Finite dimensional series and its applications in number theory and mathematical analysis

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ABSTRACT

In this study we used an algebraic method that uses elementary algebra and binomial theorem. To create finite binomial series $L_n(k, g, u) = {V_n}^n(k, g, u) + S_n(k, g)$.This is a type of series that has several properties in variables such as if $u = 1$ then $L_n(k, g, 1) = S_n(k, g)$ where ${V_n}^q(k, g, 1) = 0$. We used these series to prove results in congruences and primitive roots, Diophantine equation for example,we proved if $a^n \equiv 1(\text{mod } m)$ where $m = ad - 1$ then $d^n \equiv 1(\text{mod } m)$, $a, d, n \in \mathbb{N}$. And in primitives to roots, such as , if $m = da_j - 1$ where $a_1, a_2, a_3 \dots \dots \dots . a_j$ is a primitive roots modulo m then $\left(\frac{m+1}{a_1}\right)_1, \left(\frac{m+1}{a_2}\right)_2, \left(\frac{m+1}{a_3}\right)_3 \dots \dots \dots \left(\frac{m+1}{a_j}\right)_i$ all other primitive roots modulo m where $j = i \text{ and } 1 \leq j \leq \frac{\varphi(\varphi(m))}{2}$.We also obtained several results in finite series.

Key words: binomial theorem, Diophantine equation , finite series , primitive roots , congruences

1.INTRODUCTION

In this paper, elementary algebra and binomial theorem, and difference of tow nth power are used to created finite series in an algebraic method,to create a kind of series with specific properties, then we used series to create congruence with specific properties. Through this process, we reached the theorem. 1 theorem.2 in primitive roots and several results in finite series.

The goal of this paper is to construct a kind of finite binomial series it is a binomial and its application in the study of congruences, it was used to prove the theorem.1. and in primitive roots it was used to prove theorem.2 in primitive according binomial theorem and difference of tow nth power theorem if n a positive integer and x y real numbers then $[see\ K.H\ 22]$

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^j y^{n-j}$$

And

$$x^n - y^n = (x-y)\sum_{j=1}^{n} x^{n-j} y^{j-1}$$

2.basic series

This section first we will create the basic series

Basic series. let $k, g, u,$ *real numbers and m constant then*

$$L_n(k,g,u) = V_n{}^n(k,g,u) + S_n(k,g)$$

Where

$$L_n\ (k,g,u) = (u-k+g)^n - m(1+g)^n$$

And

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$$V_n{}^n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j$$

And

$$S_n(k,g) = mk \sum_{j=0}^{n-1} (-1)^{j+1} \frac{(k-g)^j}{g^{j+1}} \left((1+g)^n - \left(\binom{n}{0} + \binom{n}{1} g + \binom{n}{2} g^2 \ldots\ldots\ldots. \binom{n}{j} g^j \right) \right)$$

First dimension series. Let u k g real numbers and n positive integer where m constant then

$$A_n(k,u,g) = W_n(k,u,g) + E_n(k,u,g)$$

Where

$$A_n(u,k,g) = (1+u-k+g)^n - m(2+g)^n$$

And

$$W_n(k,u,g) = \sum_{j=r=0}^{n} \sum_{r=j}^{n} (-1)^j (k-g)^j \left(\binom{n}{r} \binom{r}{j} \left(u^{r-j} - m\right) \right)$$

And

$$E_n(u,k,g) = km \sum_{r=j=1}^{n} \sum_{r=j}^{n} (-1)^j \frac{(k-g)^{j-1}}{g^j} \binom{n}{r} \left((1+g)^r - \binom{r}{0} - \binom{r}{1} \mathrm{g} \ldots\ldots \binom{r}{j-1} g^{j-1} \right)$$

Second dimension series. Let u k g real numbers where n positive integer m constant then

$$R_n(u,k,g) = H_n(u,k,g) + Y_n(k,u,g)$$

Where

$$R_n(u,k,g) = (2+u-k+g)^n - m(3+g)^n$$

And

$$H_n(u,k,g) = S_n(u,m) + \sum_{h=0}^{1} \sum_{r=j=h}^{n} \sum_{r=j}^{n} \binom{n}{r} \binom{r}{j} \binom{j}{jh-h} (k-g)^{j-h} (u^h - m)$$

$$+ \sum_{h=0}^{n-2} \sum_{r=j=h+2}^{n} \sum_{r=j}^{n} \binom{n}{r} \binom{r}{j} \binom{j}{h} (k-g)^h \left(u^{j-h} - m\right)$$

And

$$Y_n(k,u,g) = km\sum_{r=j=1}^{n}\sum_{r=j}^{n}(-1)^j\binom{n}{r}\binom{r}{j}\frac{(k-g)^{j-1}}{g^j}\Big((1+g)^j-\binom{j}{0}$$
$$-\binom{j}{1}g\,\dots\dots\dots\binom{j}{j-1}g^{j-1}\Big)$$
$$+\,km\sum_{h=2}^{n}\sum_{r=j=h}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{h}\frac{(-1)^{h-1}(k-g)^{h-2}}{g^{h-1}}\Big((1+g)^h-\binom{h}{0}$$
$$-\binom{h}{1}g\,\dots\dots\dots.\binom{h}{h-2}g^{h-2}\Big)$$

Proof. let $g\ k\ u$, real numbers then according to difference of tow nth power theorem we have that

$$(k-g)^n-(-g)^n = \mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Then

$$-(-g)^n = -(k-g)^n+\mathrm{k}\sum_{j=1}^{n}f^{j-1}(k,h)g^{n-j}(-h)$$

let $q\in R$, $n\in N$ where m constant then by multiplying m and adding $u^q(k-g)^n$ from both sides

$$u^q(k-g)^n-m(-g)^n = u^q(k-g)^n-m(k-g)^n+\mathrm{m\,k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Then

(1) $$u^q(k-g)^n-m(-g)^n = (u^q-m)(k-g)^n\ +m\,\mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

According binomial theorem

$$(u-k+g)^n = u^n-\binom{n}{1}u^{n-1}(k-g)+\binom{n}{2}u^{n-2}(k-g)^2$$
$$-\binom{n}{3}u^{n-3}(k-g)^3\,\dots\dots\dots\dots\dots\dots\dots\,(k-g)^n$$

And

$$m(1+g)^n = m+m\binom{n}{1}g+m\binom{n}{2}g^2+m\binom{n}{3}g^3\,\dots\dots\dots\dots\dots.mg^n$$

By subtracting $m(k-g)^n\ from\ (u-k+g)^n$ then

$$(u-k+g)^n-m(1+g)^n$$
$$= u^n-m-\binom{n}{1}u^{n-1}(k-g)-m\binom{n}{1}g+\binom{n}{2}u^{n-2}(k-g)^2-m\binom{n}{2}g^2$$
$$-\binom{n}{3}u^{n-3}(k-g)^3-m\binom{n}{3}g^3\,\dots\dots\dots\dots\dots..(k-g)^n-mg^n$$

By extracting the common factor $\binom{n}{j}$ between the terms

(2) $$(u-k+g)^n-m(1+g)^n$$
$$= u^n-m-\binom{n}{1}(u^{n-1}(k-g)+mg)+\binom{n}{2}(u^{n-2}(k-g)^2-mg^2)$$
$$-\binom{n}{3}(u^{n-3}(k-g)^3+mg^3)\,\dots\dots\dots\dots((k-g)^n-mg^n)$$

So we note in (2) limit (1) equal $u^n - m$ and limit (2) equal $\binom{n}{1}(u^{n-1}(k-g)+mg)$ and limit 2 equal $\binom{n}{2}(u^{n-2}(k-g)^2-mg^2)$ and 3 equal $\binom{n}{3}(u^{n-3}(k-g)^3+mg^3)$ and last limit $(k-g)^n - mg^n$ then

According equation (1)

$$u^q(k-g)^n - m(-g)^n = (u^q-m)(k-g)^n + mk\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

let

(3) $$w^q{}_n(k,g,u) = u^q(k-g)^n - m(-g)^n$$

(4) $$z^q{}_n(k,g,u) = (u^q-m)(k-g)^n$$

(5) $$c_n(k,g) = m\mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

So

(4) $$W_n{}^q(k,g,u) = Z_n{}^q(\mathrm{k},g,\mathrm{u}) + C_n(k,g)$$

From (3) and limit (1) in equation (2)

$$\binom{n}{0}(u^n-m) = \binom{n}{0}W_0{}^{n-0}(k,g,u)$$

From equation (3) and Limit (2) in equation (2)

$$\binom{n}{1}(u^{n-1}(k-g)+mg) = \binom{n}{1}W_1{}^{n-1}(k,g,u)$$

Limit (3)

$$\binom{n}{2}(u^{n-2}(k-g)^2-mg^2) = \binom{n}{2}W_2{}^{n-2}(k,g,u)$$

Limit (4) in equation (2)

$$\binom{n}{3}(u^{n-3}(k-g)^3+mg^3) = \binom{n}{3}W_3{}^{n-3}(k,g,u)$$

Last limit

$$\binom{n}{n}((k-g)^n-mg^n) = \binom{n}{n}W_n{}^{n-n}(k,g,u)$$

So from (1) and (3) equations

$$(u-k+g)^n - m(1+g)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}w^{n-j}{}_j(k,g,u)$$

Let

$$L_n(k,g,u) = (u-k+g)^n - m(1+g)^n$$

Then

$$L_n(k,g,u) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}w^{n-j}{}_j(k,g,u)$$

From equation(4) $w^q{}_n(k,g,u) = z_n{}^q(k,g,u) + c_n(k,g)$ then we have that

(5) $$L_n(k,g,u) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}z_j{}^{n-j}(k,g,u) + \sum_{j=0}^{n}(-1)^j\binom{n}{j}c_j(k,g)$$

We note from the equation (4)

$$Z_n{}^q(\mathrm{k},g,\mathrm{u}) = (u^q-m)(k-g)^n$$

And

$$C_n(k,g) = m\,k\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Then

$$L_n(k,g,u) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(u^{n-j}-m\right)(k-g)^j + \mathrm{mk}\sum_{j=1}^{n}\sum_{r=1}^{j}(-1)^j\binom{n}{j}(k-g)^{r-1}(-g)^{j-r}$$

Let

$$V^n{}_n(k,g,u) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(u^{n-j}-m\right)(k-g)^j$$

And

$$S_n(k,g) = \mathrm{mk}\sum_{j=1}^{n}\sum_{r=1}^{j}(-1)^j\binom{n}{j}(k-g)^{r-1}(-g)^{j-r}$$

Then we have that

$$L_n(k,g,u) = V_n{}^n(k,g,u) + S_n(k,g) \tag{6}$$

we find in $S_n(k,g)$ tow signs $(-1)^j(-1)^{j-r} = (-1)^r$ if r j even or odd so they can by combined in $(-1)^r$ then we have

$$S_n(k,g) = \mathrm{mk}\sum_{j=1}^{n}\sum_{r=1}^{j}(-1)^r\binom{n}{j}(k-g)^{r-1}g^{j-r}$$

Where

$$s_n(k,g) = \mathrm{mk}\Bigg(\sum_{r=1}^{1}(-1)^r\binom{n}{1}(k-g)^{r-1}g^{1-r} + \sum_{r=1}^{2}(-1)^r\binom{n}{2}(k-g)^{r-1}g^{2-r} + \sum_{r=1}^{3}(-1)^r\binom{n}{3}(k-g)^{r-1}g^{3-r} \ldots\ldots\ldots\ldots. \sum_{r=1}^{n}(-1)^r\binom{n}{n}(k-g)^{r-1}g^{n-r}\Bigg)$$

In $S_n(k,g)$ a all compound terms have been dismantled note if we add for every first term in the complex term we find that $-\left(\binom{n}{1}+\binom{n}{2}g\ldots\ldots\ldots\binom{n}{n}g^{n-1}\right)$ then we adding the terms to include that $(k-g)$ finding that $(k-g)$ $\left(\binom{n}{2}+\binom{n}{3}g\ldots\ldots.\binom{n}{n}g^{n-2}\right)$ then the term that include $(k-g)^2$ we find that $(k-g)^2\left(-\left(\binom{n}{3}+\binom{n}{4}g\ldots\ldots..\binom{n}{n}g^{n-j-1}\right)\right)$ if the method is equal all the terms can be added $1 \le j \le n-1$ until we reach the last terms $(k-g)^{n-1}$ then

$$\begin{aligned} s_n(k,g) = \mathrm{mk}\Bigg(&-\left(\binom{n}{1}+\binom{n}{2}g+\binom{n}{3}g^2\ldots\ldots\ldots\binom{n}{n}g^{n-1}\right) \\ &+ (k-\mathrm{g})\left(\left(\binom{n}{2}+\binom{n}{3}g+\binom{n}{4}g^2+\binom{n}{5}g^3\ldots\ldots\ldots.\binom{n}{n}g^{n-2}\right)\right) \\ &- (k-g)^2\left(\binom{n}{3}+\binom{n}{5}g+\binom{n}{6}g^2+\binom{n}{7}g^3\ldots\ldots\ldots.\binom{n}{n}g^{n-3}\right)\ldots\ldots\ldots.\binom{n}{n}g^{n-n}\Bigg) \end{aligned}$$

Using the binomial theorem it is possible to abbreviate all the terms that include, $(k-g)$ *and* $(k-g)^2$ and $(k-g)^3$until we reach the last term $(k-g)^{n-1}$, we notice that

$$-\left(\binom{n}{1}+\binom{n}{2}g+\binom{n}{3}g^2\ldots\ldots\ldots.\binom{n}{n}g^{n-1}\right)=-\frac{(1+g)^n-\binom{n}{0}}{g}$$

$$(k-g)\left(\binom{n}{2}+\binom{n}{3}\mathrm{g}\ldots\ldots\ldots.\binom{n}{n}g^{n-2}\right)=(k-g)\left(\frac{(1+g)^n-\binom{n}{0}-\binom{n}{1}g}{g^2}\right)$$

$$-(k-g)^2\left(\binom{n}{3}+\binom{n}{4}g\ \ldots\ldots\ldots.\binom{n}{n}g^{n-3}\right)=-(k-g)^2\left(\frac{(1+g)^n-\binom{n}{0}-\binom{n}{1}g-\binom{n}{2}g^2}{g^3}\right)$$

Then

$$S_n(k,g)=\mathrm{m}k\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}g+\binom{n}{2}g^2\ldots\ldots\ldots..\binom{n}{j}g^j\right)\right)$$

we have that

(7) $$L_n(k,g,u)=(u-\mathrm{k}+\mathrm{g})^n-m(1+g)^n$$

(8) $$V^n{}_n(k,g,u)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(u^{n-j}-m\right)(k-g)^j$$

(9) $$S_n(k,g)$$

$$=\mathrm{mk}\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}g+\binom{n}{2}g^2\ldots\ldots\ldots\ldots\binom{n}{j}g^j\right)\right)$$

Proof first dimension series. from binomial theorem we find then

$$\begin{aligned}(1+u-k+g)^n-m(1+1+g)^n\\ &=1-m+\binom{n}{1}(u-k+g)-m(1+g)\binom{n}{1}+\binom{n}{2}(u-k+g)^2\\ &-m\binom{n}{2}(1+g)^2+\binom{n}{3}(u-k+g)^3\\ &-m\binom{n}{33}(1+g)^3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\binom{n}{n}(u-k+g)^n-m\binom{n}{n}(1+g)^n\end{aligned}$$

And

$$m(1+1+g)^n=m+\binom{n}{1}(1+g)+\binom{n}{2}(1+g)^2+\binom{n}{3}(1+g)^3\ldots\ldots\ldots.(1+g)^n$$

By subtracting $m(2+g)^n\ from\ (1+u-k+g)^n$ we have that

(10) $$\begin{aligned}(1+u-k+g)^n-m(2+g)^n\\ &=1-m+\binom{n}{1}\left((u-k+g)-m(1+g)\right)+\binom{n}{2}((u-k+g)^2-m(1+g)^2)\\ &+\binom{n}{3}((u-k+g)^3-m(1+g)^3)\ldots\ldots\ldots\ldots\ldots\ \ \ldots\ldots.\binom{n}{n}((u-k+g)^n\\ &-m(1+g)^n)\end{aligned}$$

So we note in equation (10) first term equal $(1-m)$ and term (2) equal $\binom{n}{1}\left((u-k+g)-m(1+g)\right)$ and term (3) equal $\binom{n}{2}((u-k+g)^2-m(1+g)^2)$ and (4) equal $\binom{n}{3}((u-k+g)^3-m(1+g)^3)$ and last term equal $\binom{n}{n}((u-k+g)^n-m(1+g)^n)$ we have according series

$$L_n(k,g,u) = (u-k+g)^n - m(1+g)^n$$
And
$$V^n{}_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j$$
And
$$S_n(k,g) = \mathrm{mk} \sum_{j=0}^{n-1} (-1)^{j+1} \frac{(k-g)^j}{g^{j+1}} \left((1+g)^n - \left(\binom{n}{0} + \binom{n}{1} g + \binom{n}{2} g^2 \dots\dots\dots\dots \binom{n}{j} g^j \right) \right)$$
Then we find that

From Term (1) in equation (10) and series

$$1 - m = L_0(k,g,u)$$
From term (2) in equation (10) and series

$$\binom{n}{1} \left((u-k+g) - m(1+g) \right) = \binom{n}{1} L_1(k,g,u)$$
From term (3) in equation (10) and series

$$\binom{n}{2} ((u-k+g)^2 - m(1+g)^2) = \binom{n}{2} L_2(k,g,u)$$
Last term in equation (10) and series

$$\binom{n}{n} ((u-k+g)^n - m(1+g)^n) = \binom{n}{n} L_n(k,u,g)$$
So
$$(1+u-k+g)^n - m(2+g)^n = 1 - m + \sum_{j=1}^{n} \binom{n}{j} L_j(u,k,g)$$
Let
$$A_n(u,k,g) = (1+u-k+g)^n - m(2+g)^n$$
Then
$$A_n(u,k,g) = 1 - m + \sum_{j=1}^{n} \binom{n}{j} L_j(u,k,g)$$
According series
$$L_n(u,k,g) = V^n{}_n(u,k,g) + S_n(k,g)$$
Then
$$A_n(u,g,k) = 1 - m + \sum_{j=1}^{n} \binom{n}{j} V^n{}_j(u,k,g) + \sum_{j=1}^{n} \binom{n}{j} S_j(k,g) \qquad (11)$$
from equation (8) and (9) we have that
$$V^n{}_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j$$
And
$$S_n(k,g) = \mathrm{mk} \sum_{j=0}^{n-1} (-1)^{j+1} \frac{(k-g)^j}{g^{j+1}} \left((1+g)^n - \left(\binom{n}{0} + \binom{n}{1} g + \binom{n}{2} g^2 \dots\dots\dots\dots \binom{n}{j} g^j \right) \right)$$
By compensation equations (8) and (9) in equation (11) we find that

$$A_n(u,g,k) = 1 - m + \sum_{j=1}^{n}\sum_{r=0}^{j}(-1)^r \binom{n}{j}\binom{j}{r}\left(u^{j-r} - m\right)(k-g)^r$$
$$+ mk \sum_{j=1}^{n}\sum_{r=0}^{j-1}(-1)^{r+1}\binom{n}{j}\left(\frac{(k-g)^r}{g^{r+1}}\right)\left((1+g)^j - \binom{j}{0} - \binom{j}{1} g \ldots\ldots\ldots..\binom{j}{r} g^r\right)$$

Let

$$W_n(u,k,g) = 1 - m + \sum_{j=1}^{n}\sum_{r=0}^{j}(-1)^r \binom{n}{j}\binom{j}{r}\left(u^{j-1} - m\right)(k-g)^r$$

And

$$E_n(u,k,g) = km \sum_{j=1}^{n}\sum_{r=0}^{j-1}(-1)^{r+1}\frac{(k-g)^r}{g^{r+1}}\binom{n}{j}\left((1+g)^j - \binom{j}{0} - \binom{j}{1} g \ldots\ldots\ldots\ldots\binom{j}{r} g^r\right)$$

Then

$$A_n(u,k,g) + W_n(u,k,g) + E_n(u,k,g) \tag{12}$$

So

$$W_n(u,k,g) = 1 - m + \sum_{r=0}^{1}(-1)^r\binom{n}{1}\binom{1}{r}(u^{1-r} - m)(k-g)^r$$
$$+ \sum_{r=0}^{2}(-1)^r\binom{n}{2}\binom{2}{r}(u^{2-r} - m)(k-g)^r + \sum_{r=0}^{3}(-1)^r\binom{n}{3}\binom{3}{r}(u^{3-r} - m)(k-g)^r$$
$$+ \sum_{r=0}^{4}(-1)^r\binom{n}{4}\binom{4}{r}(u^{4-r} - m)(k-g)^r \ldots\ldots\ldots\ldots\ldots\ldots\ldots \sum_{r=0}^{n}(-1)^r\binom{n}{n}\binom{n}{r}(u^{n-r} - m)(k-g)^r$$

In this step, we arrange the boundaries between which there is a common relationship and bring them together then we have that

Collect all limits that include $r = 0$

$$\binom{n}{1}(u-m) + \binom{n}{2}(u^2 - m)\ldots\ldots\ldots\ldots.\binom{n}{n}(u^n - m) = \sum_{r=1}^{n}\binom{n}{r}(u^r - m)$$

By adding all terms include $r = 1$ $and\ factor\ (k-g)$
then

$$-\binom{n}{1}(1-m)(k-g) - \binom{n}{2}\binom{2}{1}(u-m)(k-g)$$
$$-\binom{n}{3}\binom{3}{1}(u^2-m)(k-g)\ldots\ldots\ldots..-\binom{n}{n}\binom{n}{1}(u^{n-1}-m)(k-g)$$
$$= (k-g)\sum_{r=1}^{n}\binom{n}{r}\binom{r}{1}(u^{r-1}-m)$$

By adding all terms include $r = 2$ $and\ \ factor\ (k-g)^2$

$$\binom{n}{2}\binom{2}{2}(1-m)(k-g)^2+\binom{n}{3}\binom{3}{2}(u-m)(k-g)^2$$
$$+\binom{n}{4}\binom{4}{2}(u^2-m)(k-g)^2 \dots\dots\dots\dots \binom{n}{n}\binom{n}{2}(u^{n-2}-m)(k-g)^2$$
$$=(k-g)^2\sum_{r=2}^{n}\binom{n}{r}\binom{r}{2}(u^{r-2}-m)$$

By adding all terms include $r=3\ and\ factor\ (k-g)^3\ then$

$$-\binom{n}{3}\binom{3}{3}(1-m)(k-g)^3-\binom{n}{4}\binom{4}{3}(u-m)(k-g)^3$$
$$-\binom{n}{5}\binom{5}{3}(u^2-m)(k-g)^3 \dots\dots\dots\dots\dots\dots \binom{n}{n}\binom{n}{3}(u^{n-3}-m)(k-g)^3$$
$$=(k-g)^3\sum_{r=3}^{n}\binom{n}{r}\binom{r}{3}(u^{r-3}-m)$$

We note that we can add all the terms that include $r=0$ together and then include $r=1$ together until we reach the final term $r=n$ when
last term equal

$$(k-g)^n(1-m)=(k-g)^n\sum_{r=n}^{n}\binom{n}{n}\binom{n}{r}(u^{n-r}-m)$$

So the final equation is equal to after the order of the terms then we find that

$$W_n(k,u,g)=\sum_{j=r=0}^{n}\sum_{r=j}^{n}(-1)^j(k-g)^j\left(\binom{n}{r}\binom{r}{j}(u^{r-j}-m)\right)$$

According equation (12) we have that

$$E_n(u,k,g)=km\sum_{j=1}^{n}\sum_{r=0}^{j-1}(-1)^{r+1}\frac{(k-g)^r}{g^{r+1}}\binom{n}{j}\left((1+g)^j-\binom{j}{r}-\binom{j}{r}g-\binom{j}{r}g^2 \dots\dots\dots \binom{j}{r}g^r\right)$$

Then

$$E_n(u,k,g)=km\sum_{r=0}^{0}(-1)^{r+1}\frac{(k-g)^r}{g^{r+1}}\binom{n}{1}\left((1+g)^1-\binom{1}{0}-\binom{1}{1}g \dots\dots \binom{1}{r}g^r\right)$$
$$+km\sum_{r=0}^{1}(-1)^{r+1}\frac{(k-g)^r}{g^{r+1}}\binom{n}{2}\left((1+g)^2-\binom{2}{0}-\binom{2}{1}g \dots \binom{2}{r}g^r\right)$$
$$+km\sum_{r=0}^{2}(-1)^{r+1}\frac{(k-g)^r}{g^{r+1}}\binom{n}{3}\left((1+g)^3-\binom{3}{0}\right.$$
$$\left.-\binom{3}{1}g \dots\dots \binom{3}{r}g^r\right) \dots\dots\dots\dots\dots\dots km\sum_{r=0}^{n-1}(-1)^{r+1}\frac{(k-g)^r}{g^{r+1}}\binom{n}{n}\left((1+g)^n-\binom{n}{0}\right.$$
$$\left.-\binom{n}{1}g \dots\dots \binom{n}{r}g^r\right)$$

Then

By adding terms include $\frac{1}{g}$ we find that

$$-km\binom{n}{1}\frac{1}{g}\big((1+g)-1\big)-km\binom{n}{2}\frac{1}{g}((1+g)^2-1)-km\binom{n}{3}\frac{1}{g}((1+g)^3-1)\dots\dots\dots$$
$$-km\binom{n}{n}\frac{1}{g}((1+g)^n-1)=-km\sum_{r=1}^{n}\binom{n}{r}\frac{1}{g}((1+g)^r-1)$$

By adding terms include $\frac{(k-g)}{g^2}$ we have

$$km\binom{n}{2}\frac{(k-g)}{g^2}\left((1+g)^2-\binom{2}{0}-\binom{2}{1}g\right)+km\binom{n}{3}\frac{(k-g)}{g^2}\left((1+g)^3-\binom{3}{0}-\binom{3}{1}g\right)$$
$$+km\binom{n}{4}\frac{(k-g)}{g^2}\left((1+g)^4-\binom{4}{0}\right.$$
$$\left.-\binom{4}{1}g\right)\dots\dots\dots\dots\dots\dots km\binom{n}{n}\frac{(k-g)}{g^2}\left((1+g)^n-\binom{n}{0}-\binom{n}{1}g\right)$$
$$=km\sum_{r=2}^{n}\binom{n}{r}\frac{(k-g)}{g^2}\left((1+g)^r-\binom{r}{0}-\binom{r}{1}g\right)$$

By adding terms include $\frac{(k-g)^2}{g^3}$ we find that

$$-km\binom{n}{3}\frac{(k-g)^2}{g^3}\left((1+g)^3-\binom{3}{0}-\binom{3}{1}g-\binom{3}{2}g^2\right)$$
$$-km\binom{n}{4}\frac{(k-g)^2}{g^3}\left((1+g)^4-\binom{4}{0}-\binom{4}{1}g-\binom{4}{2}g^2\right)$$
$$-km\binom{n}{5}\frac{(k-g)^2}{g^3}\left((1+g)^5-\binom{5}{0}-\binom{5}{1}g\right.$$
$$\left.-\binom{5}{2}g^2\right)\dots\dots\dots\dots\dots\dots.km\binom{n}{n}\frac{(k-g)^2}{g^3}\left((1+g)^n-\binom{n}{0}-\binom{n}{1}g-\binom{n}{2}g^2\right)$$
$$=-km\sum_{r=3}^{n}\binom{n}{r}\frac{(k-g)^2}{g^3}\left((1+g)^r-\binom{r}{0}-\binom{r}{1}g-\binom{r}{2}g^2\right)$$

We note that we can add all the terms that include $\frac{1}{g}$ together and then include $\frac{1}{g^2}$ together until we reach the final term when $\frac{1}{g^n}$ and So the final equation is equal to after the order of the terms

$$E_n(u,k,g)=km\sum_{r=j=1}^{n}\sum_{r=j}^{n}(-1)^j\frac{(k-g)^{j-1}}{g^j}\binom{n}{r}\left((1+g)^r-\binom{r}{0}-\binom{r}{1}\mathrm{g}\dots\dots\binom{r}{j-1}g^{j-1}\right)$$

Then

$$A_n(u,k,g)+W_n(u,k,g)+E_n(u,k,g)$$

Where

$$W_n(u,k,g)=\sum_{j=r=0}^{n}\sum_{r=j}^{n}(-1)^j(k-g)^j\left(\binom{n}{r}\binom{r}{j}\left(u^{r-j}-m\right)\right)$$

And

$$E_n(u,k,g)=km\sum_{r=j=1}^{n}\sum_{r=j}^{n}(-1)^j\frac{(k-g)^{j-1}}{g^j}\binom{n}{r}\left((1+g)^r-\binom{r}{0}-\binom{r}{1}\mathrm{g}\dots\dots\binom{r}{j-1}g^{j-1}\right)$$

Proof second dimension series. from binomial theorem we find then

$$(1+1+u-k+g)^n - m(1+2+g)^n$$
$$= 1-m+\binom{n}{1}(1+u-k+g) - m\binom{n}{1}(2+g) + \binom{n}{2}(1+u-k+g)^2$$
$$- m\binom{n}{2}(2+g)^2 + \binom{n}{3}(1+u-k+g)^3$$
$$- m\binom{n}{3}(2+g)^3 \ldots\ldots\ldots\ldots\ldots\ldots \binom{n}{n}(1+u-k+g)^n - m\binom{n}{n}(2+g)^n$$

And

$$m(3+g)^n = m + \binom{n}{1}(2+g) + \binom{n}{2}(2+g)^2 + \binom{n}{3}(2+g)^3 \ldots\ldots\ldots.(2+g)^n$$

By subtracting $m(3+g)^n$ from $(2+u-k+g)^n$ we have that

$$(2+u-k+g)^n - m(3+g)^n \tag{13}$$
$$= 1-m+\binom{n}{1}\big((1+u-k+g)-m(2+g)\big)$$
$$+\binom{n}{2}((1+u-k+g)^2 - m(2+g)^2)$$
$$+\binom{n}{3}((1+u-k+g)^3 - m(2+g)^3)\ldots\ldots\ldots\ldots\ldots\binom{n}{n}((1+u-k+g)^n$$
$$- m(2+g)^n)$$

So we note in equation (13) first term equal $1-m$ and term (2) equal $\binom{n}{1}\big((1+u-k+g) - m(2+g)\big)$ and term (3) equal $\binom{n}{2}((1+u-k+g)^2 - m(2+g)^2)$ and term (4) equal $\binom{n}{3}((1+u-k+g)^3 - m(2+g)^3)$ and last term equal $\binom{n}{n}((1+u-k+g)^n - m(2+g)^n)$ then

According series (2) we find that

$$A_n(u,k,g) + W_n(u,k,g) + E_n(u,k,g)$$

where

$$A_n(u,k,g) = (1+u-k+g)^n - m(2+g)^n$$

And

$$W_n(u,k,g) = \sum_{j=r=0}^{n}\sum_{r=j}^{n}(-1)^j(k-g)^j\left(\binom{n}{r}\binom{r}{j}\left(u^{r-j}-m\right)\right)$$

And

$$E_n(u,k,g) = km\sum_{r=j=1}^{n}\sum_{r=j}^{n}(-1)^j\frac{(k-g)^{j-1}}{g^j}\binom{n}{r}\left((1+g)^r - \binom{r}{0} - \binom{r}{1}\mathrm{g}\ldots\ldots.\binom{r}{j-1}g^{j-1}\right)$$

From term (1) in equation (13) and equation series

$$1-m = \binom{n}{0}A_0(u,k,g)$$

From term (2) in equation (13) and equation series

$$\binom{n}{1}\big((1+u-k+g) - m(2+g)\big) = \binom{n}{1}A_1(u,g,k)$$

Term (3) in equation (13) and equation series

$$\binom{n}{2}((1+u-k+g)^2 - m(2+g)^2) = \binom{n}{2}A_2(k,u,g)$$

From term (4) in equation (13) and equation

$$\binom{n}{3}((1+u-k+g)^3-m(2+g)^3)=\binom{n}{3}A_3(k,u,g)$$

Last term in equation (13) and equation

$$\binom{n}{n}((1+u-k+g)^n-m(2+g)^n)=\binom{n}{n}A_n(k,u,g)$$

Then

$$(2+u-k+g)^n-m(3+g)^n=1-m+\sum_{j=1}^{n}\binom{n}{j}A_j(u,k,g)$$

Let

$$R_n(u,k,g)=(2+u-k+g)^n-m(3+g)^n$$

Then

$$R_n(u,k,g)=1-m+\sum_{h=1}^{n}\binom{n}{h}A_h(u,k,g)$$

But from first dimension series we find that

$$A_n(u,k,g)=W_n(u,k,g)+E_n(u,k,g)$$

Where

$$W_n(u,k,g)=\sum_{j=r=0}^{n}\sum_{r=j}^{n}(-1)^j(k-g)^j\left(\binom{n}{r}\binom{r}{j}\left(u^{r-j}-m\right)\right)$$

And

$$E_n(u,k,g)=km\sum_{r=j=1}^{n}\sum_{r=j}^{n}(-1)^j\frac{(k-g)^{j-1}}{g^j}\binom{n}{r}\left((1+g)^r-\binom{r}{0}-\binom{r}{1}\mathrm{g}\ldots\ldots\binom{r}{j-1}g^{j-1}\right)$$

Then

$$R_n(u,k,g)=1-m+\sum_{h=1}^{n}\binom{n}{h}W_h(u,k,g)+\sum_{h=1}^{n}\binom{n}{h}E_h(u,k,g)$$

Then

$$\begin{aligned}(14)\quad R_n(u,k,g) &= 1-m\\ &+\sum_{h=1}^{n}\binom{n}{h}\left(\sum_{j=r=0}^{h}\sum_{r=j}^{h}(-1)^j(k-g)^j\left(\binom{h}{r}\binom{r}{j}\left(u^{r-j}-m\right)\right)\right)\\ &+\sum_{h=1}^{n}\binom{n}{h}\left(km\sum_{r=j=1}^{h}\sum_{r=j}^{h}(-1)^j\binom{h}{r}\frac{(k-g)^{j-1}}{g^j}\left((1+g)^r-\binom{r}{0}\right.\right.\\ &\left.\left.-\binom{r}{1}\mathrm{g}\ldots\ldots\binom{r}{j-1}g^{j-1}\right)\right)\end{aligned}$$

Let

$$H_n(u,k,g) = 1 - m + \sum_{h=1}^{n} \binom{n}{h} \left(1 - m + \sum_{r=1}^{h} \binom{h}{r} (u^r - m) + \sum_{j=r=0}^{h} \sum_{r=j}^{h} (-1)^j (k-g)^j \left(\binom{h}{r} \binom{r}{j} (u^{r-j} - m) \right) \right)$$

And

$$Y_n(k,u,g) = \sum_{h=1}^{n} \binom{n}{h} \left(km \sum_{r=j=1}^{h} \sum_{r=j}^{h} (-1)^r \binom{h}{r} \frac{(k-g)^{j-1}}{g^j} \left((1+g)^r - \binom{r}{0} - \binom{r}{1} \mathrm{g} \dots \dots \binom{r}{j-1} g^{j-1} \right) \right)$$

So

(15) $$R_n(u,k,g) = H_n(u,k,g) + Y_n(k,u,g)$$

Then

$$H_n(u,k,g) = 1 - m + \sum_{h=1}^{n} \binom{n}{h} \left(1 - m + \sum_{r=1}^{h} \binom{h}{r} (u^r - m) + \sum_{j=r=0}^{h} \sum_{r=j}^{h} (-1)^j (k-g)^j \left(\binom{h}{r} \binom{r}{j} (u^{r-j} - m) \right) \right)$$

Let

$$S_n(u,m) = (1-m)(2^n - 1) + \sum_{h=1}^{n} \sum_{r=1}^{h} \binom{n}{h} \binom{h}{r} (u^r - m)$$

And

$$T^n{}_h(u,k,g) = \binom{n}{h} \left(\sum_{j=r=0}^{h} \sum_{r=j}^{h} (-1)^j (k-g)^j \left(\binom{h}{r} \binom{r}{j} (u^{r-j} - m) \right) \right)$$

Then

(16) $$H_n(u,k,g) = S_n(u,m) + \sum_{h=1}^{n} T^n{}_h(u,k,g)$$

Where

$$T^n(u,k,g) = \binom{n}{1} \left(\sum_{r=0}^{1} \binom{1}{r} (u^r - m) - (k-g)(1-m) \right)$$

And

$$T^n{}_2(u,k,g) = \binom{n}{2} \left(\sum_{r=0}^{2} \binom{2}{r} (u^r - m) - \sum_{r=1}^{2} \binom{2}{r} \binom{r}{1} (k-g)(u^{r-1} - m) + (k-g)^2 (1-m) \right)$$

And

$$T_3{}^n(u,k,\mathrm{g}) = \binom{n}{3}\left(\sum_{r=0}^{3}\binom{3}{r}(u^r - m) - \sum_{r=1}^{3}\binom{3}{r}\binom{r}{1}(k-g)(u^{r-1}-m) + \sum_{r=2}^{3}\binom{3}{r}\binom{r}{2}(k-g)^2(u^{r-2}-m) - (k-g)^3(1-m)\right)$$

And

$$T_4{}^n(u,k,g) = \binom{n}{4}\left(\sum_{r=0}^{4}\binom{4}{r}(u^r - m) - \sum_{r=1}^{4}\binom{4}{r}\binom{r}{1}(k-g)(u^{r-1}-m) + \sum_{r=2}^{4}\binom{4}{r}\binom{r}{2}(k-g)^2(u^{r-2}-m) - \sum_{r=3}^{4}\binom{4}{r}\binom{r}{3}(k-g)^3(u^{r-3}-m) + (k-g)^4(1-m)\right)$$

We note that we can add all the terms that include $r = 0$ in $T^n{}_j(u,k,g)$ together and then include $r = 1$ in $T^n{}_j(u,k,g)$ together until we reach the final term $r = n \ \ in \ \ T_n{}^n(u,k,g)$ when

By adding all term include $r = 0 \ and \ (1-m)$ we have that

$$(1-m)(2^n - 1)$$

By adding all terms include $r = 1 \ and \ (u-m)$ we have that

$$(u-m)\left(\binom{n}{1}\binom{1}{1} + \binom{n}{2}\binom{2}{1} + \binom{n}{3}\binom{3}{1} \dots\dots\dots\dots.\binom{n}{n}\binom{n}{1}\right) = \sum_{j=1}^{n}\binom{n}{j}\binom{j}{1}(u-m)$$

And adding all terms include $r = 2 \ \ and \ \ \binom{2}{j}\left(u^j - m\right)$ then

$$\sum_{j=r=2}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}(u^j - m)$$

By adding all terms include $(k-g) \ and \ (1-m) \ \ we \ have \ that$

$$(k-g)(1-m)\left(\binom{n}{1} + \binom{n}{2}\binom{2}{1} + \binom{n}{3}\binom{3}{1} \dots\dots\dots\dots\dots\binom{n}{n}\binom{n}{1}\right) = \sum_{j=1}^{n}(k-g)(1-m)\binom{n}{j}\binom{j}{1}$$

Adding all terms include $(k-g) \ \ and \ (u-m)$ we have that

$$(k-g)(u-m)\left(\binom{n}{2}\binom{2}{2}\binom{2}{1} + \binom{n}{3}\binom{3}{2}\binom{2}{1} + \binom{n}{4}\binom{4}{2}\binom{2}{1} \dots\dots\dots\dots\dots.\binom{n}{n}\binom{n}{2}\binom{2}{1}\right) = \sum_{j=2}^{n}(k-g)(u-m)\binom{n}{j}\binom{j}{2}\binom{2}{1}$$

By adding all terms include $r = 3 \ (k-g) \ \ and \ \left(u^{j-1} - m\right)$ we have that

$$\sum_{j=r=3}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{1}(k-g)\left(u^{j-1}-m\right)$$

By adding all terms include $(1-m)$ and $(k-g)^2$ we have that

$$(1-m)(k-g)^2\left(\binom{n}{2}\binom{2}{2}\binom{2}{2}+\binom{n}{3}\binom{3}{2}\binom{2}{2}+\binom{n}{4}\binom{4}{2}\binom{2}{2}\ldots\ldots\ldots\ldots\ldots\ldots.\binom{n}{n}\binom{n}{2}\binom{2}{2}\right)$$

$$=(1-m)(k-g)^2\sum_{r=2}^{n}\binom{n}{r}\binom{r}{2}\binom{2}{2}$$

And

$$(k-g)^2(u-m)\left(\binom{n}{3}\binom{3}{3}\binom{3}{2}+\binom{n}{4}\binom{4}{3}\binom{3}{2}+\binom{n}{5}\binom{5}{3}\binom{3}{2}\ldots\ldots\ldots\ldots.\binom{n}{n}\binom{n}{3}\binom{3}{2}\right)$$

$$=(u-m)(k-g)^2\sum_{j=3}^{n}\binom{n}{j}\binom{j}{3}\binom{3}{2}$$

Then

$$\sum_{j=r=4}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{2}(k-g)^2\left(u^{j-2}-m\right)$$

Then

So the final equation is equal to after the order of the terms

$$\sum_{h=0}^{n-2}\sum_{r=j=h+2}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{h}(k-g)^h\left(u^{j-h}-m\right)$$

And

$$\sum_{r=j=0}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}(k-g)^j(1-m)$$

And

$$\sum_{r=j=1}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{j-1}(k-g)^{j-1}(u-m)$$

Then

$$\sum_{h=0}^{1}\sum_{r=j=h}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{jh-h}(k-g)^{j-h}(u^h-m)$$

So we have that

$$(17)\qquad \sum_{h=1}^{n}T^n{}_h(u,k,g)$$

$$=\sum_{h=0}^{1}\sum_{r=j=h}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{jh-h}(k-g)^{j-h}(u^h-m)$$

$$+\sum_{h=0}^{n-2}\sum_{r=j=h+2}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{h}(k-g)^h\left(u^{j-h}-m\right)$$

But from equation (5)

$$H_n(u,k,g) = S_n(u,m) + \sum_{h=1}^{n} T^n{}_h(u,k,g)$$

So from equation (17) we find that

$$H_n(u,k,g) = S_n(u,m) + \sum_{h=0}^{1} \sum_{r=j=h}^{n} \sum_{r=j}^{n} \binom{n}{r}\binom{r}{j}\binom{j}{jh-h}(k-g)^{j-h}(u^h - m)$$
$$+ \sum_{h=0}^{n-2} \sum_{r=j=h+2}^{n} \sum_{r=j}^{n} \binom{n}{r}\binom{r}{j}\binom{j}{h}(k-g)^h\left(u^{j-h} - m\right)$$

From equation (15) we have

$$Y_n(k,u,g) = \sum_{h=1}^{n}\binom{n}{h}\left(km\sum_{r=j=1}^{h}\sum_{r=j}^{h}(-1)^r\binom{h}{r}\frac{(k-g)^{j-1}}{g^j}\left((1+g)^r - \binom{r}{0}\right.\right.$$
$$\left.\left. - \binom{r}{1}\text{g} \dots\dots\binom{r}{j-1}g^{j-1}\right)\right)$$

Let

$$Q^n{}_h(k,g) = \binom{n}{h}\left(\sum_{r=j=1}^{h}\sum_{r=j}^{h}(-1)^r\binom{h}{r}\frac{(k-g)^{j-1}}{g^j}\left((1+g)^r - \binom{r}{0} - \binom{r}{1}\text{g}\dots\dots\binom{r}{j-1}g^{j-1}\right)\right)$$

Then

$$Y_n(k,u,g) = km\sum_{h=1}^{n} Q_h{}^n(k,g)$$

When $h = 1$ *we have*

$$Q^n{}_1(k,g) = \binom{n}{1}km\left(-\frac{\binom{1}{1}((g+1)-1)}{g}\right)$$

When $h = 2$ we have that

$$Q_2{}^n(k,g) = \binom{n}{2}\left(km\sum_{r=1}^{2}(-1)^r\binom{2}{r}\frac{1}{g}((1+g)-1)\right.$$
$$\left. + km\left(\left(\frac{\binom{2}{2}(k-g)}{g^2}\right)\left((1+g)^2 - 1 - \binom{2}{1}\right)\right)\right)$$

When $h = 3$ we have

$$Q^n{}_3(k,g)=\binom{n}{3}\left(km\sum_{r=1}^{3}(-1)^r\binom{3}{r}\frac{1}{g}\big((1+g)-1\big)\right.$$
$$+km\sum_{r=2}^{3}(-1)^r\binom{3}{r}\frac{(k-g)}{g^2}\left((1+g)^2-1-\binom{2}{1}\right)$$
$$\left.-\binom{3}{3}\frac{(k-g)^2}{g^2}\left((1+g)^3-1-\binom{3}{1}-\binom{3}{2}\right)\right)$$

So last term when $h=n$ we have

$$Q_n{}^n(k,g)=\binom{n}{n}\left(\binom{n}{n}(-1)^n(k-g)^n\right)$$

We note that we can add all the terms that include $r=0\ in\ all\ Q^n{}_j(k,g)\ j=1,2,3\ldots\ldots n$ together and then include $r=1\ in\ Q_j{}^n(k,g)\ \ all\ j=1,2,3\ldots\ldots\ldots n$ together until we reach the final term when $r=n\ in\ Q_n{}^n(k,g)$

Then

$$\sum_{h=1}^{n}Q_h{}^n(k,g)=km\sum_{r=j=1}^{n}\sum_{r=j}^{n}(-1)^j\binom{n}{r}\binom{r}{j}\frac{(k-g)^{j-1}}{g^j}\left((1+g)^j-\binom{j}{0}\right.$$
$$\left.-\binom{j}{1}g\ldots\ldots\ldots\binom{j}{j-1}g^{j-1}\right)$$
$$+km\sum_{h=2}^{n}\sum_{r=j=h}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{h}\frac{(-1)^{h-1}(k-g)^{h-2}}{g^{h-1}}\left((1+g)^h-\binom{h}{0}\right.$$
$$\left.-\binom{h}{1}g\ldots\ldots\ldots.\binom{h}{h-2}g^{h-2}\right)$$

Then

$$Y_n(k,u,g)=km\sum_{r=j=1}^{n}\sum_{r=j}^{n}(-1)^j\binom{n}{r}\binom{r}{j}\frac{(k-g)^{j-1}}{g^j}\left((1+g)^j-\binom{j}{0}-\binom{j}{1}g\ldots\ldots\ldots\binom{j}{j-1}g^{j-1}\right)$$
$$+km\sum_{h=2}^{n}\sum_{r=j=h}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{h}\frac{(-1)^{h-1}(k-g)^{h-2}}{g^{h-1}}\left((1+g)^h-\binom{h}{0}\right.$$
$$\left.-\binom{h}{1}g\ldots\ldots\ldots.\binom{h}{h-2}g^{h-2}\right)$$

Then
According equation(15)

$$R_n(u,k,g)=H_n(u,k,g)+Y_n(k,u,g)$$

Where

$$R_n(u,k,g)=(2+u-k+g)^n-m(3+g)^n$$

And

$$H_n(u,k,g)=S_n(u,m)+\sum_{h=0}^{1}\sum_{r=j=h}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{jh-h}(k-g)^{j-h}(u^h-m)$$
$$+\sum_{h=0}^{n-2}\sum_{r=j=h+2}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}\binom{j}{h}(k-g)^h(u^{j-h}-m)$$

And

$$Y_n(k,u,g) = km \sum_{r=j=1}^{n} \sum_{r=j}^{n} (-1)^j \binom{n}{r}\binom{r}{j} \frac{(k-g)^{j-1}}{g^j}\left((1+g)^j - \binom{j}{0}\right.$$
$$\left. - \binom{j}{1} g \dots\dots\dots \binom{j}{j-1} g^{j-1}\right)$$
$$+ km \sum_{h=2}^{n} \sum_{r=j=h}^{n} \sum_{r=j}^{n} \binom{n}{r}\binom{r}{h} \frac{(-1)^{h-1}(k-g)^{h-2}}{g^{h-1}}\left((1+g)^h - \binom{h}{0}\right.$$
$$\left. - \binom{h}{1} g \dots\dots\dots. \binom{h}{h-2} g^{h-2}\right)$$

3.proof theorem.1

In this section we will use the basic series $L_n(k,g,u) = V^n{}_n(k,g,u) + S_n(k,g)$ in prove the theorem.1 then according basic infinite series if $u = 1$ in $V_n{}^n(k,g,u)$ we have that

$$V_n{}^n(k,g,1) = \sum_{j=1}^{n} (-1)^j \binom{n}{j}\left((1)^{n-j} - 1\right)(k-g)^j = 0$$

Then according equations

$$L_n(k,g,1) = V_n{}^n(k,g,1) + S_n(k,g)$$

Then

$$L_n(k,g,1) = S_n(k,g)$$

Then according to the equations, (2,7, 2.8 ,2.9) we find that

$$L_n(k,g,1) = S_n(k,g)$$

$$(1-k+g)^n - (1+g)^n = \mathrm{k}\sum_{j=0}^{n-1} (-1)^{j+1} \frac{(k-g)^j}{g^{j+1}}\left((1+g)^n - \left(\binom{n}{0} + \binom{n}{1}\mathrm{g} \dots\dots \binom{n}{j} g^j\right)\right)$$

Let a d a positive integers where

$$g = a - 1$$
$$k = -a\mathrm{d} + a$$

Then

$$(1+ad-a+a-1)^n - (1+a-1)^n$$
$$= (-ad+a)\sum_{j=0}^{n-1} (-1)^{j+1} \frac{(-ad+a-a+1)^j}{(a-1)^{j+1}}\left((1+a-1)^n\right.$$
$$\left. - \left(\binom{n}{0} + \binom{n}{1}(a-1) \dots\dots\dots \binom{h}{j}(a-1)^j\right)\right)$$

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We have that

$$(ad)^n - a^n$$
$$= (-ad+a)\sum_{j=0}^{n-1} (-1)^{j+1+j} \frac{(ad-a)^j}{(a-1)^{j+1}}\left(a^n - \left(\binom{n}{0} + \binom{n}{1}(a-1) \dots\dots\dots \binom{n}{j}(a-1)^j\right)\right)$$

Then

$$a^{n-1}(d^n - 1) = \frac{-d+1}{a-1}(a^n - 1)$$

$$+(\mathrm{d}-1)\sum_{j=1}^{n-1}\frac{(-1)^{2j+2}(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots\binom{n}{j}(a-1)^j\right)\right)$$

Now we have that

$$a^{n-1}(d^n-1)=\frac{d-1}{a-1}(a^n-1)$$

$$+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots.\binom{n}{j}(a-1)^j\right)\right)$$

Note the negative sign in the equation

3.1

$$a^{n-1}(d^n-1)$$

$$=\frac{d-1}{a-1}(a^n-1)$$

$$+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots\dots\binom{n}{j}(a-1)^j\right)\right)$$

Then

$$a^{n-1}(d^n-1)=\frac{d-1}{a-1}(a^n-1)$$

$$+(ad-1)\left((d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^{j-1}}{(a-1)^j}\left(a^n-\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots..\binom{n}{j}(a-1)^j\right)\right)$$

Then we note $(d-1,ad-1)=(a^{n-1},ad-1)=1$ then we find that if $a^n\equiv 1(mod\ m)$ where $m=ad-1$ then $d^n\equiv 1(mod\ m)$

Lemma.1 if $2^n\equiv 1(mod\ m)$ and $m=2x-1$ where n x a positive integers then

$$x^n\equiv 1(mod\ \ m\ )$$

Proof let in theorem.1 $a=2\ \ and\ \ \ d=x$

Remark. We call the number primes the form $M_p=2^p-1$ mersenne number discovered in 2005 by Martin nowak the largest prime number of mersenne $\mathrm{M}_{25964951}$ and 42 in the list $[see\ James\ J\ Tattersall\ 143]$. we know about Mersenne's number if M_p it is not prime then there is a prime number $q=2pr+1$ where $q\setminus M_P$ example M_{11} of a non-prime. Also there is a relationship between Mersenne prime and the perfect numbers. The number n is called a perfect number if $n=2^{p-1}\left(M_p\right)$ where Mersenne prime number $[see\ K.H\ Rosen\ \ 159]$

Lemma.2 if p primer number and $q=2pr+1$ where M_p morsesen number then

$$\begin{cases}(pr+1)^p\equiv 1(mod\ q\ )\ \ if\ \ \ 2^p\equiv 1(mod\ \ \ q)\\(pr+1)^p\not\equiv 1(mod\ \ q)\ \ \ all\ \ q<\sqrt{M_p}\ \ \ if\ M_p\ prime\ number\end{cases}$$

Proof according mersenne number if M_p non-prime then we find $q=2pr+1$ where $q\setminus M_p$ then according lemma.1 if $2^2\equiv 1(mod\ m)$ and $m=2x-1$ then $x^n\equiv 1(mod\ m)$ then let $n=p$ and $m=2px+1$ then $x=pr+1$ then if M_P non-prime then $(pr+1)^p\equiv 1(mod\ q)$ and if $(pr+1)^p\not\equiv 1(mod\ q)$ all $q<\sqrt{M_p}$ then M_p prime number according mersenne number

Lemma.3 let n a d a positive integers then

$$a^{n-1}(d^n-1)=\frac{d-1}{a-q}(a^n-1)$$
$$+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^j}\left(a^n-\binom{n}{0}-\binom{n}{1}(a-1)\ldots\ldots\binom{n}{j}(a-1)^j\right)$$

Proof from last equation in proof theorem.1

Lemma.4 let M_p moresen number and n prefect number then

$$n=M_p+\sum_{j=1}^{p-1}(3)^j\left(2^p-\binom{n}{0}-\binom{n}{1}\ldots\ldots\binom{n}{j}\right)$$

Proof let in lemma.3 $a=d=2\ and\ n=p$

4.Diophantine equation and proof theorem.2

In this section we will prove theorem.2 using theorem.1 but before that we mention according to Euler's theorem $a^{\varphi(n)}\equiv 1(mod\,n)$ where $(a,n)=1$ and $\varphi(n)$ is Euler function see proof Euler theorem in $[K.M.244]$

Theorem.2 if $n=yx^k-1$ and $\varphi(n)$ Euler function where $\frac{\varphi(n)}{u}$ and $1\le i\le j$ where $1\le u\le k$ then

$$(yx^{k-u})^{\frac{\varphi(n)}{u}}\equiv 1(mod\,n)$$

Proof according Euler theorem if $(a,n)=1$ then $a^{\varphi(n)}\equiv 1(mod\,n)$ and according theorem.1 if $a^n\equiv 1(mod\,m)$ where $m=ad-1$ then $d^n\equiv 1(mod\ \ m)$

Then let $\varphi(n)\,Eulere\,function$ where $1\le u\le k$ and $n=yx^k-1$ then we find according Euler theorem

$$\left((x^u)^{\frac{\varphi(n)}{u}}\equiv 1(mod\,n)\right)$$

Then according theorem.1 $\mathrm{d}=\frac{n+1}{x^u}=\mathrm{yx}^{\mathrm{k-u}}$ then we have that

$$(yx^{k-u})^{\frac{\varphi(n)}{u}}\equiv 1(mod\,n\,)$$

Lemma.1 let $x\,y\,k\in\mathbb{N}$ where $n=yx^m-1$ and $\varphi(n)$ Euler function where $m\backslash\varphi(n)$ then

$$y^{\frac{\varphi(n)}{m}}\equiv 1(mod\,n)$$

Proof let in theorem.2 $m=u_i=k$ then $(yx^{k-k})^{\frac{\varphi(n)}{m}}\equiv 1(mod\ \ n)$ then we have that $y^{\frac{\varphi(n)}{m}}\equiv 1(mod\ \ n)$

In this we will explain the relationship of the solutions of the Diophantine equation to the solutions of some types of congruences , and before that we mention fermat and Euler,s equation lesson diophantine in the form of $x^2-dy^2=\pm 1$ $[see\ James\,J\,Tattersall\ 247]$ it is called the bell's equation lagrange studied the solutions , as well as the equation $y^2=x^3-k$ which is shaped like mordel's equation which has infinite solutions depends on the value of k $[see\ K.\ M.\ Rosen\ 289]$.

Theorem.3 if (g,x) of solutions equation $g^m - da^n = -1$ and $\varphi(g^m)$ Euler function where $1 \le k \le \varphi(g^m)$ and $n \setminus \varphi(g^m)$ then

$$d^{\frac{\varphi(g^m)}{n}} \equiv 1(mod\ \ g^m)$$

Proof let $n = g^m\ and\ x =\ a^n and\ \ d = y\ where\ k = n$ then $g^n - da^n = -1$ in theorem.1

Theorem.4 Let q p m n k a positive integers where and where $q^2 - dp^2 = -1$ then

$$d^{\frac{\varphi(q^2)}{2}} \equiv 1(mod\ q^2)$$

Proof let in theorem.3 let $m = n = 2$

Lemma.4 let p prime number and x m a positive integer if $p = dx^m - 1$ and $m \setminus p - 1$ then

$$d^{\frac{p-1}{m}} \equiv 1(mod\ \ p)$$

Proof let $n = p\ \ and\ \ \ y = d$ in lemma.1

5.Primitive roots

Theorem.2 in primitive roots let $m = da_j - 1$ all a_j and $1 \le j \le \frac{\varphi(\varphi(m))}{2}$ where $\varphi(m)$ Euler function and $1 \le i \le j$ where $\{a_1, a_2\ \ , a_3 \dots \dots \dots \dots \dots . . a_j\}$ is a primitive roots modulo m then $\left\{\left(\frac{m+1}{a_1}\right)_1\ ,\left(\frac{m+1}{a_2}\right)_2\ ,\ \left(\frac{m+1}{a_3}\right)_3\ ,\ \left(\frac{m+1}{a_4}\right)_4 \dots \dots \dots \dots \dots \dots \dots . \left(\frac{m+1}{a_i}\right)_j\right\}$ is a primitive roots modulo m

Proof theorem.2 in primitive roots. to prove the second theorem, first, we will mention the theorem that determines the number of possible primitive roots for each number that has roots. See proof in [$see\ K.H.Rosen, 244$] Theorem.4 if the positive integer m has a primitive root, then it has a total of $\varphi(\varphi(n))$ incongruent primitive roots n , then

According definition of primitive roots [$see\ K.H, 245$] if r and are relatively prime integers with $n > 0$ and if $ord_n r = \varphi(n)$, then r is called a primitive root modulo n.

Then

According **theorem.1** if $a^n \equiv 1(mod\ m)$ where $m = ad - 1$ then $d^n \equiv 1(mod\ m)$ then

let $n = \varphi(m)$ then if $a^{\varphi(m)} \equiv 1(mod\ m)$ where $m = ad - 1$ then $d = \frac{m+1}{a}$

$so\ \left(\frac{m+1}{a}\right)^{\varphi(m)} \equiv 1(mod\ m)$ according theorem.1

so we find if according theorem.1 if $ord_m a = \varphi(m)\ then\ ord_m \frac{m+1}{a} = \varphi(m)$

then if $m = da_j - 1$ all a_j and $1 \leq j \leq \frac{\varphi(\varphi(m))}{2}$ where all $a_1,\ a_2, a_3, \dots \dots . . a_j$ is a primitive root modulo m then if $ord_m a_j = \varphi(m)$ $then\ ord_m \frac{m+1}{a_j} = \varphi(m)$ all $1 \leq j \leq \frac{\varphi(\varphi(m))}{2}$ then let $j = i$ then $\left[a_1, a_2, a_3, \dots \dots . . a_j, \left(\frac{m+1}{a_1}\right)_1, \left(\frac{m+1}{a_2}\right)_2 \dots \dots \dots \left(\frac{m+1}{a_j}\right)_i\right]$ is a primitive roots modulo m

generalization of Artin's conjecture in proportion to a according Artin,s conjecture that given an integer a that is not a square there are infinitely many primes for which a is a primitive root $\{see\ K.M.Rosen\ 47\}$ then according theorem.2 in primitive roots if a primitive root p then $\frac{p+1}{a}$ primitive root p then there are an infinite number of primes p_j , $j = 1$,2 ,3 … … … …. where $\left[a,\ \frac{p_j+1}{a}\right]$ there are primitive roots for them.

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