' = 3p_1$. Δ becomes:

$$\Delta = (y^2 - q)^2 - 4p_1^3 \tag{42}$$

Case $p_1 = 1$. We obtain $\Delta = (y^2 - q)^2 - 4$. $\Delta = -m^2$ with m integer, then $m^2 = 4 - (y^2 - q)^2 \Rightarrow m^2 + (y^2 - q)^2 = 2^2$, the solutions are:
** $m^2 = 4, y^2 - q = 0 \Rightarrow y^2 = q$. If q is a square, let $q = q_1^2$, then $y = \pm q_1$. We have also $x^3 - 3x = 0$. The only integer coordinates of the elliptic curve are:

$$(0, q_1), \quad (0, -q_1) \tag{43}$$

** $m^2 = 1, \quad y^2 - q = \sqrt{3}$ *or* $y^2 - q = -\sqrt{3}$
**-1- $y^2 - q = \sqrt{3}$, If $q = \sqrt{3}$, we have the equation $y^2 = x^3 - 3x + \sqrt{3}$ and $X^2 - \sqrt{3}X + 1 = 0$ and :

$$X_1 = \frac{\sqrt{3} + i}{2} = e^{\frac{i\pi}{6}} \tag{44}$$

$$X_2 = \frac{\sqrt{3} - i}{2} = e^{-\frac{i\pi}{6}} \tag{45}$$

u, v verify $u^3 = e^{\frac{i\pi}{6}}; v^3 = e^{-\frac{i\pi}{6}} \Longrightarrow |u_i| = 1$ and $|v_j| = 1$, $|x_k| = |u_i + v_k| = |2cos\frac{\pi}{18}| < 2 \Longrightarrow$ no integer coordinates if $q = \sqrt{3}$.

**-2- $y^2 - q = -\sqrt{3}$, we suppose that $q = -\sqrt{3}$ then $X^2 + \sqrt{3}X + 1 = 0$. We obtain :

$$X_1 = \frac{-\sqrt{3} + i}{2} = e^{\frac{i5\pi}{6}} \tag{46}$$

$$X_2 = \frac{-\sqrt{3} - i}{2} = e^{-\frac{i5\pi}{6}} \tag{47}$$

Using the same remark as above, we arrive to $|x_k| < 2$, with $|x_k| \neq 1$, then there are no integer coordinates when $q = -\sqrt{3}$.

Case $p_1 > 1$. We obtain $m^2 = 4p_1^3 - (y^2 - q)^2 \Longrightarrow m^2 + (y^2 - q)^2 = 4p_1^3$, then $\pm m, \pm(y^2 - q)$ are solutions of the Diophantine equation :

$$A^2 + B^2 = N \tag{48}$$

with $N = 4p_1^3$. The following theorem (theorem 36.3,[2]) gives the conditions to be verified by N:

Theorem 1. *The Diophantine equation:*

$$A^2 + B^2 = N \tag{49}$$

has a solution if and only if :

$$N = 2^\alpha p_1'^{h_1} ... p_k'^{h_k} . q_1^{2\beta_1} ... q_n^{2\beta_n} \tag{50}$$

where the p_i' are primes congruent to 1 modulo 4, and the q_j are prime congruent to 3 modulo 4. When N is of this form, equation (49) has :

$$N_S = \left[\frac{(h_1 + 1) \cdots (h_k + 1) + 1}{2} \right] \tag{51}$$

inequivalent solutions ($[x]$ is the largest integer less or equal to x.)

From the conditions given by the theorem above, $2 \nmid p_1$ and p_1 must be written as:

$$p_1 = p_1'^{3h_1} ... p_k'^{3h_k} . q_1^{6\beta_1} ... q_n^{6\beta_n} \tag{52}$$

$$and \quad p_1 \equiv 1 (\mathrm{mod} 4) \tag{53}$$

We suppose in the following, that equation (52) is true. We obtain:

$$\begin{cases} X_{1l} = \dfrac{y_l^2 - q + i m_l}{2} \\ \\ X_{2l} = \dfrac{y_l^2 - q - i m_l}{2} \end{cases} \quad l = 1, 2, .., N_S \tag{54}$$

To simplify the notation, we remove the indices l. The roots of the equation (8) are :

$$\begin{cases} X_1 = \dfrac{y^2 - q + i m}{2} \\ \\ X_2 = \dfrac{y^2 - q - i m}{2} \end{cases} \tag{55}$$

We have to resolve:

$$\begin{cases} u^3 = X_1 = \dfrac{y^2 - q + im}{2} \\ \\ v^3 = X_2 = \overline{X}_1 = \dfrac{y^2 - q - im}{2} \end{cases} \tag{56}$$

We write X_1 as $X_1 = \rho e^{i\theta}$ with:

$$\rho = \frac{\sqrt{(y^2 - q)^2 + m^2}}{2} = p_1\sqrt{p_1}; \quad sin\theta = \frac{\sqrt{-\Delta}}{2\rho} = \frac{m}{2\rho} > 0; \quad cos\theta = \frac{y^2 - q}{2\rho}$$

If $y^2 - q > 0 \Longrightarrow cos\theta > 0 \Longrightarrow 0 < \theta < \dfrac{\pi}{2}[2\pi] \Longrightarrow \dfrac{1}{4} < cos^2\dfrac{\theta}{3} < 1$.
If $y^2 - q < 0 \Longrightarrow cos\theta < 0$, then :

$$\frac{\pi}{2} < \theta < \pi[2\pi] \Longrightarrow \frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4} \tag{57}$$

A. We suppose that $y^2 - q > 0 \Longrightarrow 0 < \dfrac{\theta}{3} < \dfrac{\pi}{6}[2\pi] \Longrightarrow \dfrac{1}{4} < cos^2\dfrac{\theta}{3} < 1$**.**

Then the expression of X_2: $X_2 = \rho e^{-i\theta}$. Let :

$$u = re^{i\psi}, \quad \text{and } j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}}$$

The parameters u and v are:

$$\begin{cases} u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \\ u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \\ u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \end{cases}$$

$$\begin{cases} v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \\ v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \\ v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \end{cases}$$

We choose u_k and v_h so that $u_k + v_h$ is real. In this case, we have necessary :

$$v_1 = \overline{u}_1; \quad v_2 = \overline{u}_2; \quad v_3 = \overline{u}_3$$

Then, the three real solutions of the equation (2) are:

$$\begin{cases} x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3} \\ x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\ x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases} \tag{58}$$

The discussion of the integrity of x_1, x_2, x_3***:***
We suppose that x_1 is an integer, then x_1^2 is an integer. We obtain:

$$x_1^2 = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4p_1 cos^2\frac{\theta}{3} \tag{59}$$

We write $cos^2\dfrac{\theta}{3}$ as :

$$cos^2\frac{\theta}{3} = \frac{1}{a} \quad or \quad \frac{a}{b} \tag{60}$$

where a, b are relatively coprime integers.

$**$ $cos^2\dfrac{\theta}{3} = \dfrac{1}{a}$. In this case, $\dfrac{1}{4} < \dfrac{1}{a} < 1 \Longrightarrow 1 < a < 4 \Longrightarrow a = 2$ or $a = 3$.

Case $a = 2$, we obtain $x_1^2 = 4\sqrt[3]{\rho^2}cos^2\dfrac{\theta}{3} = 2p_1 \Longrightarrow 2|p_1$, but from (53) $2 \nmid p_1$, then the contradiction. We verify easily that x_2 and x_3 are irrationals.

Case $a = 4$, we obtain $x_1^2 = 4\sqrt[3]{\rho^2}cos^2\dfrac{\theta}{3} = 4p_1.\dfrac{1}{3}$. If $3 \nmid p_1 \Longrightarrow x_1^2$ is a rational. We suppose that $3|p_1$, then p_1 must be written as $p_1 = 3\omega^2$. From the equation (52), $p_1 \equiv 1 (\text{mod} 4)$, we deduce that $\omega^2 \equiv 3(\text{mod}4)$, as ω^2 is a square, $\omega^2 \equiv 0(\text{mod}4)$ or $\omega^2 \equiv 1(\text{mod}4)$, then x_1 can not be an integer. We verify easily that x_2, x_3 are also not integers.

$**$ $cos^2\dfrac{\theta}{3} = \dfrac{a}{b}$, *$a, b$ coprime with $a > 1$.* We obtain :

$$x_1^2 = 4p_1 cos^2\frac{\theta}{3} = \frac{4p_1 a}{b}$$

where b verifies the condition:

$$\boxed{b|4p_1} \tag{61}$$

and using the (57), we obtain a second condition:

$$\boxed{b < 4a < 3b} \tag{62}$$

A-1- $b=2 \Longrightarrow a=1 \Longrightarrow x_1^2=2p_1 \Longrightarrow 2|p_1$, but $p_1 \equiv 1 (\bmod 4)$ then case to reject.

A-2- $b=4 \Longrightarrow a=2$, a,b no coprime. Case to reject.

A-3- $b=2b'$ avec $2 \nmid b'$, then we obtain:

$$x_1^2 = \frac{4p_1 a}{b} = \frac{2p_1 a}{b'} \Rightarrow b'|p_1 \tag{63}$$

then $p_1 = b'^{\alpha} p_2$ with $\alpha \geq 1$ and $b' \nmid p_2$, we obtain $x_1^2 = 2b'^{\alpha-1}.p_2.a \Rightarrow 2|(p_2.a)$, but from (52) $2 \nmid p_1 \Rightarrow 2 \nmid p_2$ and $2 \nmid a$, if not a,b are not coprime. Then x_1^2 cannot be an square integer, the case $b=2b'$ is to reject.

A-4- $b=4b'$ avec $4 \nmid b'$, then we obtain:

$$x_1^2 = \frac{4p_1 a}{b} = \frac{p_1 a}{b'} \Rightarrow b'|p_1 \tag{64}$$

then $p_1 = b'^{\alpha} p_2$ with $\alpha \geq 1$ and $b' \nmid p_2$, we obtain $x_1^2 = b'^{\alpha-1}.p_2.a$.

* if $b'^{\alpha-1}.p_2.a = f^2$ a square then $x_1 = \pm f$, if not x_1 is not an integer. We consider that $x_1 = \epsilon f$ is an integer with $\epsilon = \pm 1$. As $x_1+x_2+x_3=0 \Longrightarrow x_2+x_3=-x_1$. From the equations given by (58) the product $x_2.x_3 = f^2-3p_1$, then x_2,x_3 are solutions of the equation:

$$\lambda^2 - \epsilon f \lambda + f^2 - 3p_1 = 0 \tag{65}$$

The discriminant of (65) is:

$$\delta = f^2 - 4(f^2-3p_1) = 12p_1 - 3f^2 = 3(4p_1 - f^2) = 3p_2 b'^{\alpha-1}(b-a) > 0$$

If δ is not a square, then x_2,x_3 are not integers. We suppose that $\delta = g^2$ a square. The real roots of (65) are:

$$\lambda_1 = \frac{\epsilon f + g}{2} \tag{66}$$

$$\lambda_2 = \frac{\epsilon f - g}{2} \tag{67}$$

From the expressions of f and g, we deduce that $2|f$ and $2|g$, then λ_1, λ_2 are integers.
We recall that y^2-q is supposed >0 and are determined by the equations (48-49-51), we obtain the integer coordinates $\in$ to the elliptic curve (E) :

$$\begin{aligned} &\text{For } l=1,2,...,N_S \\ &(f,y_l),(-f,y_l),(f,-y_l),(-f,-y_l), \\ &(\lambda_1,y_l),(\lambda_2,y_l),(\lambda_1,-y_l),(\lambda_2,-y_l), \\ &(-\lambda_1,y_l),(-\lambda_2,y_l),(-\lambda_1,-y_l),(-\lambda_2,-y_l) \end{aligned} \tag{68}$$

B. We suppose that $y^2 - q < 0 \Longrightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3}[2\pi]$
that gives :

$$\frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2} \Longrightarrow \frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}$$

$cos^2\frac{\theta}{3} = \frac{1}{a}$. In this case, $\frac{3}{4} < \frac{1}{a} < 1 \Longrightarrow 3a < 4$ which is impossible case to reject.

$cos^2\frac{\theta}{3} = \frac{a}{b}$. In this case, $\frac{3}{4} < \frac{a}{b} < 1 \Longrightarrow 3b < 4a$. Then we obtain:

$$x_1^2 = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4p_1cos^2\frac{\theta}{3} = \frac{4p_1a}{b} \Rightarrow b|(4p_1) \tag{69}$$

B-1- $b = 2 \Longrightarrow a = 1 \Longrightarrow 8 < 4$ case to reject.

B-2- $b = 4 \Longrightarrow 3 < a < 4$ case to reject.

B-3- $b = 2b'$ avec $2 \nmid b'$, then we obtain:

$$x_1^2 = \frac{4p_1a}{b} = \frac{2p_1a}{b'} \Rightarrow b'|p_1 \tag{70}$$

then $p_1 = b'^{\alpha}p_2$ with $\alpha \geq 1$ and $b' \nmid p_2$, we obtain $x_1^2 = 2b'^{\alpha-1}.p_2.a$.

* if $2b'^{\alpha-1}.p_2.a = f^2$ a square then $x_1 = \pm f$, if not x_1 is not an integer. We consider that $x_1 = \epsilon f$ is an integer with $\epsilon = \pm 1$. As $x_1 + x_2 + x_3 = 0 \Longrightarrow x_2 + x_3 = -x_1$. The product $x_2.x_3 = f^2 - 3p_1$, then x_2, x_3 are solutions of the equation:

$$\lambda^2 - \epsilon f\lambda + f^2 - 3p_1 = 0 \tag{71}$$

The discriminant of (71) is:

$$\delta = f^2 - 4(f^2 - 3p_1) = 12p_1 - 3f^2 = 3(4p_1 - f^2) = 2p_2b'^{\alpha-1}(b-a) > 0$$

If δ is not a square, then x_2, x_3 are not integers. We suppose that $\delta = g^2$ a square. The real roots of (71) are:

$$\lambda_1 = \frac{\epsilon f + g}{2} \tag{72}$$

$$\lambda_2 = \frac{\epsilon f - g}{2} \tag{73}$$

From the expressions of f and g, we deduce that $2|f$ and $2|g$, then λ_1, λ_2 are integers.

B-4- $b = 4b'$ avec $4 \nmid b'$, then we obtain:

$$x_1^2 = \frac{4p_1 a}{b} = \frac{p_1 a}{b'} \Rightarrow b'|p_1 \tag{74}$$

then $p_1 = b'^{\alpha} p_2$ with $\alpha \geq 1$ and $b' \nmid p_2$, we obtain $x_1^2 = b'^{\alpha-1}.p_2.a$.

* if $b'^{\alpha-1}.p_2.a = f^2$ a square then $x_1 = \pm f$, if not x_1 is not an integer. We consider that $x_1 = \epsilon f$ is an integer with $\epsilon = \pm 1$. As $x_1 + x_2 + x_3 = 0 \Longrightarrow x_2 + x_3 = -x_1$. The product $x_2.x_3 = f^2 - 3p_1$, then x_2, x_3 are solutions of the equation:

$$\lambda^2 - \epsilon f \lambda + f^2 - 3p_1 = 0 \tag{75}$$

The discriminant of (75) is:

$$\delta = f^2 - 4(f^2 - 3p_1) = 12p_1 - 3f^2 = 3(4p_1 - f^2) = 2p_2 b'^{\alpha-1}(b-a) > 0$$

If δ is not a square, then x_2, x_3 are not integers. We suppose that $\delta = g^2$ a square. The real roots of (75) are:

$$\lambda_1 = \frac{\epsilon f + g}{2} \tag{76}$$

$$\lambda_2 = \frac{\epsilon f - g}{2} \tag{77}$$

From the expressions of f and g, we deduce that $2|f$ and $2|g$, then λ_1, λ_2 are integers.

We recall that $y^2 - q$ is supposed < 0 and are determined by the equations (48-49-51), we obtain the integer coordinates $\in$ to the elliptic curve (E) :

$$\begin{aligned} &\text{For } l = 1, 2, ..., N_S \\ &(f, y_l), (-f, y_l), (f, -y_l), (-f, -y_l), \\ &(\lambda_1, y_l), (\lambda_2, y_l), (\lambda_1, -y_l), (\lambda_2, -y_l), \\ &(-\lambda_1, y_l), (-\lambda_2, y_l), (-\lambda_1, -y_l), (-\lambda_2, -y_l) \end{aligned} \tag{78}$$

2.3.2 We suppose $3 \nmid p'$:

Then $\Delta = (y^2 - q)^2 - \dfrac{4p'^3}{27} = -m^2$ where $m > 0$ is a real. As in paragraph 2.2.2 above, we find the same results there are no integers coordinates of the elliptic curve (E). □

Declarations:

- The author declares no conflicts of interest.
- No funds, grants, or other support was received.
- The author declares he has no financial interests.

- ORCID - ID:0000-0002-9633-3330.

References

[1] B.M. Stewart : Theory of numbers. 2sd ed. The Macmillan Company, New-York (1964).

[2] E.D. Bolker : Elementary number theory: an algebraic approach. W.A. Benjamin, Inc., New-York (1970).