One-to-One Map of Dirac Equation between Matrix Algebra and Geometric Algebra $Cl_{3,0}$

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Abstract

In this paper, it is created a one-to-one map between the Dirac Equation in Matrix Algebra and the Dirac Equation in Geometric Algebra $Cl_{3,0}$. The form of the Dirac equation in Geometric Algebra $Cl_{3,0}$ is found to be:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi - m\psi_{even}\hat{z} + m\psi_{odd}\hat{z} = 0$$

Where:

$$\psi_{even} = \psi_0 + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx}$$
$$\psi_{odd} = \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\hat{z}\psi_{xyz}$$
$$\psi = \psi_{even} + \psi_{odd} = \psi_0 + \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} + \hat{x}\hat{y}\hat{z}\psi_{xyz}$$

Considering the wavefunction solution in standard Matrix Algebra as:

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\psi_{1r} + i\psi_{1i}\\ \psi_{2r} + i\psi_{2i}\\ \psi_{3r} + i\psi_{3i}\\ \psi_{4r} + i\psi_{4i}\end{pmatrix}$$

It is obtained is a one-to-one mapping between both representations, being:

$$\psi_{1r} = -\psi_y$$
$$\psi_{1i} = -\psi_x$$
$$\psi_{2r} = \psi_{xyz}$$
$$\psi_{2i} = \psi_z$$
$$\psi_{3r} = -\psi_{yz}$$
$$\psi_{3i} = \psi_{zx}$$
$$\psi_{4r} = \psi_{xy}$$
$$\psi_{4i} = \psi_0$$

Keywords

Geometric Algebra, Matrix Algebra, Dirac Equation, Quantum Mechanics, wavefunction, basis vectors

1. Introduction

In this paper, we will use derive the Dirac Equation in Geometric Algebra $Cl_{3,0}$. Then, after some transformations, we will create a one-to-one map of the Dirac Equation and the wave-function between Matrix Algebra and Geometric Algebra.

2. Introduction to Geometric Algebra

If you do not know anything regarding geometric algebra, I strongly recommend you [1]. You have a complete study of Geometric Algebra in [3].

We will use Geometric Algebra $Cl_{3,0}$. This means, it has three basis vectors with positive signature and zero basis vectors with negative signature. We will explain this in a minute. In Geometric Algebra $Cl_{3,0}$, we have the vectors:

$$\hat{x} \quad \hat{y} \quad \hat{z}$$

In this case, we will consider them orthonormal, and, in the appendix A1, I will explain which would be the difference in the calculations if they were not orthonormal.

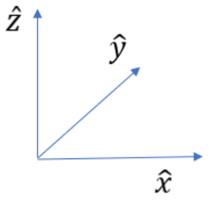

Fig.1 Orthonormal basis vectors in $Cl_{3,0}$

The square of these vectors in Geometric Algebra is its norm to the square. The norm of a vector is a scalar (not a vector anymore). As we have considered the basis as orthonormal, its square is the scalar 1.

$$\hat{x}^2 = \hat{x}\hat{x} = ||\hat{x}||^2 = 1 \qquad (1)$$
$$\hat{y}^2 = \hat{y}\hat{y} = ||\hat{y}||^2 = 1 \qquad (2)$$
$$\hat{z}^2 = \hat{z}\hat{z} = ||\hat{z}||^2 = 1 \qquad (3)$$

In the nomenclature $Cl_{3,0}$, the 3 stands for the number of vectors which square is positive and the 0 for the number of basis vectors which squares is negative. In this case, no basis vectors have a negative square (also known as negative signature), so all of them have a positive square (positive signature), that equals +1 in an orthonormal basis.

The basis vectors can be multiplied by each other (this operation is called Geometric Product). For orthonormal or orthogonal bases, this product follows the anticommutative property, this is:

$$\hat{x}\hat{y} = -\hat{y}\hat{x} \qquad (4)$$
$$\hat{y}\hat{z} = -\hat{z}\hat{y} \qquad (5)$$
$$\hat{z}\hat{x} = -\hat{x}\hat{z} \qquad (6)$$

This combination of two vectors via this product is called a bivector. The bivector instead of representing a vector (an oriented segment), it represents an oriented plane. So $\hat{x}\hat{y}$ represents the plane xy with its normal in a certain direction. And $\hat{y}\hat{x}$ represents the same plane xy but with its normal in the opposite direction.

In Geometric Algebra we do not talk about normal vectors anymore. Instead, we talk about the orientation of a theoretical rotation in that plane. See Fig.2 for a visual explanation. Also, in [1][2] and [3], you can find more information about the meaning or interpretation of the bivectors.

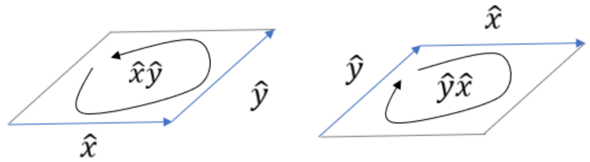

Fig.2 Representation of the bivectors $\hat{x}\hat{y}$ and $\hat{y}\hat{x}$. They represent the same plane with opposite orientation. In fact, $\hat{x}\hat{y} = -\hat{y}\hat{x}$.

If we multiply the three vectors, we obtain the trivector (also called pseudoscalar in the literature [1][3]):

$$\hat{x}\hat{y}\hat{z} = -\hat{y}\hat{x}\hat{z} = \hat{y}\hat{z}\hat{x} = -\hat{z}\hat{y}\hat{x} = \hat{z}\hat{x}\hat{y} = -\hat{x}\hat{z}\hat{y} \quad (7)$$

You can check that the same relations as in equations (4)(5)(6) apply. So, every time you swap the position of two vectors you have to put a minus sign (or multiply by -1, as you prefer).
The meaning of the trivector is an oriented volume. The same as the bivector is a plane with two possible orientations. The trivector is a volume with two possible orientations. You can see visual representation in Fig. 3. Again in [1] [2] and [3] you can find a more information regarding trivectors.

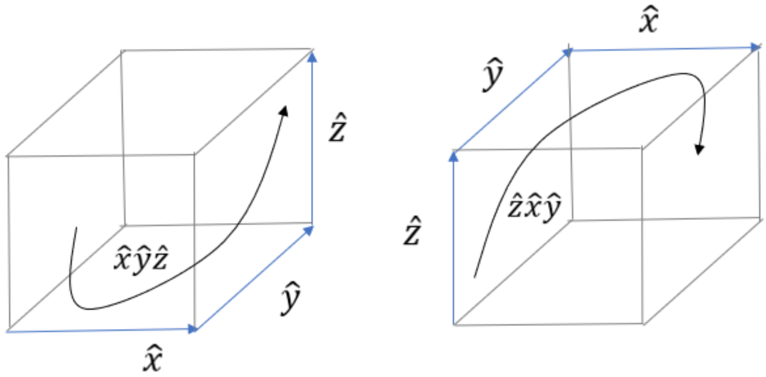

Fig.3 Representation of the two possible orientations of the trivector.
We can check that $\hat{x}\hat{y}\hat{z} = -\hat{y}\hat{x}\hat{z}$.

3. Operations in Geometric Algebra

One of the most surprising characteristics of Geometric Algebra is that you can mix scalars with vectors, bivectors and trivectors. You represent this a sum. For example, a typical element in Geometric Algebra could have the form:

$$A = 3 + 5\hat{x} + 3\hat{y} + 4\hat{y}\hat{z} - 2\hat{x}\hat{y}\hat{z}$$

The same that is done with polynomials or complex numbers, that is to leave the sum among different components indicated, it is done in Geometric Algebra. This type of element in Geometric Algebra that has different components as scalars, vectors, bivectors etc… is called a multivector. So, the A element in the example above is a multivector.

In a multivector, the vectors and the trivector are called odd-grade elements. The reason is because they are composed by one vector or by three vectors (odd grade number).

In a multivector, the scalars and the bivectors are called even grade elements. The reason is because the elements have 0 vectors (the scalars) or 2 vectors (the bivectors). We consider the 0 and 2 even for this purpose.

And if you want to make a product between two multivectors in Geometric Algebra you just have to follow the laws (1) to (6). For example:

$$(2+3\hat{x})(5\hat{y}+7\hat{x}+\hat{y}\hat{z}+\hat{z}\hat{x})$$

The first thing we have to do is to multiply component by component as we would do in a polynomial for example. But the very important thing is that you have to keep the order of the product as we have seen that it is not commutative, so:

$$(2+3\hat{x})(5\hat{y}+7\hat{x}+\hat{y}\hat{z}+\hat{z}\hat{x}) \\ =10\hat{y}+14\hat{x}+2\hat{y}\hat{z}+2\hat{z}\hat{x}+15\hat{x}\hat{y}+21\hat{x}\hat{x}+3\hat{x}\hat{y}\hat{z}+3\hat{x}\hat{z}\hat{x}=$$

Now, with the relations (1) to (6) we will operate the square of $\hat{x}$ and we will swap the $\hat{x}$ and the $\hat{z}$ in the last component:

$$=10\hat{y}+14\hat{x}+2\hat{y}\hat{z}+2\hat{z}\hat{x}+15\hat{x}\hat{y}+21(+1)+3\hat{x}\hat{y}\hat{z}-3\hat{x}\hat{x}\hat{z}=$$

Now, we have again a square of $\hat{x}$ in the last component, so we can operate:

$$=10\hat{y}+14\hat{x}+2\hat{y}\hat{z}+2\hat{z}\hat{x}+15\hat{x}\hat{y}+21+3\hat{x}\hat{y}\hat{z}-3(+1)\hat{z}= \\ =10\hat{y}+14\hat{x}+2\hat{y}\hat{z}+2\hat{z}\hat{x}+15\hat{x}\hat{y}+21+3\hat{x}\hat{y}\hat{z}-3\hat{z}=$$

If we order the terms, starting by the scalar, vectors, bivectors and finally the trivector we have:

$$=21+14\hat{x}+10\hat{y}-3\hat{z}+15\hat{x}\hat{y}+2\hat{y}\hat{z}+2\hat{z}\hat{x}+3\hat{x}\hat{y}\hat{z}$$

Let's see another example:

$$(3+\hat{x}+2\hat{y})(5\hat{x}+7\hat{y})=$$

We start multiplying the components but keeping always the order of the vectors.

$$(3+\hat{x}+2\hat{y})(5\hat{x}+7\hat{y})=15\hat{x}+21\hat{y}+5\hat{x}\hat{x}+7\hat{x}\hat{y}+10\hat{y}\hat{x}+14\hat{y}\hat{y}=$$

Now we apply the (1) to (6) to the squares:

$$15\hat{x}+21\hat{y}+5(+1)+7\hat{x}\hat{y}+10\hat{y}\hat{x}+14(+1)=15\hat{x}+21\hat{y}+5+7\hat{x}\hat{y}+10\hat{y}\hat{x}+14 \\ =$$

We can see that now we have two scalars (5 and 14) that have appeared coming from vector products that can be summed, so:

$$=15\hat{x}+21\hat{y}+19+7\hat{x}\hat{y}+10\hat{y}\hat{x}=$$

Also, we see that we have the same bivector $\hat{x}\hat{y}$ in two different forms, so we apply (1) to (6) to get:

$$=15\hat{x}+21\hat{y}+19+7\hat{x}\hat{y}-10\hat{x}\hat{y}=15\hat{x}+21\hat{y}+19-3\hat{x}\hat{y}=$$

Ordering the terms:

$$=19+15\hat{x}+21\hat{y}-3\hat{x}\hat{y}$$

To sum up, we can say that the geometric product keeps the associative and the distributive properties but not the commutative property. In an orthonormal basis the commutative property is substituted by the anticommutative property as can be seen in (4) to (6). For n on orthonormal basis, the thing is not so simple but we will not treat this case in this paper. You can see a hint about it in Appendix A1.

4. Square of the bivectors and the trivector

If we multiply a bivector by itself (applying (1) to (6)):

$$\hat{x}\hat{y}\hat{x}\hat{y} = -\hat{x}\hat{y}\hat{y}\hat{x} = -\hat{x}(1)\hat{x} = -\hat{x}\hat{x} = -1$$
$$\hat{y}\hat{z}\hat{y}\hat{z} = -1$$
$$\hat{z}\hat{x}\hat{z}\hat{x} = -1$$

We see that the result is -1. The same happens with the trivector:

$$\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = -\hat{x}\hat{y}\hat{z}\hat{x}\hat{z}\hat{y} = \hat{x}\hat{y}\hat{z}\hat{z}\hat{x}\hat{y} = \hat{x}\hat{y}(1)\hat{x}\hat{y} = \hat{x}\hat{y}\hat{x}\hat{y} = -\hat{x}\hat{y}\hat{y}\hat{x} = -\hat{x}(1)\hat{x} = -1$$

In Geometric Algebra the imaginary or complex numbers are not used and are not necessary. The reason is that there are elements that are already in fact the square root of -1, as the bivectors or the trivector. We will see the importance of this in Quantum Mechanics. Instead of using imaginary numbers, these will be substituted by bivectors and trivectors with geometric meaning.

The imaginary unit *i* was defined as "something unknown" (whatever it is) that is the square root of -1. Now, that we have elements that are in fact, known, and are the square root of -1 (the bivectors and the trivector), we can be more specific and use these elements to play this role. We will use this conversion from the *i* imaginary unit into bivectors and trivector, mainly in Quantum Mechanics.

When the *i* does not have any preferred spatial direction will be related to the trivector $\boldsymbol{\hat{x}\hat{y}\hat{z}}$.This happens for example when the *i* appears related to mass, energy or time.

If the *i* is related to something with a preferred direction like speed or momentum, normally the *i* is related to a bivector. Do not worry, we will see how to work with this in the next chapters.

Summing up, even if we are in $Cl_{3,0}$ with the three basis vectors with positive signature (positive square), the algebra itself has created two type of elements more (bivectors and trivector) which square is negative.

In a multivector we will have these two types of elements depending on its square, the scalars and the vectors which square is +1 (positive signature) and the bivectors and the trivector which square is -1 (negative signature).

5. Inverse of vector in Geometric Algebra

Another interesting property in Geometric Algebra is that you can take the inverse a vector. We can calculate its value for a basis vector the following way. We start with equation (1):

$$\hat{x}\hat{x} = 1$$

We premultiply both equations by the inverse of $\hat{x}$:

$$\hat{x}^{-1}\hat{x}\hat{x} = \hat{x}^{-1}(1)$$

By definition, the product of the inverse of an element by itself is equal to 1.

$$(1)\hat{x} = \hat{x}^{-1}$$

So,

$$\hat{x} = \hat{x}^{-1}$$
$$\hat{x}^{-1} = \hat{x}$$

The inverse of a basis vector in an orthonormal basis is the vector itself. So:

$$\hat{x}^{-1} = \hat{x}$$
$$\hat{y}^{-1} = \hat{y}$$
$$\hat{z}^{-1} = \hat{z}$$

When we have to take the inverse a product of vectors (bivectors or trivectors) you can check in [3] that apart from inverting each element you have to reverse the order of them, this way:

$$(\hat{x}\hat{y})^{-1} = \hat{y}^{-1}\hat{x}^{-1} = \hat{y}\hat{x} = -\hat{x}\hat{y}$$

or

$$(\hat{x}\hat{y}\hat{z})^{-1} = \hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \hat{z}\hat{y}\hat{x} = -\hat{x}\hat{y}\hat{z}$$

Remember that every time you swap two vectors, you add a minus sign (4) to (6). To convert $\hat{z}\hat{y}\hat{x}$ into $-\hat{x}\hat{y}\hat{z}$ you have two make three swaps, that is the reason of the final negative sign.

We will use the convention that the division by a vector is to postmultiply by the inverse of that vector. This means for example, if we want to do the following operation, this will be the result:

$$\frac{\hat{x}}{\hat{y}} = \hat{x}(\hat{y})^{-1} = \hat{x}\hat{y}$$

Remind that we are always talking about orthonormal bases. To have a hint about not orthonormal bases, you check Annex A1.

6. Reverse operation and reverse product

There is another operation we can make in Geometric Algebra that is the reversion of a multivector. I will represent this with a line above the multivector. This operation reverses all the internal order of bivectors and trivectors. As an example:

$$A = 3 + 5\hat{x} + 3\hat{y} + 4\hat{y}\hat{z} - 2\hat{x}\hat{y}\hat{z}$$
$$\bar{A} = \overline{(3 + 5\hat{x} + 3\hat{y} + 4\hat{y}\hat{z} - 2\hat{x}\hat{y}\hat{z})} = 3 + 5\hat{x} + 3\hat{y} + 4\hat{z}\hat{y} - 2\hat{z}\hat{y}\hat{x}$$
$$= 3 + 5\hat{x} + 3\hat{y} - 4\hat{y}\hat{z} + 2\hat{x}\hat{y}\hat{z}$$

You can see that it is similar to a conjugate in complex numbers. It changes the sign of the elements which square is -1 (in this case, are the bivectors and the trivector).

With this we can define the reverse product. It consists of the product of a multivector by the reverse of itself.

The main characteristic of the reverse product is that if the multivector only has one type of elements with positive square and only one type of elements of negative square, the result of this product is a scalar.

This means, if the multivector only has scalars (positive square) and bivectors (negative square) the reverse product of the multivector will be a scalar. The same if it only has vectors (positive square) and bivectors (negative square). Or vectors (positive square) and trivector (negative square).

But when the multivector has scalars and vectors (both positive square) and a bivector for example, the result could be not scalar. The same if it has scalars and both bivectors and the trivector (both negative square).

This is, the multivector has to have only scalars or vectors (not both) mixed with only bivectors or trivectors (not both).

Let's see some examples. B only has vectors (positive square) and the trivector (negative square), the result must be scalar:

$$B = 5\hat{x} + 3\hat{y} + 4x\hat{y}\hat{z}$$
$$\bar{B} = \overline{(5\hat{x} + 3\hat{y} + 4x\hat{y}\hat{z})} = 5\hat{x} + 3\hat{y} - 4x\hat{y}\hat{z}$$

$$\begin{aligned} B\bar{B} = (5\hat{x} + 3\hat{y} + 4\hat{x}\hat{y}\hat{z})(5\hat{x} + 3\hat{y} - 4\hat{x}\hat{y}\hat{z}) \\ = 25 + 15\hat{x}\hat{y} - 20\hat{x}\hat{x}\hat{y}\hat{z} + 15\hat{y}\hat{x} + 9 - 12\hat{y}\hat{x}\hat{y}\hat{z} + 20\hat{x}\hat{y}\hat{z}\hat{x} + 12\hat{x}\hat{y}\hat{z}\hat{y} \\ - 16\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} \\ = 25 + 15\hat{x}\hat{y} - 20\hat{y}\hat{z} - 15\hat{x}\hat{y} + 9 + 12\hat{x}\hat{y}\hat{y}\hat{z} - 20\hat{x}\hat{y}\hat{x}\hat{z} - 12\hat{x}\hat{y}\hat{y}\hat{z} \\ + 16\hat{x}\hat{y}\hat{x}\hat{z}\hat{y}\hat{z} = \end{aligned}$$

We sum the scalars, we see that the elements in xy sum zero, we square to +1 the vectors that are the same and consecutive and we continue swapping vectors to try to simplify:

$$\begin{aligned} = 34 - 20\hat{y}\hat{z} + 12\hat{x}\hat{z} + 20\hat{x}\hat{x}\hat{y}\hat{z} - 12\hat{x}\hat{z} - 16\hat{x}\hat{x}\hat{y}\hat{z}\hat{y}\hat{z} \\ = 34 - 20\hat{y}\hat{z} + 12\hat{x}\hat{z} + 20\hat{y}\hat{z} - 12\hat{x}\hat{z} + 16\hat{x}\hat{x}\hat{y}\hat{y}\hat{z}\hat{z} = 34 + 16 = 50 \end{aligned}$$

The result, 50, is a scalar as we expected.

Another example. C has only scalars and the trivector, the result should be scalar:

$$C = 5 + 4\hat{x}\hat{y}\hat{z}$$
$$\bar{C} = \overline{(5 + 4\hat{x}\hat{y}\hat{z})} = 5 - 4\hat{x}\hat{y}\hat{z}$$
$$C\bar{C} = (5 + 4\hat{x}\hat{y}\hat{z})(5 - 4\hat{x}\hat{y}\hat{z}) = 25 - 20\hat{x}\hat{y}\hat{z} + 20\hat{x}\hat{y}\hat{z} - 16\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} =$$

The elements in xyz sum zero. Swapping vectors in the last element we get:

$$= 25 + 16\hat{x}\hat{y}\hat{x}\hat{z}\hat{y}\hat{z} = 25 - 16\hat{x}\hat{x}\hat{y}\hat{z}\hat{y}\hat{z} = 25 + 16\hat{x}\hat{x}\hat{y}\hat{y}\hat{z}\hat{z} = 25 + 16 = 31$$

31 is a scalar as expected.

New example. D has scalars but has a mix of bivectors and trivectors (the result could be not a scalar):

$$D = 5 + 2\hat{x}\hat{y} + 3\hat{x}\hat{y}\hat{z}$$
$$\bar{D} = \overline{(5 + 2\hat{x}\hat{y} + 4\hat{x}\hat{y}\hat{z})} = 5 - 2\hat{x}\hat{y} - 3\hat{x}\hat{y}\hat{z}$$

$$\begin{aligned} D\bar{D} = (5 + 2\hat{x}\hat{y} + 3\hat{x}\hat{y}\hat{z})(5 - 2\hat{x}\hat{y} - 3\hat{x}\hat{y}\hat{z}) \\ = 25 - 10\hat{x}\hat{y} - 15\hat{x}\hat{y}\hat{z} + 10\hat{x}\hat{y} - 4\hat{x}\hat{y}\hat{x}\hat{y} - 6\hat{x}\hat{y}\hat{x}\hat{y}\hat{z} + 15\hat{x}\hat{y}\hat{z} \\ - 6\hat{x}\hat{y}\hat{z}\hat{x}\hat{y} - 9\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = \end{aligned}$$

We see that the terms in $\hat{x}\hat{y}$ and $\hat{x}\hat{y}\hat{z}$ vanish. Also, we swap some vectors:

$$\begin{aligned} = 25 - 4\hat{x}\hat{y}\hat{x}\hat{y} - 6\hat{x}\hat{y}\hat{x}\hat{y}\hat{z} - 6\hat{x}\hat{y}\hat{z}\hat{x}\hat{y} - 9\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = 25 + 4\hat{x}\hat{x}\hat{y}\hat{y} + 6\hat{x}\hat{x}\hat{y}\hat{y}\hat{z} + 6\hat{x}\hat{y}\hat{z}\hat{y}\hat{x} \\ = 25 + 4 + 6\hat{z} - 6\hat{x}\hat{y}\hat{y}\hat{z}\hat{x} = 29 + 6\hat{z} - 6\hat{x}\hat{z}\hat{x} = 29 + 6\hat{z} + 6\hat{x}\hat{x}\hat{z} \\ = 29 + 6\hat{z} + 6\hat{z} = 29 + 12\hat{z} \end{aligned}$$

We see that the result is not a scalar as we had both bivectors and the trivector (both of negative signature) in the same multivector.

One important thing to comment about the reverse product is that acts very similar to the scalar product of an element with himself (the square) in the bra-ket notation of Dirac Algebra.

In the bra-ket notation of Dirac algebra, when you want to calculate the square of a complex function or vector, you multiply this function or vector by the conjugate of itself, so you always get a real scalar result. This reverse product makes the same, you multiply a multivector by a version of itself where the sign of different elements of this multivector have changed with the aim of obtaining a real scalar as a result.

7. Summary of Geometric Algebra $Cl_{3,0}$

We have seen that the Geometric Algebra have some elements called multivectors that are composed by scalars, vectors, bivectors and a trivector. In fact, although the Geometric Algebra $Cl_{3,0}$ has only three basis vectors, it has really 8 degrees of freedom. A general multivector in Geometric Algebra could have the form (being all the coefficients α_i real scalars):

$$A = \alpha_0 + \alpha_x \hat{x} + \alpha_y \hat{y} + \alpha_z \hat{z} + \alpha_{xy} \hat{x}\hat{y} + \alpha_{yz} \hat{y}\hat{z} + \alpha_{zx} \hat{z}\hat{x} + \alpha_{xyz} \hat{x}\hat{y}\hat{z}$$

This means, although we have only three special dimensions (x, y and z) we have really 8 degrees of freedom (or 8 expanded dimensions in a meta sense) coming from this original three special dimensions.

These eight degrees of freedom are represented by these 8 α_i scalars. These scalars are always real. As commented, we do not need imaginary numbers in Geometric Algebra as we have two type of elements (the bivectors and the trivector) which square is -1 and fulfills this necessity.

One comment for the people that has some experience in Geometric Algebra used in Physics. If you are new in Geometric Algebra, please do not read it, so you do not start running.

In most of the literature regarding the use of Geometric Algebra both in Quantum Mechanics and General Relativity the $Cl_{1,3}$ or $Cl_{3,1}$ is used. This means, there are three basis vectors (the spatial dimensions) with one signature and another one (the time) with opposite signature.

The issue is that these 4 dimensions expand to 16 degrees of freedom. But in reality, only the sub-even algebra of these 16 degrees of freedom is used (only 8 degrees of the 16 possible are used). So why is this $Cl_{1,3}$ or $Cl_{3,1}$ used in the first place? We know that with $Cl_{3,0}$ we already have the 8 degrees of freedom we need.

The need of $Cl_{1,3}$ and $Cl_{3,1}$ is to accommodate the time dimension in Geometric Algebra. But we will explain in the next chapter why this is not necessary anymore.

8. So, where is the time?

If you have worked with Quantum Mechanics or with the Dirac equation (whether you have done it using Geometric Algebra or not) you might be asking where the time is.

We have $\hat{x}$, $\hat{y}$ and $\hat{z}$. But where is the $\hat{t}$? As I have commented in some papers already [2][5] we can use the trivector as the basis vector of the dimension of time. Does this mean that the dimension of time does not exist? No, the dimension of time has its own freedom (its own scalar coefficient t) but the basis vector $\hat{t}$ that accompanies this coefficient is a combination of the space vectors.

I know, it is very difficult to believe but if you continue reading the next chapters, you will see that this works perfectly. Leading to a one-to one map of the Dirac equation solutions in standard algebra to Geometric Algebra without the need of a specific vector of time.

In fact, we will work with the following definition:

$$\hat{t}^{-1} = \hat{x}\hat{y}\hat{z} \qquad (7)$$

The reason of why we define the inverse of the basis vector instead of the basis vector itself, we will see later. Anyhow, following the rules in chapter 5 you can see that for an orthonormal basis (not in general for other bases):

$$\hat{t} = (\hat{x}\hat{y}\hat{z})^{-1} = \hat{z}^{-1}\hat{y}^{-1}x^{-1} = \hat{z}\hat{y}\hat{x} = -\hat{x}\hat{y}\hat{z} \quad (7.1)$$

So:

$$\hat{x}\hat{y}\hat{z} = -\hat{t}$$

So, a general multivector will be of the type:

$$\begin{aligned} A &= \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} + \alpha_{xyz}\hat{x}\hat{y}\hat{z} \\ &= \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} - \alpha_{xyz}\hat{t} \end{aligned}$$

Even, we reorder putting the time consecutive to the spatial dimensions we would have:

$$A = \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} - \alpha_{xyz}\hat{t} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x}$$

We can even recall the α_{xyz} as α_t:

$$\alpha_t = \alpha_{xyz}$$

This leads to,

$$\begin{aligned} A &= \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} - \alpha_t\hat{t} \\ &= \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} - \alpha_t\hat{t} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} \end{aligned}$$

You can see that the α_t assures that the time has its own freedom compared to the spatial dimensions. But we do not need an original dimension more to accommodate it, it appears naturally in Geometric Algebra. In fact, we will not use it is it is above, we will use the more convenient definition we put in the beginning for a multivector:

$$A = \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} + \alpha_{xyz}\hat{x}\hat{y}\hat{z}$$

And we will explain how to work with these multivectors when time is involved.

If you have worked with Geometric Algebra before in $Cl_{1,3}$ or $Cl_{3,1}$, I give you in advance the following relations we will use. If you do not what we are talking about, just skip the following equations and continue reading:

$$\begin{gathered} i = I = \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = \hat{t}^{-1} = \hat{x}\hat{y}\hat{z} \\ \sigma_1 = \gamma_1\gamma_0 = \hat{x} \\ \sigma_2 = \gamma_2\gamma_0 = \hat{y} \\ \sigma_3 = \gamma_3\gamma_0 = \hat{z} \\ \gamma_0 \to \hat{t}^{-1} = \hat{x}\hat{y}\hat{z} \end{gathered}$$

As commented before, not always the *i* will be equal to the trivector, but sometimes to the bivectors also. But we will check this case by case.

You can check more thing regarding $\hat{t}$ as a composition of special vectors in Annex A2. And more information regarding non-orthonormal bases in Annex A1.

9. The Dirac Equation in Geometric Algebra $Cl_{3,0}$

We will do exactly the same thing that Dirac did to discover his famous equation. We will start form this relativistic equation that relates energy with momentum and mass [6].

$$E^2 = p^2c^2 + m^2c^4$$

For simplicity, we will do what is commonly done in these cases, we will consider a system of units where the speed of light c=1 and reduced constant Planck $\hbar$=1. This is commonly done, in [4] is already done de facto, with no loss of generality:

$$E^2 = p^2 + m^2 \quad (8)$$

Now, if we use the commonly operator for Energy used in Quantum Mechanics, see [4] we have:

$$E = i\hbar\frac{\partial}{\partial t}$$

As we have commented before when we see an *i* imaginary unit in an equation with no preferred spatial direction (like Energy, mass, time…) we can directly convert it to the trivector so we have:

$$E = \hat{x}\hat{y}\hat{z}\hbar\frac{\partial}{\partial t}$$

And, as we have said considering $\hbar$=1 we have:

$$E = \hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} \qquad (9)$$

If you want more details of how to obtain this equation instead of using the conversion of *i*, deriving it directly form the wave equation you can see [5]

That's it for the Energy. Let's go with the momentum. For simplicity we will consider only the direction of x and later we will generalize it. We start from the momentum operator in Quantum Mechanics [4]:

$$p_x = -i\hbar\frac{\partial}{\partial x}$$

Here we cannot convert the *i* directly to the trivector as it has a preferred direction (x). So we will do it in another way. We know that the momentum units are mass multiplied by distance divided by time (in SI this is kg, m, s)

$$p_{x\_units} = kg\frac{m}{s}$$

Considering mass, a scalar, we have that the vectors applying should be for the distance the direction x (the $\hat{x}$ vector) and be divided by time ($\hat{t}$ vector).

$$p_{x\_vectors} = \frac{\hat{x}}{\hat{t}} = \hat{x}(\hat{t})^{-1}$$

Where for the last step we have used the convention of division by vectors commented in chapter 5. Now we use the equation (7) to convert the inverse of the time vector in the trivector:

$$p_{vectors} = \hat{x}(\hat{t})^{-1} = \hat{x}\hat{x}\hat{y}\hat{z} = (1)\hat{y}\hat{z} = \hat{y}\hat{z} \quad (10)$$

So, we have seen that for equation (10) where x is a preferred direction, the vectors that apply are $\hat{y}\hat{z}$ (not $\hat{x}$). This is logic as we need to have an element (in this case the bivector) which square is -1 to substitute the *i*.

So, we would have:

$$p_x = \hat{y}\hat{z}\hbar\frac{\partial}{\partial x}$$

And the minus sign? The minus sign is a convention about what direction of the wave is considered positive. We could add it directly or we could reverse the $\hat{y}\hat{z}$ bivector to $-\hat{z}\hat{y}$ so we get it anyhow. But we will not add it at this stage, as it will appear in a later step in a natural way. We will see it later. So again, considering $\hbar$ is equal to 1, we have:

$$p_x = \hat{y}\hat{z}\frac{\partial}{\partial x}$$

Doing the same operation for p_y and p_z we will obtain:

$$p_y = \hat{z}\hat{x}\frac{\partial}{\partial y}$$
$$p_z = \hat{x}\hat{y}\frac{\partial}{\partial z}$$

So, summing all, we have:

$$p = \hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} \qquad (11)$$

Again, do not worry about the minus sign not appearing, it will appear naturally later.

Regarding equation (8) the only pending element is the mass that we consider as a scalar and we will not apply any vector to it. So, we have:

$$E^2 = p^2 + m^2 \qquad (8)$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right)^2 = \left(\hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z}\right)^2 + m^2 \quad ?? \qquad (12)$$

This above should be a kind of Klein-Gordon equation [4] in Geometric Algebra in $Cl_{3,0}$. But first thing to comment is that in equation (8) we have scalars to the square (E, P and m). But if you make the squares directly as they are in the geometric algebra equation (12) you would not obtain scalars. So that equation (12) is not properly ok.

We have to convert those squares to an operation that gives as results scalars and continue being true to its nature. Something similar and when you conjugate a vector or wavefunction to perform the bra-ket square in Dirac notation and obtain a scalar.

In Geometric Algebra, you guessed it, this operation is the reverse product.

In fact, if we do:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right)\left(\hat{z}\hat{y}\hat{x}\frac{\partial}{\partial t}\right) = \left(\hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right)\left(\hat{z}\hat{y}\frac{\partial}{\partial x} + \hat{x}\hat{z}\frac{\partial}{\partial y} + \hat{y}\hat{x}\frac{\partial}{\partial z} + m\right)$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right)\left(-\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right) = \left(\hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right)\left(-\hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right) \quad (13)$$

You will see that all the cross products vanish (the dream of Dirac) and you will get:

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + m^2 \qquad (14)$$

Which yes, we obtain all scalars (vectors have disappeared) as we wanted to be in line with equation (8). But what we have obtained (14) is not the Dirac equation. The reason is that we are playing with operators, but we are missing the wavefunction.

We will use the wavefunction in Geometric Algebra that has this form:

$$\psi = \psi_0 + \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} + \hat{x}\hat{y}\hat{z}\psi_{xyz}$$

You can see that it has 8 degrees of freedom that are represented by the 8 factors ψ_i that multiply each of the elements that exist in Geometric Algebra $Cl_{3,0}$ (the scalars, 3 vectors, 3 bivectors and the trivector). These factors ψ_i are real functions. So, no generality is lost, as the solution to Dirac equation are four complex functions (that if they could be divided in real and imaginary part, would map to 8 functions).

Now, we will include the wavefunction in the equation (13). In Geometric Algebra, when an operator has two parts and one is the reverse of the other, the function that is affected by the operator is always between both. You can check this in [1][3]. A typical operator like this are the rotors, but this is another story. We introduce our wave function between the two parts of the operator:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right)\psi\left(-\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right) = \left(\hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right)\psi\left(-\hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right)$$

If we take everything to the left side of the equation, we have:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right)\psi\left(-\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}\right) - \left(\hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right)\psi\left(-\hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right) = 0$$

As the geometric product is associative and distributive, we can take the wavefunction as common factor of both sides of the operator and get:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \left(\hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right)\right)\psi\left(-\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \left(-\hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z} + m\right)\right) = 0$$

Operating the signs, we have:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z} - m\right)\psi\left(-\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} + \hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} - m\right) = 0 \qquad (15)$$

As the equation is zero, it could be that the product of the first two elements of the triple product is zero:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}-m\right)\psi=0 \qquad (16)$$

Or it could be that the product of the last two elements is zero:

$$\psi\left(-\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}+\hat{y}\hat{z}\frac{\partial}{\partial x}+\hat{z}\hat{x}\frac{\partial}{\partial y}+\hat{x}\hat{y}\frac{\partial}{\partial z}-m\right)=0 \qquad (17)$$

As convention and what Dirac did was to take the first half of the equation (16) (but in standard algebra not in Geometric Algebra).

One first comment of that equation (16) is that the element that represents energy (the partial derivative with respect to time) is positive while the ones regarding momentum (the partial derivatives with respect to spatial coordinates) are negative. Remember the minus sign of the momentum? It appears naturally here. Even if we chose the equation (17) you can check that the sign of the momentum would be the opposite of the sign of the energy again.

Continuing, if we put the complete equation with all the elements of the wavefunction we have:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}-m\right)\left(\psi_0+\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\hat{x}\hat{y}\psi_{xy}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx}+\hat{x}\hat{y}\hat{z}\psi_{xyz}\right)=0 \quad (18)$$

If we operate it,we have:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}-m\right)\left(\psi_0+\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\hat{x}\hat{y}\psi_{xy}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx}+\hat{x}\hat{y}\hat{z}\psi_{xyz}\right)=0$$

$$\begin{aligned}
&\hat{x}\hat{y}\hat{z}\frac{\partial\psi_0}{\partial t}+\hat{y}\hat{z}\frac{\partial\psi_x}{\partial t}+\hat{z}\hat{x}\frac{\partial\psi_y}{\partial t}+\hat{x}\hat{y}\frac{\partial\psi_z}{\partial t}-\hat{z}\frac{\partial\psi_{xy}}{\partial t}-\hat{x}\frac{\partial\psi_{yz}}{\partial t}-\hat{y}\frac{\partial\psi_{zx}}{\partial t}-\frac{\partial\psi_{xyz}}{\partial t}-\\
&-\hat{y}\hat{z}\frac{\partial\psi_0}{\partial x}-\hat{x}\hat{y}\hat{z}\frac{\partial\psi_x}{\partial x}+\hat{z}\frac{\partial\psi_y}{\partial x}-\hat{y}\frac{\partial\psi_z}{\partial x}-\hat{z}\hat{x}\frac{\partial\psi_{xy}}{\partial x}+\frac{\partial\psi_{yz}}{\partial x}+\hat{x}\hat{y}\frac{\partial\psi_{zx}}{\partial x}+\hat{x}\frac{\partial\psi_{xyz}}{\partial x}-\\
&-\hat{z}\hat{x}\frac{\partial\psi_0}{\partial y}-\hat{z}\frac{\partial\psi_x}{\partial y}-\hat{x}\hat{y}\hat{z}\frac{\partial\psi_y}{\partial y}+\hat{x}\frac{\partial\psi_z}{\partial y}+\hat{y}\hat{z}\frac{\partial\psi_{xy}}{\partial y}-\hat{x}\hat{y}\frac{\partial\psi_{yz}}{\partial y}+\frac{\partial\psi_{zx}}{\partial y}+\hat{y}\frac{\partial\psi_{xyz}}{\partial y}-\\
&-\hat{x}\hat{y}\frac{\partial\psi_0}{\partial z}+\hat{y}\frac{\partial\psi_x}{\partial z}-\hat{x}\frac{\partial\psi_y}{\partial z}-\hat{x}\hat{y}\hat{z}\frac{\partial\psi_z}{\partial z}+\frac{\partial\psi_{xy}}{\partial z}+\hat{z}\hat{x}\frac{\partial\psi_{yz}}{\partial z}-\hat{y}\hat{z}\frac{\partial\psi_{zx}}{\partial z}+\hat{z}\frac{\partial\psi_{xyz}}{\partial z}-\\
&-m\psi_0-\hat{x}m\psi_x-\hat{y}m\psi_y-\hat{z}m\psi_z-\hat{x}\hat{y}m\psi_{xy}-\hat{y}\hat{z}m\psi_{yz}-\hat{z}\hat{x}m\psi_{zx}-\hat{x}\hat{y}\hat{z}m\psi_{xyz}\\
&=0 \qquad (19)
\end{aligned}$$

Now, for this equation to be zero, it has to be zero the sum of all the elements that multiply the same vector, bivector or trivector. This is, if we take all the elements that multiply the trivector $\hat{x}\hat{y}\hat{z}$ we get this equation:

$$\frac{\partial\psi_0}{\partial t}-\frac{\partial\psi_x}{\partial x}-\frac{\partial\psi_y}{\partial y}-\frac{\partial\psi_z}{\partial z}-m\psi_{xyz}=0 \quad (20)$$

If we take the elements that multiply the $\hat{y}\hat{z}$ bivector, we have:

$$\frac{\partial\psi_x}{\partial t}-\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_{xy}}{\partial y}-\frac{\partial\psi_{zx}}{\partial z}-m\psi_{yz}=0 \quad (21)$$

And so on. Doing this for the scalars, the vectors, bivectors and the trivector, we get these eight equations:

$$\frac{\partial\psi_0}{\partial t}-\frac{\partial\psi_x}{\partial x}-\frac{\partial\psi_y}{\partial y}-\frac{\partial\psi_z}{\partial z}-m\psi_{xyz}=0 \quad (20)$$
$$\frac{\partial\psi_x}{\partial t}-\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_{xy}}{\partial y}-\frac{\partial\psi_{zx}}{\partial z}-m\psi_{yz}=0 \quad (21)$$
$$\frac{\partial\psi_y}{\partial t}-\frac{\partial\psi_{xy}}{\partial x}-\frac{\partial\psi_0}{\partial y}+\frac{\partial\psi_{yz}}{\partial z}-m\psi_{yz}=0 \quad (22)$$
$$\frac{\partial\psi_z}{\partial t}+\frac{\partial\psi_{zx}}{\partial x}-\frac{\partial\psi_{yz}}{\partial y}-\frac{\partial\psi_0}{\partial z}-m\psi_{xy}=0 \quad (23)$$
$$-\frac{\partial\psi_{xy}}{\partial t}+\frac{\partial\psi_y}{\partial x}-\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_{xyz}}{\partial z}-m\psi_z=0 \quad (24)$$
$$-\frac{\partial\psi_{yz}}{\partial t}+\frac{\partial\psi_{xyz}}{\partial x}+\frac{\partial\psi_z}{\partial y}-\frac{\partial\psi_y}{\partial z}-m\psi_x=0 \quad (25)$$
$$-\frac{\partial\psi_{zx}}{\partial t}-\frac{\partial\psi_z}{\partial x}+\frac{\partial\psi_{xyz}}{\partial y}+\frac{\partial\psi_x}{\partial z}-m\psi_y=0 \quad (26)$$
$$-\frac{\partial\psi_{xyz}}{\partial t}+\frac{\partial\psi_{yz}}{\partial x}+\frac{\partial\psi_{zx}}{\partial y}+\frac{\partial\psi_{yx}}{\partial z}-m\psi_0=0 \quad (27)$$

But is this correct? And what is the mapping of this wavefunction to the wavefunction obtained by the Dirac equation? So, let's check in the next chapter.

10. Solution to Dirac Equation in standard matrix algebra

In this chapter we will not use Geometric Algebra. We will just follow paper [4] to get the equations needed to solve the general Dirac Equation.

In matrix algebra the solution to Dirac equation has this form:

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \quad (28)$$

Where the ψ_k are complex functions. If we consider that they can be divided in the real and the imaginary part of the function, the wavefunction would have the form:

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}=\begin{pmatrix}\psi_{1r}+i\psi_{1i}\\ \psi_{2r}+i\psi_{2i}\\ \psi_{3r}+i\psi_{3i}\\ \psi_{4r}+i\psi_{4i}\end{pmatrix} \quad (29)$$

Now, we apply the Dirac equation in matrix algebra according [4]:

$$\begin{pmatrix} i\frac{\partial}{\partial t}-m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t}-m & i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial t}-m & 0 \\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t}-m \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Applying the division in real and imaginary parts commented, we have:

$$\begin{pmatrix} i\frac{\partial}{\partial t}-m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t}-m & i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & -i\frac{\partial}{\partial t}-m & 0 \\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t}-m \end{pmatrix}\begin{pmatrix} \psi_{1r}+i\psi_{1i} \\ \psi_{2r}+i\psi_{2i} \\ \psi_{3r}+i\psi_{3i} \\ \psi_{4r}+i\psi_{4i} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (30)$$

And now, performing the matrix multiplication, we have for the first line:

$$\left(i\frac{\partial}{\partial t}-m\right)(\psi_{1r}+i\psi_{1i})+\left(i\frac{\partial}{\partial z}\right)(\psi_{3r}+i\psi_{3i})+\left(i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(\psi_{4r}+i\psi_{4i})=0 \quad (31)$$

$$i\frac{\partial\psi_{1r}}{\partial t}-\frac{\partial\psi_{1i}}{\partial t}-m\psi_{1r}-im\psi_{1i}+i\frac{\partial\psi_{3r}}{\partial z}-\frac{\partial\psi_{3i}}{\partial z}+i\frac{\partial\psi_{4r}}{\partial x}-\frac{\partial\psi_{4i}}{\partial x}+\frac{\partial\psi_{4r}}{\partial y}+i\frac{\partial\psi_{4i}}{\partial y}=0 \quad (32)$$

Dividing in two equations, one for the real part and another one for the imaginary part, we get:

$$-\frac{\partial\psi_{1i}}{\partial t}-m\psi_{1r}-\frac{\partial\psi_{3i}}{\partial z}-\frac{\partial\psi_{4i}}{\partial x}+\frac{\partial\psi_{4r}}{\partial y}=0 \quad (33)$$

$$\frac{\partial\psi_{1r}}{\partial t}-m\psi_{1i}+\frac{\partial\psi_{3r}}{\partial z}+\frac{\partial\psi_{4r}}{\partial x}+\frac{\partial\psi_{4i}}{\partial y}=0 \quad (34)$$

For the second line of the matrix, we get:

$$\left(i\frac{\partial}{\partial t}-m\right)(\psi_{2r}+i\psi_{2i})+\left(i\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(\psi_{3r}+i\psi_{3i})+\left(-i\frac{\partial}{\partial z}\right)(\psi_{4r}+i\psi_{4i})=0$$

$$i\frac{\partial\psi_{2r}}{\partial t}-\frac{\partial\psi_{2i}}{\partial t}-m\psi_{2r}-im\psi_{2i}+i\frac{\partial\psi_{3r}}{\partial x}-\frac{\partial\psi_{3i}}{\partial x}-\frac{\partial\psi_{3r}}{\partial y}-i\frac{\partial\psi_{3i}}{\partial y}-i\frac{\partial\psi_{4r}}{\partial z}+\frac{\partial\psi_{4i}}{\partial z}=0$$

Again, dividing in two equations (real and imaginary part):

$$-\frac{\partial\psi_{2i}}{\partial t}-m\psi_{2r}-\frac{\partial\psi_{3i}}{\partial x}-\frac{\partial\psi_{3r}}{\partial y}+\frac{\partial\psi_{4i}}{\partial z}=0 \quad (35)$$

$$\frac{\partial\psi_{2r}}{\partial t}-m\psi_{2i}+\frac{\partial\psi_{3r}}{\partial x}-\frac{\partial\psi_{3i}}{\partial y}-\frac{\partial\psi_{4r}}{\partial z}=0 \quad (36)$$

For the third line of the equation, you have

$$\left(-i\frac{\partial}{\partial z}\right)(\psi_{1r}+i\psi_{1i})+\left(-i\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(\psi_{2r}+i\psi_{2i})+\left(-i\frac{\partial}{\partial t}-m\right)(\psi_{3r}+i\psi_{3i})=0$$

$$-i\frac{\partial\psi_{1r}}{\partial z}+\frac{\partial\psi_{1i}}{\partial z}-i\frac{\partial\psi_{2r}}{\partial x}+\frac{\partial\psi_{2i}}{\partial x}-\frac{\partial\psi_{2r}}{\partial y}-i\frac{\partial\psi_{2i}}{\partial y}-i\frac{\partial\psi_{3r}}{\partial t}+\frac{\partial\psi_{3i}}{\partial t}-m\psi_{3r}-im\psi_{3i}=0$$

Dividing in real and imaginary part, we get:

$$\frac{\partial\psi_{1i}}{\partial z}+\frac{\partial\psi_{2i}}{\partial x}-\frac{\partial\psi_{2r}}{\partial y}+\frac{\partial\psi_{3i}}{\partial t}-m\psi_{3r}=0 \quad (37)$$

$$-\frac{\partial\psi_{1r}}{\partial z}-\frac{\partial\psi_{2r}}{\partial x}-\frac{\partial\psi_{2i}}{\partial y}-\frac{\partial\psi_{3r}}{\partial t}-m\psi_{3i}=0 \quad (38)$$

And for the fourth line:

$$\left(-i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(\psi_{1r}+i\psi_{1i})+\left(i\frac{\partial}{\partial z}\right)(\psi_{2r}+i\psi_{2i})+\left(-i\frac{\partial}{\partial t}-m\right)(\psi_{4r}+i\psi_{4i})=0$$

$$-i\frac{\partial\psi_{1r}}{\partial x}+\frac{\partial\psi_{1i}}{\partial x}+\frac{\partial\psi_{1r}}{\partial y}+i\frac{\partial\psi_{1i}}{\partial y}+i\frac{\partial\psi_{2r}}{\partial z}-\frac{\partial\psi_{2i}}{\partial z}-i\frac{\partial\psi_{4r}}{\partial t}+\frac{\partial\psi_{4i}}{\partial t}-m\psi_{4r}-im\psi_{4i}$$
$$=0$$

Getting these two equations:

$$\frac{\partial\psi_{1i}}{\partial x}+\frac{\partial\psi_{1r}}{\partial y}-\frac{\partial\psi_{2i}}{\partial z}+\frac{\partial\psi_{4i}}{\partial t}-m\psi_{4r}=0 \quad (39)$$

$$-\frac{\partial\psi_{1r}}{\partial x}+\frac{\partial\psi_{1i}}{\partial y}+\frac{\partial\psi_{2r}}{\partial z}-\frac{\partial\psi_{4r}}{\partial t}-m\psi_{4i}=0 \quad (40)$$

Putting all the equations together:

$$-\frac{\partial\psi_{1i}}{\partial t}-m\psi_{1r}-\frac{\partial\psi_{3i}}{\partial z}-\frac{\partial\psi_{4i}}{\partial x}+\frac{\partial\psi_{4r}}{\partial y}=0 \quad (33)$$

$$\frac{\partial\psi_{1r}}{\partial t}-m\psi_{1i}+\frac{\partial\psi_{3r}}{\partial z}+\frac{\partial\psi_{4r}}{\partial x}+\frac{\partial\psi_{4i}}{\partial y}=0 \quad (34)$$

$$-\frac{\partial\psi_{2i}}{\partial t}-m\psi_{2r}-\frac{\partial\psi_{3i}}{\partial x}-\frac{\partial\psi_{3r}}{\partial y}+\frac{\partial\psi_{4i}}{\partial z}=0 \quad (35)$$

$$\frac{\partial\psi_{2r}}{\partial t}-m\psi_{2i}+\frac{\partial\psi_{3r}}{\partial x}-\frac{\partial\psi_{3i}}{\partial y}-\frac{\partial\psi_{4r}}{\partial z}=0 \quad (36)$$

$$\frac{\partial\psi_{1i}}{\partial z}+\frac{\partial\psi_{2i}}{\partial x}-\frac{\partial\psi_{2r}}{\partial y}+\frac{\partial\psi_{3i}}{\partial t}-m\psi_{3r}=0 \quad (37)$$

$$-\frac{\partial\psi_{1r}}{\partial z}-\frac{\partial\psi_{2r}}{\partial x}-\frac{\partial\psi_{2i}}{\partial y}-\frac{\partial\psi_{3r}}{\partial t}-m\psi_{3i}=0 \quad (38)$$

$$\frac{\partial\psi_{1i}}{\partial x}+\frac{\partial\psi_{1r}}{\partial y}-\frac{\partial\psi_{2i}}{\partial z}+\frac{\partial\psi_{4i}}{\partial t}-m\psi_{4r}=0 \quad (39)$$

$$-\frac{\partial\psi_{1r}}{\partial x}+\frac{\partial\psi_{1i}}{\partial y}+\frac{\partial\psi_{2r}}{\partial z}-\frac{\partial\psi_{4r}}{\partial t}-m\psi_{4i}=0 \quad (40)$$

The issue is that with this form, you cannot get a one-to-one map between these equations (33) to (40) coming from matrix algebra and the ones (20) to (27) coming from Geometric Algebra.

But we will get to it, do not worry.

11. Finding the one-to-one map from Dirac solution in matrix algebra to Geometric Algebra $Cl_{3,0}$

In the current times, when you have a problem, normally the best option is to look for someone that has solved it before. And in fact, this is the case. In [3] (8.20) Doran and Lasenby are able to create a one-to-one map for the Pauli equations. For the Dirac equation they do not explicitly give any solution. But they comment that the form of the equation should be [3] (8.87):

$$\nabla\psi I\sigma_3 = m\psi\gamma_0 \quad (41)$$

Their notation is different to the one I have used and so I will explain it. The ∇ symbol represents the operator that is applied to the wavefunction. It is equivalent to the first parenthesis in our equations (16) and (18).

The ψ represents the wavefunction (our second parenthesis in the equation (18)).

The I represents the trivector. And the σ_3 is the equivalent to our z. These equivalences I have commented already at the end of chapter 8.

So why in equation (41) they are multiplying by a specific direction like $\hat{z}$ (σ_3) (and not $\hat{x}$ or $\hat{y}$ for example)? They have the theory that in standard algebra with Pauli or Dirac equations, there is a preferred direction that is z. If you check the matrix in (30) of Dirac standard Algebra, y will see that the z direction is treated differently than the x and the y direction. There are some elements that mix x and y and other that only have z. Let's say that the matrix is somehow oriented and z direction has a different treatment than x and y in that matrix.

What Doan and Lasenby do multiplying by $\hat{z}$ (σ_3) is orienting the Pauli or Dirac equation in the same direction that the standard algebra is. So, they can create a one-to-one map. They say, it is equivalent as choosing another equivalent set of matrices for Pauli or Dirac standard algebra. The final result will not change, but the equations will be presented in another form.

The issue is that, as expected, they are right, as we will see now.

We will modify our Dirac equation to take into account this prefered direction in z. We will use the table of conversions at the end of chapter 8 to apply it to our equation, so:

$$\nabla\psi I\sigma_3 = m\psi\gamma_0$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi I\sigma_3 = m\psi\gamma_0$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi\hat{x}\hat{y}\hat{z}\hat{z} = m\psi\hat{x}\hat{y}\hat{z}$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi\hat{x}\hat{y} = m\psi\hat{x}\hat{y}\hat{z}$$

Now we postmultiply by $\hat{y}\hat{x}$ to simplify both sides of the equation:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi\hat{x}\hat{y}\hat{y}\hat{x} = m\psi\hat{x}\hat{y}\hat{z}\hat{y}\hat{x} = -m\psi\hat{x}\hat{y}\hat{y}\hat{z}\hat{x} = -m\psi\hat{x}\hat{z}\hat{x}$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi = m\psi\hat{z} \quad (42)$$

So, in the end, the equation (42) is the same as equation (16) but postmultiplying the element of the mass by $\hat{z}$.

The issue is that if you do exactly as this equation, it will not work also. I leave you as exercise.

The only way to find a real one-to-one map to the standard matrix algebra is to do one step more. And this step is to divide the wavefunction in two wavefunctions: one for the even grade elements and another one for the odd-grade elements. And to apply a different sign for each of these parts. This is something similar to consider two different projections depending on the helicity or chirality of the solution.

We will see the process here. First, we divide the ψ in two parts, one with even grade elements and another one with odd elements. So, the sum of both is the same psi we have been using until now.

$$\psi_{even} = \psi_0 + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} \quad (43)$$
$$\psi_{odd} = \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\hat{z}\psi_{xyz} \quad (44)$$

$$\psi = \psi_{even} + \psi_{odd} = \psi_0 + \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} + \hat{x}\hat{y}\hat{z}\psi_{xyz}$$

(45)

And now, we apply a similar equation as (41) but changing the same of the projection to $\hat{z}$ depending on if we are in the ψ_{even} or the ψ_{odd}. We start with the even part (making a positive projection in $\hat{z}$), leading to:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{even}I\sigma_3 = m\psi_{even}\gamma_0$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{even}\hat{x}\hat{y}\hat{z}\hat{z} = m\psi_{even}\hat{x}\hat{y}\hat{z}$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{even}\hat{x}\hat{y} = m\psi_{even}\hat{x}\hat{y}\hat{z}$$

We post multiply both sides by $\hat{y}\hat{x}$ to simplify

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{even}\hat{x}\hat{y}\hat{y}\hat{x} = m\psi_{even}\hat{x}\hat{y}\hat{z}\hat{y}\hat{x} = -m\psi_{even}\hat{x}\hat{y}\hat{y}\hat{z}\hat{x}$$
$$= -m\psi_{even}\hat{x}\hat{z}\hat{x}$$
$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{even} = m\psi_{even}\hat{z} \quad (46)$$

Foe the odd part we make a negative projection to $\hat{z}$.

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{odd}(-I\sigma_3) = m\psi_{odd}\gamma_0$$
$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{odd}(-\hat{x}\hat{y}\hat{z}\hat{z}) = m\psi_{odd}\gamma_0$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{odd}(-\hat{x}\hat{y}) = m\psi_{odd}\gamma_0$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{odd}\hat{y}\hat{x} = m\psi_{odd}\hat{x}\hat{y}\hat{z}$$

We postmultply both sides by xy to simplify the equation:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{odd}\hat{y}\hat{x}\hat{x}\hat{y} = m\psi_{odd}\hat{x}\hat{y}\hat{z}\hat{x}\hat{y} = m\psi_{odd}\hat{x}\hat{x}\hat{y}\hat{z}\hat{y}$$
$$= -m\psi_{odd}\hat{y}\hat{y}\hat{z}$$
$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{odd} = -m\psi_{odd}\hat{z} \quad (47)$$

Now, if we sum equations (46) and (47) we get:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{even} + \left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi_{odd}$$
$$= m\psi_{even}\hat{z} - m\psi_{odd}\hat{z}$$
$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)(\psi_{even}+\psi_{odd}) = m\psi_{even}\hat{z} - m\psi_{odd}\hat{z}$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi = m\psi_{even}\hat{z} - m\psi_{odd}\hat{z}$$

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi - m\psi_{even}\hat{z} + m\psi_{odd}\hat{z} = 0 \quad (48)$$

We see that above equation (48) is similar to original ones (16) and (18) but with a projection in positive z axis or negative z axis depending on the parity of the elements of the wavefunction.

Now, continuing operating, we get:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)(\psi_0+\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\hat{x}\hat{y}\psi_{xy}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx}+\hat{x}\hat{y}\hat{z}\psi_{xyz})$$
$$-m(\psi_0+\hat{x}\hat{y}\psi_{xy}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx})\hat{z} + m(\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\hat{x}\hat{y}\hat{z}\psi_{xyz})\hat{z}$$
$$= 0 \quad (49)$$

Making the multiplication element by element, we get:

$$\hat{x}\hat{y}\hat{z}\frac{\partial\psi_0}{\partial t}+\hat{y}\hat{z}\frac{\partial\psi_x}{\partial t}+\hat{z}\hat{x}\frac{\partial\psi_y}{\partial t}+\hat{x}\hat{y}\frac{\partial\psi_z}{\partial t}-\hat{z}\frac{\partial\psi_{xy}}{\partial t}-\hat{x}\frac{\partial\psi_{yz}}{\partial t}-\hat{y}\frac{\partial\psi_{zx}}{\partial t}-\frac{\partial\psi_{xyz}}{\partial t}-$$
$$-\hat{y}\hat{z}\frac{\partial\psi_0}{\partial x}-\hat{x}\hat{y}\hat{z}\frac{\partial\psi_x}{\partial x}+\hat{z}\frac{\partial\psi_y}{\partial x}-\hat{y}\frac{\partial\psi_z}{\partial x}-\hat{z}\hat{x}\frac{\partial\psi_{xy}}{\partial x}+\frac{\partial\psi_{yz}}{\partial x}+\hat{x}\hat{y}\frac{\partial\psi_{zx}}{\partial x}+\hat{x}\frac{\partial\psi_{xyz}}{\partial x}-$$
$$-\hat{z}\hat{x}\frac{\partial\psi_0}{\partial y}-\hat{z}\frac{\partial\psi_x}{\partial y}-\hat{x}\hat{y}\hat{z}\frac{\partial\psi_y}{\partial y}+\hat{x}\frac{\partial\psi_z}{\partial y}+\hat{y}\hat{z}\frac{\partial\psi_{xy}}{\partial y}-\hat{x}\hat{y}\frac{\partial\psi_{yz}}{\partial y}+\frac{\partial\psi_{zx}}{\partial y}+\hat{y}\frac{\partial\psi_{xyz}}{\partial y}-$$
$$-\hat{x}\hat{y}\frac{\partial\psi_0}{\partial z}+\hat{y}\frac{\partial\psi_x}{\partial z}-\hat{x}\frac{\partial\psi_y}{\partial z}-\hat{x}\hat{y}\hat{z}\frac{\partial\psi_z}{\partial z}+\frac{\partial\psi_{xy}}{\partial z}+\hat{z}\hat{x}\frac{\partial\psi_{yz}}{\partial z}-\hat{y}\hat{z}\frac{\partial\psi_{zx}}{\partial z}+\hat{z}\frac{\partial\psi_{xyz}}{\partial z}-$$
$$-\hat{z}m\psi_0-\hat{z}\hat{x}m\psi_x+\hat{y}\hat{z}m\psi_y+m\psi_z-\hat{x}\hat{y}\hat{z}m\psi_{xy}-\hat{y}m\psi_{yz}+\hat{x}m\psi_{zx}+\hat{x}\hat{y}m\psi_{xyz}$$
$$= 0 \quad (50)$$

If we separate the equations (as we did at the end of chapter 9) depending on the element they are multiplying (the vector, bivector, trivector or scalars) we get these 8 equations:

$$\frac{\partial\psi_0}{\partial t}-\frac{\partial\psi_x}{\partial x}-\frac{\partial\psi_y}{\partial y}-\frac{\partial\psi_z}{\partial z}-m\psi_{xy} = 0 \quad (51)$$
$$\frac{\partial\psi_x}{\partial t}-\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_{xy}}{\partial y}-\frac{\partial\psi_{zx}}{\partial z}+m\psi_y = 0 \quad (52)$$

$$\frac{\partial\psi_y}{\partial t}-\frac{\partial\psi_{xy}}{\partial x}-\frac{\partial\psi_0}{\partial y}+\frac{\partial\psi_{yz}}{\partial z}-m\psi_x=0 \quad (53)$$
$$\frac{\partial\psi_z}{\partial t}+\frac{\partial\psi_{zx}}{\partial x}-\frac{\partial\psi_{yz}}{\partial y}-\frac{\partial\psi_0}{\partial z}+m\psi_{xyz}=0 \quad (54)$$
$$-\frac{\partial\psi_{xy}}{\partial t}+\frac{\partial\psi_y}{\partial x}-\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_{xyz}}{\partial z}-m\psi_0=0 \quad (55)$$
$$-\frac{\partial\psi_{yz}}{\partial t}+\frac{\partial\psi_{xyz}}{\partial x}+\frac{\partial\psi_z}{\partial y}-\frac{\partial\psi_y}{\partial z}+m\psi_{zx}=0 \quad (56)$$
$$-\frac{\partial\psi_{zx}}{\partial t}-\frac{\partial\psi_z}{\partial x}+\frac{\partial\psi_{xyz}}{\partial y}+\frac{\partial\psi_x}{\partial z}-m\psi_{yz}=0 \quad (57)$$
$$-\frac{\partial\psi_{xyz}}{\partial t}+\frac{\partial\psi_{yz}}{\partial x}+\frac{\partial\psi_{zx}}{\partial y}+\frac{\partial\psi_{yx}}{\partial z}+m\psi_z=0 \quad (58)$$

And now yes. I will write again below, the equations we got with matrix algebra at the end of chapter 10, (33) to (40) in a slight different order, So you can compare them with the ones above (51) to (58) (obtained with geometric algebra).

$$\frac{\partial\psi_{4i}}{\partial t}+\frac{\partial\psi_{1i}}{\partial x}+\frac{\partial\psi_{1r}}{\partial y}-\frac{\partial\psi_{2i}}{\partial z}-m\psi_{4r}=0 \quad (39)$$
$$-\frac{\partial\psi_{1i}}{\partial t}-\frac{\partial\psi_{4i}}{\partial x}+\frac{\partial\psi_{4r}}{\partial y}-\frac{\partial\psi_{3i}}{\partial z}-m\psi_{1r}=0 \quad (33)$$
$$\frac{\partial\psi_{1r}}{\partial t}+\frac{\partial\psi_{4r}}{\partial x}+\frac{\partial\psi_{4i}}{\partial y}+\frac{\partial\psi_{3r}}{\partial z}-m\psi_{1i}=0 \quad (34)$$
$$-\frac{\partial\psi_{2i}}{\partial t}-\frac{\partial\psi_{3i}}{\partial x}-\frac{\partial\psi_{3r}}{\partial y}+\frac{\partial\psi_{4i}}{\partial z}-m\psi_{2r}=0 \quad (35)$$
$$-\frac{\partial\psi_{4r}}{\partial t}-\frac{\partial\psi_{1r}}{\partial x}+\frac{\partial\psi_{1i}}{\partial y}+\frac{\partial\psi_{2r}}{\partial z}-m\psi_{4i}=0 \quad (40)$$
$$-\frac{\partial\psi_{3r}}{\partial t}-\frac{\partial\psi_{2r}}{\partial x}-\frac{\partial\psi_{2i}}{\partial y}-\frac{\partial\psi_{1r}}{\partial z}-m\psi_{3i}=0 \quad (38)$$
$$\frac{\partial\psi_{3i}}{\partial t}+\frac{\partial\psi_{2i}}{\partial x}-\frac{\partial\psi_{2r}}{\partial y}+\frac{\partial\psi_{1i}}{\partial z}-m\psi_{3r}=0 \quad (37)$$
$$\frac{\partial\psi_{2r}}{\partial t}+\frac{\partial\psi_{3r}}{\partial x}-\frac{\partial\psi_{3i}}{\partial y}-\frac{\partial\psi_{4r}}{\partial z}-m\psi_{2i}=0 \quad (36)$$

You can see that there is a one-to-one map that corresponds to:

$$\psi_{1r}=-\psi_y \quad (59)$$
$$\psi_{1i}=-\psi_x \quad (60)$$
$$\psi_{2r}=\psi_{xyz} \quad (61)$$
$$\psi_{2i}=\psi_z \quad (62)$$
$$\psi_{3r}=-\psi_{yz} \quad (63)$$
$$\psi_{3i}=\psi_{zx} \quad (64)$$
$$\psi_{4r}=\psi_{xy} \quad (65)$$
$$\psi_{4i}=\psi_0 \quad (66)$$

This means considering the solution in geometric algebra as:

$$\psi=\psi_0+\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\hat{x}\hat{y}\psi_{xy}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx}+\hat{x}\hat{y}\hat{z}\psi_{xyz} \quad (45)$$

And the solution in matrix algebra as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_{1r} + i\psi_{1i} \\ \psi_{2r} + i\psi_{2i} \\ \psi_{3r} + i\psi_{3i} \\ \psi_{4r} + i\psi_{4i} \end{pmatrix} \qquad (29)$$

There is a one-to one relation between both ways of representing the wave function (the (45) and the (29), which are the equations above (59) to (66).

For the equation of Dirac in Geometric Algebra, we have used is:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi - m\psi_{even}\hat{z} + m\psi_{odd}\hat{z} = 0 \qquad (48)$$

Where:

$$\psi_{even} = \psi_0 + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} \qquad (43)$$

$$\psi_{odd} = \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\hat{z}\psi_{xyz} \qquad (44)$$

$$\psi = \psi_{even} + \psi_{odd} = \psi_0 + \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} + \hat{x}\hat{y}\hat{z}\psi_{xyz} \ (45)$$

So, putting all the elements in equation (), we have:

$$\begin{aligned}\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)&\left(\psi_0 + \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} + \hat{x}\hat{y}\hat{z}\psi_{xyz}\right) \\ &- m\left(\psi_0 + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx}\right)\hat{z} + m\left(\hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\hat{z}\psi_{xyz}\right)\hat{z} \\ &= 0 \qquad (49)\end{aligned}$$

12. Conclusions

In this paper, we have created a one-to-one map between the Dirac Equation in Matrix Algebra and the Dirac Equation in Geometric Algebra $Cl_{3,0}$. To do that, the most surprising definition that was necessary was to define the time basis vector as function of the space vectors (only holding in this form in an orthonormal basis in Euclidean metric):

$$\hat{t}^{-1} = \hat{x}\hat{y}\hat{z} \qquad (7)$$

Once this is defined, the calculations are straight forward leading to a form of the Dirac equation in Geometric Algebra $Cl_{3,0}$ as:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi - m\psi_{even}\hat{z} + m\psi_{odd}\hat{z} = 0 \qquad (48)$$

Where:

$$\psi_{even} = \psi_0 + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} \qquad (43)$$

$$\psi_{odd} = \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\hat{z}\psi_{xyz} \qquad (44)$$

$$\psi = \psi_{even} + \psi_{odd} = \psi_0 + \hat{x}\psi_x + \hat{y}\psi_y + \hat{z}\psi_z + \hat{x}\hat{y}\psi_{xy} + \hat{y}\hat{z}\psi_{yz} + \hat{z}\hat{x}\psi_{zx} + \hat{x}\hat{y}\hat{z}\psi_{xyz} \ (45)$$

If the wavefunction solution in Matrix Algebra is defined as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_{1r} + i\psi_{1i} \\ \psi_{2r} + i\psi_{2i} \\ \psi_{3r} + i\psi_{3i} \\ \psi_{4r} + i\psi_{4i} \end{pmatrix} \qquad (29)$$

There is a one-to-one mapping of both representations:

$$\psi_{1r} = -\psi_y \quad (59)$$
$$\psi_{1i} = -\psi_x \quad (60)$$
$$\psi_{2r} = \psi_{xyz} \quad (61)$$
$$\psi_{2i} = \psi_z \quad (62)$$
$$\psi_{3r} = -\psi_{yz} \quad (63)$$
$$\psi_{3i} = \psi_{zx} \quad (64)$$
$$\psi_{4r} = \psi_{xy} \quad (65)$$
$$\psi_{4i} = \psi_0 \quad (66)$$

Bilbao, 29th October 2022 (viXra-v1).

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If you consider this helpful, do not hesitate to drop your BTC here:

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A1. Annex A1. Considering non-orthonormal basis

In the paper we have considered all the time an orthonormal basis in Euclidean metric. I will give here some hints of what we should do in case we do not have an orthonormal basis or even if we work in a non-Euclidean metric in geometric algebra.

You will find more information in the papers [2] and [5].

If the basis is orthogonal but not orthonormal, the difference is in equations (1) to (3) that now, would read:

$$\hat{x}^2 = \hat{x}\hat{x} = \|\hat{x}\|^2 \quad (A1.1)$$
$$\hat{y}^2 = \hat{y}\hat{y} = \|\hat{y}\|^2 \quad (A2.2)$$

$$\hat{z}^2 = \hat{z}\hat{z} = \|\hat{z}\|^2 \qquad (A3.3)$$

Where in general the norm is different to 1. And depending on the signature of the metric could the square of the norm could be positive or negative.

So, for example, imagine a basis where:

$$\hat{x}^2 = \hat{x}\hat{x} = \|\hat{x}\|^2 = g_{xx} = 3^2 \qquad (A1.4)$$
$$\hat{y}^2 = \hat{y}\hat{y} = \|\hat{y}\|^2 = g_{yy} = -5^2 \qquad (A1.5)$$
$$\hat{z}^2 = \hat{z}\hat{z} = \|\hat{z}\|^2 = g_{zz} = 2^2 \qquad (A1.6)$$

You can see that we have added the nomenclature g_{ii} typical for a diagonal element of the metric tensor in a non-Euclidean metric, typically in general relativity for example. In Geometric Algebra these g_{ii} are the same as the square of the norm of the basis vectors. Check [2] for more information.

Imagine we have to perform the following operation that represents whatever physics calculation in that basis/metric:

$$(2 + \hat{x})(5\hat{x}\hat{y} + 7\hat{x})$$

We will perform the product as usual:

$$(2 + \hat{x})(5\hat{x}\hat{y} + 7\hat{x}) = 10\hat{x}\hat{y} + 14\hat{x} + 5\hat{x}\hat{x}\hat{y} + 7\hat{x}\hat{x} =$$

Now, we have to apply (A1.1) to perform the calculation of the square of $\hat{x}$.

$$= 10\hat{x}\hat{y} + 14\hat{x} + 5(3^2)\hat{y} + 7(3^2) = 10\hat{x}\hat{y} + 14\hat{x} + 45\hat{y} + 63$$

As the basis is still orthogonal (but not orthonormal), if we would need to make a reversion of vectors, we would have used the equations (4) to (6) as we have done all along the paper.

But if the basis is not orthogonal? Here is where the things get more complicated. In that case, we cannot use the reverse equations (4) to (6). Instead, we have to use the following equations, to make a reversion [2]:

$$\hat{x}\hat{y} = 2g_{xy} - \hat{y}\hat{x} \qquad (A1.7)$$
$$\hat{y}\hat{z} = 2g_{yz} - \hat{z}\hat{y} \qquad (A1.8)$$
$$\hat{z}\hat{x} = 2g_{zx} - \hat{x}\hat{z} \qquad (A1.9)$$

Where the g_{ij} correspond to the cross component of the metric tensor between x and y in a nin-Euclidean metric. These components g_{ij} can be considered also as the scalar product of the two basis vectors $\hat{x}$ and $\hat{y}$.

In fact, an easy to demonstrate relations (A1.7) to (A1.9) is via the definition of the scalar product in Geometric Algebra. You can find this definition in [1] and [3] (2.3).

$$\hat{x} \cdot \hat{y} = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2}$$

Considering the element g_{xy} of the metric tensor ans the scalar product of the two basis vectors:

$$\hat{x} \cdot \hat{y} = g_{xy} = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2}$$

And now, operating:

$$g_{xy} = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2}$$

$$2g_{xy} = \hat{x}\hat{y} + \hat{y}\hat{x}$$
$$2g_{xy} - \hat{y}\hat{x} = \hat{x}\hat{y}$$

$$\hat{x}\hat{y} = 2g_{xy} - \hat{y}\hat{x}$$

So, you get the relations (A1.7) to (A1.9).

Now, imagine a non-orthonormal and non-orthogonal metric where the relations (A1.4) to (A1.6) apply and also we know that:

$$g_{xy} = 3 \quad (A1.10)$$
$$g_{yz} = 2 \quad (A1.11)$$
$$g_{zx} = 7 \quad (A1.12)$$

And we want to calculate:

$$(2\hat{y} + \hat{x})(5\hat{x}\hat{y} + 7\hat{x} + 3\hat{y}) =$$

$$(2\hat{y} + \hat{x})(5\hat{x}\hat{y} + 7\hat{x} + 3\hat{y}) = 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} + 6\hat{y}\hat{y} + 5\hat{x}\hat{x}\hat{y} + 7\hat{x}\hat{x} + 3\hat{x}\hat{y} =$$

First, we operate the squares using equation (A1.4) to (A1.6).

$$= 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} + 6(-5^2) + 5(3^2)\hat{y} + 7(3^2) + 3\hat{x}\hat{y} =$$
$$= 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} - 150 + 45\hat{y} + 63 + 3\hat{x}\hat{y} =$$
$$= 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$

Now, we reverse two vectors of the first element, so we can get a square of $\hat{y}$.But, we cannot do it as we always have done, just changing the sign. Now, we are in a non-orthogonal basis, so we have to use (A1.7) to (A1.12).

$$= 10\hat{y}\left(2g_{xy} - \hat{y}\hat{x}\right) + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$
$$= 10\hat{y}(2(3) - \hat{y}\hat{x}) + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$
$$= 60\hat{y} - 10\hat{y}\hat{y}\hat{x} + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$

Now, we use (A1.4) to (A1.6) for the square of $\hat{y}$. And we sum the elements that multiply the vector $\hat{y}$.

$$= 60\hat{y} - 10(-5^2)\hat{x} + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$
$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 3\hat{x}\hat{y} =$$

Now, we reverse the last element (using (A1.7) to (A1.12).), so we can sum it to the third element.

$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 3\left(2g_{xy} - \hat{y}\hat{x}\right) =$$
$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 3(2(3) - \hat{y}\hat{x}) =$$
$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 18 - 3\hat{y}\hat{x} =$$

Now, we sum the scalars and the third and the last element.

$$= 105\hat{y} + 250\hat{x} + 11\hat{y}\hat{x} - 69 =$$

We cannot simplify more, so this would be the result. In case that for convention we should have to leave $\hat{x}\hat{y}$ instead of $\hat{y}\hat{x}$ in a certain discipline, we could have used the following equation that is another form for the equation (A1.7), to leave everything in $\hat{x}\hat{y}$ form. You can obtain the equation, just changing the side where $\hat{x}\hat{y}$ and $\hat{y}\hat{x}$ are.

$$\hat{y}\hat{x} = 2g_{xy} - \hat{x}\hat{y}$$

Another important point is the inverse of the vectors in a non-orthonormal basis. If we take (A1.1):

$$\hat{x}\hat{x} = \|\hat{x}\|^2 \qquad (A1.1)$$

And you premultiply by $\hat{x}^{-1}$both sides of the equation, you have:

$$\hat{x}^{-1}\hat{x}\hat{x} = \hat{x}^{-1}\|\hat{x}\|^2$$

By definition, the product of the inverse of a vector by the vector itself is 1.

$$(1)\hat{x} = \hat{x}^{-1}\|\hat{x}\|^2$$

$$\hat{x} = \hat{x}^{-1}\|\hat{x}\|^2$$

Now, the square of the norm is a scalar (it is a number, not a vector), so we can pass it to the other side dividing:

$$\frac{\hat{x}}{\|\hat{x}\|^2} = \hat{x}^{-1}$$

Exchanging sides:

$$\hat{x}^{-1} = \frac{\hat{x}}{\|\hat{x}\|^2} \qquad (A1.13)$$

Doing the same for the other vectors, we get:

$$\hat{x}^{-1} = \frac{\hat{x}}{\|\hat{x}\|^2} \qquad (A1.13)$$

$$\hat{y}^{-1} = \frac{\hat{y}}{\|\hat{y}\|^2} \qquad (A1.14)$$

$$\hat{z}^{-1} = \frac{\hat{z}}{\|\hat{z}\|^2} \qquad (A1.15)$$

A2. Annex A2. Time as the trivector

In this chapter I will develop a little more regarding time being the trivector. Also, how it is used when we are in a non-othonormal basis (and/or non-Euclidean metric)

First, we will comment regarding the time vector $\hat{t}$ and its inverse $\hat{t}^{-1}$. In general, it is more practical to work and to give the original definition to $\hat{t}^{-1}$ instead of $\hat{t}$. The reason is in physics (including Quantum Mechanics) the time appears normally dividing. As in general, it is the magnitude that is used to take the derivatives. See for example equation (10) and the ones before it, in chapter 9.

So, we start defining:

$$\hat{t}^{-1} = \hat{x}\hat{y}\hat{z}$$

If we premultiply by $\hat{t}$ in both sides:

$$\hat{t}\,\hat{t}^{-1} = \hat{t}\,\hat{x}\hat{y}\hat{z}$$

By definition, the product of the inverse of a vector by the vector itself is 1.

$$1 = \hat{t}\,\hat{x}\hat{y}\hat{z}$$

Now, we postmultiply by $\hat{z}^{-1}$ both sides.

$$\hat{z}^{-1} = \hat{t}\,\hat{x}\hat{y}\hat{z}\hat{z}^{-1}$$

Again, the product of a vector by its inverse is 1.

$$\hat{z}^{-1} = \hat{t}\,\hat{x}\hat{y}$$

Now, we postmultiply by $\hat{y}^{-1}$ both sides and we operate.

$$\hat{z}^{-1}\hat{y}^{-1} = \hat{t}\,\hat{x}\hat{y}\hat{y}^{-1}$$
$$\hat{z}^{-1}\hat{y}^{-1} = \hat{t}\,\hat{x}$$

In the last step we will post multiply by $\hat{x}^{-1}$.

$$\hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \hat{t}\,\hat{x}\hat{x}^{-1}$$
$$\hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \hat{t}$$

So,

$$\hat{t} = \hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1}$$

In a non-orthonormal basis, we have to use the equations (A1.13) to (A1.15) to calculate the inverses:

$$\hat{t} = \hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \frac{\hat{z}}{\|\hat{z}\|^2}\frac{\hat{y}}{\|\hat{y}\|^2}\frac{\hat{x}}{\|\hat{x}\|^2}$$

In an orthonormal basis, the norms are equal to 1, so we get the relation that has been commented in the paper (7.1) for orthonormal bases:

$$\hat{t} = \hat{z}\hat{y}\hat{x} = -\hat{x}\hat{y}\hat{z} = -\hat{t}^{-1}$$

One thing to comment is the time basis vector in $Cl_{1,3}$ that is commented in the literature [1][3] normally denoted as γ_0. γ_0 has positive signature and its norm is 1 so:

$$\gamma_0\gamma_0 = \|\gamma_0\|^2 = 1$$

So, its inverse is itself. We can prove it premultiplying by its inverse:

$$\gamma_0\gamma_0 = 1$$
$${\gamma_0}^{-1}\gamma_0\gamma_0 = {\gamma_0}^{-1}$$
$$(1)\gamma_0 = {\gamma_0}^{-1}$$
$$\gamma_0 = {\gamma_0}^{-1}$$
$${\gamma_0}^{-1} = \gamma_0$$

Our $\hat{t}$ and $\hat{t}^{-1}$ have negative signature (they are the trivector, see chapter 8 for its definition and chapter 4 to check the negative signature of the trivector).

This means, we can choose γ_0 to be $\hat{t}$ or $\hat{t}^{-1}$, what we prefer as a convention, if we keep the same definition all the time. Choosing one or another will only change the sign of $\hat{t}$ in all the subsequent equations but all of them will be coherent among them if we keep the convention in all the equations.

In the paper we have chosen to consider γ_0 to $\hat{t}^{-1}$.

Another thing I commented in Annex 3 of [5] is that time instead of being exact inverse of the spatial dimensions they could be related by a constant k that could be Ricci scalar, trace of the metric tensor, product of the diagonal elements of the metric tensor, determinant of the metric tensor… This is, a scalar related to the metric, a constant that is necessary to normalize the value of $\hat{t}$ or $\hat{t}^{-1}$ compared with the space elements.

$$\hat{t} = \frac{1}{\hat{t}^{-1}} = \frac{k}{\hat{x}\hat{y}\hat{z}} = k\cdot\hat{x}^{-1}\hat{y}^{-1}\hat{z}^{-1} = k\,\frac{\hat{x}\hat{y}\hat{z}}{\|\hat{x}\|^2\|\hat{y}\|^2\|\hat{z}\|^2}$$

It is important to remark, as I did in [2] and [5], that if the basis vector $\hat{t}$ is composed by the space basis vectors, it does not mean that the dimension time is not independent from the space ones. The parameter t (without hat) that multiplies the basis vector $\hat{t}$ (with hat) is completely free and independent. The dimension of time exists although its basis vector is somehow related to the space ones. In fact, in geometric algebra, having three space vectors imply the existence of 8 dimensions (scalars, 3 basis vectors, 3 bi-vectors and one pseudoscalar (the time in this approach)). You can check this in [3] for example. So, time would be just one of these 8 dimensions (the trivector/pseudoscalar) appearing from the three space dimensions.

A3. Annex A3. The Electromagnetic Field Strength and the Lorentz Force in Geometric Algebra $Cl_{3,0}$. The Electromagnetic Trivector

In a following paper I will comment about the Electromagnetic Field and the Lorentz force in Geometric Algebra. One of the important points there is that another effects that are not considered in standard Electromagnetic calculations affect the particles.

These effects are normally invisible or neglected as they affect only the orientation of the particle or create oscillatory movements which average value in the trajectory is zero. But they have to be taken into account if we want to define more precisely the trajectory in a very local (or small) frame and if we want to understand why some measurements of spin or orientation of the particles have certain results.

We will see that if the Electric field is a vector E_i and the magnetic field a bivector B_i. It exists an Electromagnetic trivector B_{xyz} which can create the effects commented above.

$$F_{Electromagnetic\ Field\ Strength} = E_0 + E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + B_{xyz}\hat{x}\hat{y}\hat{z}$$

The reason why it has not been considered until now is because their effects do not have any consequence in the macroscopic world, they do not change the velocity or trajectory of the particles but yes they have consequences in small frames (oscillatory movements, change of orientations -spin- etc…).

Also, it exists the electromagnetic scalar E_0 but its meaning is more related to a general scalation of all the magnitudes in a certain frame. So, no local effect will be seen as all the magnitudes are escalated in the same way. The local interactions will not see any change among them. But from another frame distant from that, this escalation would be perceived as an escalation in the metric, lengths, time etc…So it seems more related to gravitation than Electromagnetism? This opens another door; I will comment also this in Annex A4.

I will comment about this new definition of Electromagnetic Field in in a next paper.

Same thing could be commented for example in equation (48) or (49) of this paper.

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi - m\psi_{even}\hat{z} + m\psi_{odd}\hat{z} = 0 \qquad (48)$$

The first parenthesis of the Dirac Equation in GA $Cl_{3,0}$ (48), has only the trivector and bivectors. Could it have any meaning to add vectors or scalar to check the influence in the results? This has to be studied. But in this case, it is important to recall how the equation was obtained. It has that form because we needed the cross products to disappear. If we add more elements, this could be not the case (that does not mean that is wrong, it only means that a lot of new effects should be considered and check experimentally if they really exist or not).

A4. Annex A4. Normalization of the wavefunction in Quantum Mechanics. A loss of information?

As I have commented in A3, it could be that the scalar component of the electromagnetic field E_0 has an effect of escalation of all the magnitude in a frame. So inside the local frame no change will be noted as all magnitudes (lengths, time…) change in the same proportion for all the elements of the interaction. But for a distant observer probably he could see the difference of these values compared to its own local frame.

This means, the absolute value of the magnitudes is not important in a local frame, only the relative differences between them to understand the interactions. But a distant observer could see not only the relative differences of the magnitudes in that frame but the absolute values as he can compare with its own frame values. He can see that in that far frame the things are slower or bigger than in his own frame. But the ones that are in the distant frame if they only see their own interactions, he cannot see any difference as he measures with elements inside his own frame affected also for whatever escalation is happening.

So, what does this have to do with the normalization of the wavefunction?

The standard process is to get the value of unknown constants in the wavefunction, normalizing it. So, the square of the wavefunction is always 1. And the square of the partial coefficients of the wave function have only a value between 0 and 1 representing the probability.

If the reason of normalizing is only that, it is not really necessary. You can define the probability as the square of the partial coefficient divided by the square of the wavefunction (even if it is not 1 and has whatever other value). The result will also be a number between 0 and 1 representing the probability.

When you have normalized you have lost information of the real square of the wavefunction. That you could keep there and use as denominator when you want to calculate probabilities between 0 and 1.

The answer here, would be that normally you have free constants where can select the value we want and we decide to normalize the function, so really, we have not lost information.

But if is this not really the case? Could it not be that we have a lack of equations that we still do not know (and we see that Geometric Algebra can create a lot of them) that would apply a specific value to these constants? And the square of the wavefunction instead of being always 1 have a value that represents something? For example, a kind of escalation in its own frame similar of what we have commented in Annex A3 for the scalar of the Electromagnetic Field? I keep the question open, but it is something that it should be checked in the future.