The General Hohmann Transfer

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An analytical method is presented for tangent transfers (Hohman type transfers) between non-coaxial elliptical orbits. Since Hohmann transfers are thought not to apply to non-coaxial orbits, this method generalizes the Hohmann transfer, typically used only between circular orbits. Since tangent transfers are less complex, as they require no change in direction, they offer an alternative to other orbital transfer and rendezvous methods.

1 Introduction

A profound pioneering effect for space travel came in 1925, when Walter Hohmann published his book, "The Attainability of Heavenly Bodies" [1]. The book describes many aspects of space travel including liftoffs and landings, passenger considerations and destinations. More importantly, he describes the route taken to these destinations. The route taken from Earth to Venus, say, is an elliptical path, connecting the orbit of Earth around the sun, to the orbit of Venus around the sun, i.e. connecting circular orbits. The points of connection require, in the words of Walter Hohmann, "changes in velocity, but no changes in direction".

Since then, Hohmann transfers have been considered to apply only to circular orbits or coaxial elliptical orbits (see [2, chap. 6], for example). Here, we generalize Hohmann transfers between any two orbits in a plane, coaxial or not - offering an alternative procedure for orbital transfers and rendezvous. The advantages are similar to the standard Hohmann transfer - a less complex transfer, tangent to both orbits. As activities in space increase (space tourism, for instance), these transfers may be of interest.

2 The Hohmann Transfer

A typical Hohmann transfer is shown in Fig. 1. An elliptical orbit is generated connecting the initial and final orbits through an impulse velocity at point A. Then, a circular orbit is generated through an impulse velocity at point B. At the apse points of the transfer ellipse, the velocities are tangent to the paths of the orbits. The procedure is listed in the appendix.

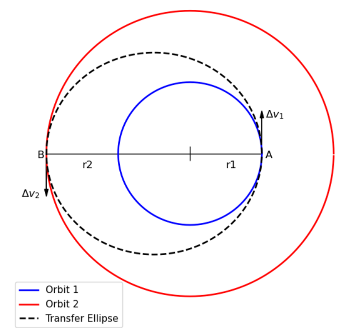

Fig. 1: Hohmann transfer example.

In contrast, two same plane non-coaxial orbits are shown in Fig. 2. The usual Hohmann transfer will not work. Any generated intermediate ellipse will not be tangent at the point on the target orbit that crosses the apse line of the initial orbit.

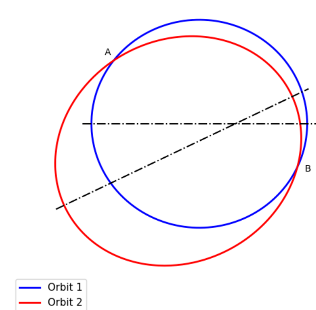

Fig. 2: Two non-coaxial orbits.

In Table 1 we define parameters for the initial and target orbits, designated by the subscripts 1 and 2 respectively, used to generate the plot in Fig. 2. The angle, θ_0, is the rotation of the target orbit apse line relative to the initial orbit apse line. The data will be used later to calculate the transfer.

Table 1: Orbital definitions used in Fig. 2.

Orbit	Orbital Parameters	Values
1	e_1	1/3.0
	r_{1p} [km]	8000.0
	r_{1a} [km]	16000.0
	h_1 [km]	$\sqrt{r_{1p}u(1+e_1)}$
2	e_2	1/2.0
	r_{2p} [km]	7000.0
	r_{2a} [km]	21000.0
	h_2 [km]	$\sqrt{r_{2p}u(1+e_2)}$
	θ_0 [rad]	25.0 $(\pi/180)$

3 General Transfer Method

There are at least two angles (or true anomalies) that designate points where the flight path angles of two orbits are equal, i.e. the orbits are tangent to each other along the radial line at these angles. If an (transfer) ellipse coincides with these points, then two orbital equations for the known distances and a third equation for the known flight angles would determine the three unknowns of the ellipse: the eccentricity (e), the angular momentum (h) and the degree of apse line rotation (ϕ) relative to the initial orbit.

3.1 The Tangent Points

Tangent points between any two orbits with a common focus can be found analytically by equating the flight path angles as follows:

$$\tan(\gamma_1) = \tan(\gamma_2) \tag{1}$$

where the flight path angle, γ, is the angle between the orbiting body's velocity vector (the vector tangent to the instantaneous orbit) and the local horizontal, the line perpendicular to radial line passing through the object in orbit, as shown in Fig. 3.

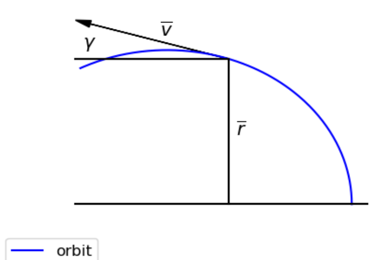

Fig. 3: Flight Path Angle (γ)

The equations for the flight path angles can be written

$$e_1 \frac{\sin(\theta)}{(1 + e_1\cos(\theta))} = e_2 \frac{\sin(\theta - \theta_0)}{(1 + e_2\cos(\theta - \theta_0))} \tag{2}$$

resulting in

$$a\cos(\theta) + b\sin(\theta) = c \tag{3}$$

where

$$\begin{aligned} e &= \text{eccentricity} \\ \theta &= \text{true anomaly} \\ \theta_0 &= \text{rotation of target apse line} \\ a &= e_2\sin(\theta_0) \\ b &= (e_1 - e_2\cos(\theta_0)) \\ c &= e_1 e_2 \sin(\theta_0) \end{aligned}$$

and we finally solve for two angles where the two orbits are tangent (77° and 224° in Fig. 4 below).

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(c \quad \frac{\cos\left(\tan^{-1}\left(\frac{b}{a}\right)\right)}{a}\right). \tag{4}$$

3.2 Analytic Derivation

For convenience, we reference the initial, target and transfer orbits, orbits 1,2 and 3, respectively. We also reference the two angles (true anomalies) where the flight path angles are equal, θ_1 and θ_2. The distances of orbits 1 and orbit 2 are known at these angles. Since orbit 3 is tangent and touching both of these points, we have $r_3(\theta_1) = r_1(\theta_1)$ and $r_3(\theta_2) = r_2(\theta_2)$. This gives us two of the three equations needed to solve for orbit 3:

$$r_3(\theta_1) = \frac{h_3^2}{\mu(1 + e_3\cos(\theta_1 - \phi))} \tag{5}$$

and

$$r_3(\theta_2) = \frac{h_3^2}{\mu(1 + e_3\cos(\theta_2 - \phi))} \tag{6}$$

where $\mu = \mu_{\text{earth}} = \text{GM} = 398600\ \text{Km}^3/\text{sec}^2$.

The third equation is obtained by equating flight path angles. First, the flight path angles, γ's, at θ_1 are equal and known, and the flight path angles at θ_2 are equal and known. Second, a simplification can be obtained by recognizing that the flight path angles at θ_1 are equal and opposite to the flight path angles at θ_2. For example, in Fig. 1, the flight path angles at $\theta = 0°$ are equal and opposite to the flight path angles at $\theta = 180°$. This is not obvious between non-coaxial orbits. However, it can be demonstrated as a general property by rotating orbit 2 relative to orbit 1 and evaluating the flight angles at various angular displacements. This was done at 45^o intervals and tabulated in Table 2 from the data in Table 1 .

The flight angle equation (see equation 2) for orbit 3, at the appropriate true anomalies, θ_1 and θ_2, can then be placed in the following relation:

$$e_3 \frac{\sin(\theta_1 - \phi)}{(1 + e_3\cos(\theta_1 - \phi))} = (-)e_3 \frac{\sin(\theta_2 - \phi)}{(1 + e_3\cos(\theta_2 - \phi))} \tag{7}$$

Table 2: Flight angles (γ) at various rotations are equal and opposite. (in Deg.)

Apse Line Rotation	θ_1	θ_2	γ_1	γ_2
45	106.16	247.28	20.21	-20.21
90	139.79	287.58	16.53	-16.53
135	161.53	323.98	8.84	-8.84
180	180	0	0	0
225	36.01	198.46	8.84	-8.84
270	72.41	220.20	16.53	-16.53
315	112.71	253.83	20.21	-20.21

Using equations 5 and 6 gives us:

$$e_3 \frac{r_3(\theta_1)\mu \sin(\theta_1 - \phi)}{h_3^2} = (-)e_3 \frac{r_3(\theta_2)\mu \sin(\theta_2 - \phi)}{h_3^2} \tag{8}$$

To solve for ϕ, rearrange Eq. 8 using a trig identity and A $= (-)r_3(\theta_2)/r_3(\theta_1)$:

$$\tan(\phi) = \frac{A \sin(\theta_2) - \sin(\theta_1)}{A \cos(\theta_2) - \cos(\theta_1)} \tag{9}$$

Using equations 5 and 6 and the definition of A above:

$$e_3 = \frac{(-)(A + 1)}{\cos(\theta_1 - \phi) + A \cos(\theta_2 - \phi)} \tag{10}$$

$$h_3 = \sqrt{r_3(\theta_1)\mu(1 + e_3 \cos(\theta_1 - \phi))} \tag{11}$$

The transfer ellipse, plotted in Fig. 4, is now fully defined and represented by:

$$r_3(\theta) = \frac{h_3^2}{\mu(1 + e_3 \cos(\theta - \phi))} \tag{12}$$

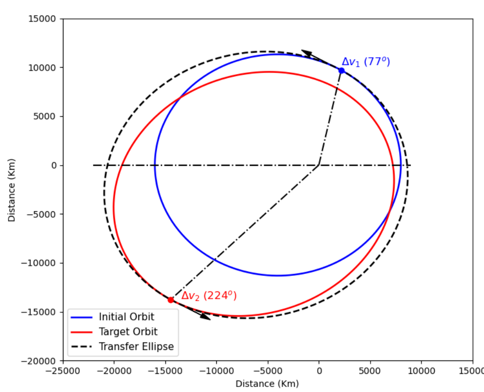

Fig. 4: Analytically generated transfer ellipse

3.3 Impulse Velocities

We now calculate the impulse velocities to produce the transfer. First, calculate the orbital parameters for the transfer ellipse. From eq. 5 and eq. 6, calculate the distances to the transfer points, used to calculate A in eq. 9 and eq. 10:

at 77°: $r_3 = r_1 = 9923$ Km

at 224°: $r_3 = r_2 = 19915$ Km

A = -(19915/9923) = -2.007

from eq's 9, 10 and 11 and the data in Table 1:

$\phi_3 = 19°$

$e_3 = 0.429$

$h_3 = 69670$ Km^2/sec

The tangent velocities at the transfer points are calculated as follows:

$v_1 = \mu(1 + e_1 \cos(\theta))/h_1$

$v_2 = \mu(1 + e_2 \cos(\theta - \theta_\circ))/h_2$

$v_3 = \mu(1 + e_3 \cos(\theta - \phi))/h_3$

The impulse velocities are calculated similar to the standard Hohmann transfer (see appendix 1):

at 77°: $\Delta v_1 = v_3 - v_1 = (7.02 - 6.57)$ Km/sec

at 224°: $\Delta v_2 = v_2 - v_3 = (3.77 - 3.50)$ Km/sec

$\Delta v_{\text{Total}} = \Delta v_1 + \Delta v_2 = 0.72$ Km/sec

4 Discussion

Implementation of the general Hohmann transfer is similar to the standard Hohmann transfer. The impulse is always in the direction of the orbits. This makes transfers less complicated. However, the standard Hohmann transfer is also an optimal transfer [3]. Optimal transfers conserve fuel. General Hohmann transfers are fuel efficient, but not optimal.

An optimal transfer is a numeric calculation that minimizes fuel (velocities). A comparison between an optimal and tangent transfer is shown in appendix 2. The optimal transfer, $\Delta v_{\text{Total}} = 711$ m/sec, is less than the tangent transfer, $\Delta v_{\text{Total}} = 719$ m/sec - a difference of about 1%.

This is a general result. However, the optimal transfer is not a tangent transfer. The impulse is not in the direction of the orbits. This makes transfers more complicated. The tangent transfer offers an alternative, at a minimal increase in fuel.

The tangent transfer is close to optimal. If there is no practical difference in velocity between the two transfers, the tangent transfer should be considered. Conserving fuel is important, but safety and reliability of transfer may be items of concern, particularly if space travel becomes popular.

5 Conclusion

This procedure generalizes the Hohmann transfer. These transfers are less complex and easier to implement than other transfer methods. They are not optimal concerning fuel, but they are close.

This is a new method of transfer. It should be evaluated in areas of transfer applications, particularly where optimal transfers are used. If this method makes a transfer or rendezvous more safe or reliable, it should be considered.

Similar tangent transfers can be generated numerically at most points on the initial orbit where they exist (for example, a transfer would not exist at points of intersection). Numerically we can demonstrate the analytic method shown here is the lowest energy transfer of these types.

6 Appendix 1: Hohmann Transfer

Calculation of Hohmann transfers for circular orbits, as shown in Fig. 1, is straightforward. In a circular orbit, the centrifugal force is equal to the gravitational force, and the velocity can be obtained from the relation

$$\frac{v^2}{r} = \frac{GM}{r^2} \quad \text{or} \quad v = \sqrt{\frac{GM}{r}} \tag{13}$$

where

r is the radius of the orbit

v is the velocity of the object in orbit

G is the gravitational constant

M is the mass of the primary

In an elliptical orbit, motion is governed by the equation

$$r(\theta) = \frac{h^2}{GM(1 + e\cos(\theta))} \tag{14}$$

where

θ is the angle of the orbit (true anomaly)

e is the eccentricity, $(r_2 - r_1)/(r_2 + r_1)$

h is the specific angular momentum

At periapsis, the angle is zero, and the velocity, with no radial component, can be determined by

$$v_{ellipse@A} = \sqrt{\frac{GM(1+e)}{r_1}} \tag{15}$$

At apoapsis, since the angular momentum of an orbit around a central force is a constant, the velocity can be determined by

$$v_{ellipse@B} = r_1 \frac{v_{ellipse@A}}{r_2} \tag{16}$$

The impulsive thrusts are then calculated as follows:

$$\Delta v_1 = v_{ellipse@A} - v_1 \tag{17}$$

$$\Delta v_2 = v_2 - v_{ellipse@B} \tag{18}$$

$$\Delta v_{\text{Total}} = \Delta v_1 + \Delta v_2 \tag{19}$$

7 Appendix 2: Optimal Transfer

A comparison of impulse velocities, between an optimal and tangent transfer, is shown below. Optimization minimizes an item of interest in a set of orbital equations, such as time or fuel.

The parameters in Table 1, used to generate the orbits in Fig. 2, were input into optimization software with the results listed below. The optimization routine is based off the papers by McCue [4], Lee [5] and Bender and McCue [6].

The Δv_{Total} = 710.9 m/sec for the optimal transfer. The Δv_{Total} = 719.2 m/sec for the tangent transfer. This is an increase in fuel (velocity) of about 1%.

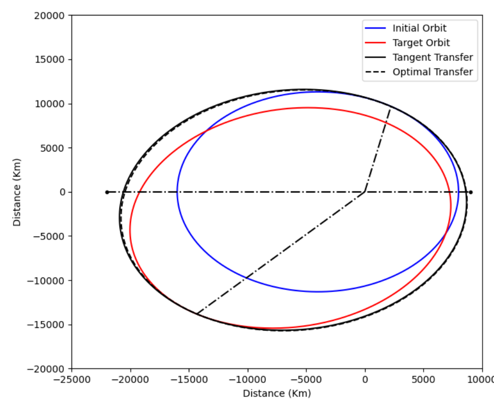

Fig. 5: Comparison of optimal versus tangent transfers. Optimal ellipse (obscured) is dashed black - tangent ellipse is solid black.

Changes in Velocity

	Total ΔV [m/sec]	710.9
ΔV_1	Total ΔV_1 [m/sec]	459.1
	ΔV_1 Along-track	456.2
	ΔV_1 Across-track	-0.016
	ΔV_1 Radial	51.570
	MA initial Orb [deg]	40.532
ΔV_2	Total ΔV_2 [m/sec]	251.8
	ΔV_2 Along-track	-248.7
	ΔV_2 Across-track	-0.010
	ΔV_2 Radial	38.917
	MA final Orb [deg]	225.476

Table 3: Optimal impulse velocity for Fig. 5

Transfer Orbit Data	
SMA [km]	14914.59851
ECC	0.428353482
Inc [deg]	10.0
Arg of Peri [deg]	19.12800631
RAAN [deg]	359.999
Periapse [km]	2147.73831
Apoapse [km]	14925.17871
MA at Departure [deg]	22.00286475
MA at Arrival [deg]	231.8594478
Transfer Angle [deg]	148.6738861
Transfer Time [min]	176.1153365
Phi 1 [deg]	254.40112862
Phi 2 [deg]	43.07517233

Table 4: Optimal orbit parameters in Fig. 5 (required for orbit: Arg of Peri, Periapse, Apoapse)

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