Calculation of the Cosmological Constant for Fermion and Boson Electric Charges Swap Lie-Groupoid

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Abstract: Following a previous article [1], in this paper, we investigate a version of the Standard Model (SM) algebroid with the anchor map depending on the Electric Charge Swap (ECS) angle θs. We find that many SM algebras depend on θs. We call these ECSM algebras. Furthermore, the SM algebroid is integrable to the SM groupoid; our results, therefore, potentially extend well beyond this case. We then investigate how the massive ECS particle can be derived from the breaking of the symmetry of the SM groupoid. We find that the ECS particle mass is related to the SM particle mass through the ECS angle θs. We investigate the finite subgroups of the ECS Möbius transformations. The ECS angle θs could originate from the ECS dihedral group that refers to the symmetry of the Particle polygon (P-gon). This angle can then be determined through the multi-triangulation of a convex-particle P-gon. Finally, we find that, at loop-level, the ECS Physics is different from the SM physics, and the ECSM mass is suppressed by the particle Catalan numbers C_P. For 24-fermions [1], and 6-vector gauge bosons, the calculated one-loop radiative correction to the bare cosmological constant Λ_0 is 10^{-47}GeV^4—very close to the experimental value.

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1. Introduction

The simplest example of dark energy (DE) is a cosmological constant [2-10], introduced by A. Einstein in 1917 [11-15]. In the presence of the cosmological constant, the Einstein equations acquire the form:

$$R_{mn} - \frac{1}{2} g_{mn} R = \frac{8\pi G}{c^4} T_{mn} + \Lambda g_{mn} . \tag{1}$$

Following Varun Sahni [16], in the Friedman-Robertson-Walker (FRW) Universe, considering of pressureless dust and Λ, the Rychaudhury equation which follows from equation (1) takes the form:

$$\ddot{a} = -\frac{4\pi G}{3} a \rho_m + \frac{\Lambda}{3} . \tag{2}$$

The repulsive nature of the term Λ could be responsible for the accelerating expansion of the Universe, as demonstrating in Equation (2).

In the 1960s, it was realized that the zero-point vacuum fluctuations must respect Lorenz invariance, and, therefore, have the form $\langle T_{mn} \rangle = \Lambda g_{mn}$ [17]. The vacuum expectation value of the energy-momentum tensor is, therefore, divergent for bosonic and fermionic fields. This gives rise to the 'cosmological constant problem'. The effective cosmological constant generated by the vacuum fluctuations is as follows:

$$\frac{\Lambda}{8\pi G} = \langle T_{00} \rangle_{vac} \simeq \int_0^\infty k^2 \sqrt{k^2 + m^2}\, dk \quad [16]. \tag{3}$$

Since the integral divergence is k^4, we get an infinite value for vacuum energy. Even if we impose an ultraviolet (UV) cutoff at the Planck scale of energy ($M_{Planck} = 10^{19}\, GeV$), we are still left with a very large value for vacuum energy $\langle T_{00} \rangle_{vac} \simeq c^5 / G^2 \hbar \sim 10^{76}\, GeV^4$ is 123 orders of magnitude larger than the currently observed value, $\rho_\Lambda = 10^{-47}\, GeV^4$ [16].

Some supersymmetric theories require a cosmological constant that is exactly zero, which further complicates things [18, 19]. This 'cosmological constant problem' may be the most challenging problem of fine-tuning in physics: there is no known natural way to derive the tiny cosmological constant from particle physics.

Furthermore, no vacuum in the string-theory landscape is known to support a metastable, positive cosmological constant. In 2018, a group of four physicists advanced a controversial conjecture implying that no such Universe exists [20]. The very small window of Λ values that allows life to emerge led some cosmologists to propose the anthropic argument for the existence of a small cosmological constant [21-24].

In view of the Hubble tension and the CMB dipole, however, it was recently proposed that the cosmological principle is no longer valid in the late Universe, and that the FRW metric breaks down [25][26][27]. It is thus possible that observations usually attributed to an accelerating universe are simply a result of the cosmological principle being non-applicable in the late Universe [28]. As recentl work by Hooft, Susskind and others has shown, a positive cosmological constant has surprising consequences, such as a finite maximum entropy of the observable Universe (the 'holographic principle') [29].

A different, but less widespread view is that there might not be anything in need of an explanation [30]. According to this view, the 'cosmological constant problem' may be a pseudo-problem: the value of the (observed) cosmological constant is a new fundamental constant which may not be derivable from some fundamental theory. As Bludman and Ruderman note, one can get any value (including zero) for the vacuum energy in QFT by adding suitable counter terms to the Lagrangian [30].

For quarks and leptons, Electric Charge Swap (ECS) symmetry was proposed by the author [31-34]. ECS transformation between ordinary families of leptons produces heavy, neutral, non-regular leptons of an O-order mass (TeV). These particles may form cold dark matter [31]. Furthermore, the ECS symmetry could explain certain properties of lepton families within the framework of superstring theories [35-39].

From the mathematical point of view, in R-Category (a category theory with invertible morphisms [40]), the geometric structures under consideration are always associated with local Lie brackets [,] on sections of some vector bundles (Lie algebroids [41,42]). Based on [41-43], in this article, we study the structure of transitive Lie algebroids as a mathematical framework for generalising the formulation of a gauge theory through an action functional:

the integral of a differential form on the algebroid [44]. On Atiyah Lie algebroids [45], the space of ordinary-connection 1-forms corresponds to the Ehresmann connections on a principal fiber bundle P (Lazzarini and Masson in [46]). Cédric Fournel (2013) [44] proved that transitive Lie algebroids equipped with generalised connections contain scalar fields as algebraic parameters. These parameters, absent in differential geometry, have a role similar to that of the scalar field in the Higgs mechanism [47]. In higher-dimensional spacetime, the Double Field Theory (DFT) is a gravity theory with manifest T duality (Hull-Zwiebach, 2009 [48]). The DFT has gauge symmetry (described by the C bracket ($[\ ,\ ]_C$: see[48])), which defines the Vaisman-algebroid (Vaisman, 2013[49]).

In a previous article [1], based to the R-Category theory, we found that, at loop-level, the ECS Physics differs from the SM physics, and the ECSM mass is suppressed by the fermionic Catalan numbers, C_F. For 24-fermions, the calculated one-loop radiative correction to the bare Higgs mass μ^2 is 125 GeV—a value very close to the experimental one.

Following that previous article [1], in this paper, we investigate a version of the SM algebroid with the anchor map dependent on the ECS angle, θs. We find that many SM algebras depend on θs; we call these ECSM algebras. Furthermore, the SM algebroid is integrable to the SM groupoid. Our results, therefore, potentially extend well beyond this case. Then, we investigate how the breaking of the SM groupoid symmetry gives the massive ECS particle. We find that the mass of the ECS particle mass is related to that of the SM particle through the ECS angle θs. We investigate the finite subgroups of the ECS Möbius transformations. In this case, the ECS angle θs could originate from the ECS dihedral group that refers to the symmetry of the Particle polygon (P-gon). The ECS angle θs can then be determined through the multitriangulation of a convex P-gon.

2. The global ECS symmetry for particles and fields in R-Category

Hypothetical non-regular quarks are, a) a 1/3-electric charged version of the up (α) quark types, $\tilde{\alpha}$ and, b) a -2/3- electric charged version of the down (κ) quark types, $\tilde{\kappa}$. Non-regular quarks may, therefore, be obtained by the swap of electric charges between up and down quark types. We call these proposed non-regular quarks electric charge swap (ECS) quarks [31]. Such automorphisms are rotations of $S^2 = SU(2)/U(1)$ [31]. These non-regular quarks can be expressed as ECS transformations from $S^2 = SU(2)_{B_s}/U(1)_{Y_s}$,whose declarations has been gauged in [31], and where, B_s is the non-regular quark (non-ordinary) baryonic number, and, Y_s is the swap hypercharge.

If follows that the automorphism from the $S^2 = SU(2)/U(1)$ to itself, which brings the electric charge swap between the up (α) and down (κ) quark types, is derived from the isometries that form a proper subgroup of the group of projective linear transformations $PGL_2(\hat{\mathbb{C}})_{(charge)}$, $PSU_2(\hat{\mathbb{C}})$. Subgroup $PSU_2(\hat{\mathbb{C}})$ is isomorphic to the rotation group $SO(3)^{(ECS)}$ [31], which is the isometric group of the unit sphere in three-dimensional real space $\mathbb{R}^3$. The automophism of the Riemann sphere $\hat{\mathbb{C}}$, is given by:

$$rot\left(\hat{\mathbb{C}}\right)_{ECS} = PSU_2^{(charge)}(\hat{\mathbb{C}}) = SO(3)^{(ECS)}, \tag{4}$$

where $\hat{\mathbb{C}}$, is the extended complex plane, $PSU_2(\hat{\mathbb{C}})$ is the proper subgroup of the projective linear transformations, and swap symmetry, $SO(3)^{(ECS)}$, is the group of rotations in 3-

dimensional vector space $\mathbb{R}^3$. This can be consigned in the double fibration on a vector bundle of lines $\Im^2$, in the extended internal space (ad infinitum), that is to say, $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. The universal cover of $SO(3)^{(ECS)}$, is the special unitary group $SU(2)^{(ECS)}$ [31]. This group is also diffeomorphic to the unit 3-sphere S^3 . Non-regular quarks obtained by the swap of electric charges between up (α) and down (κ) quark types, are given by Equation (4). We regard ordinary and ECS quarks as different electric charge states of the same particle – analogous, that is, to the proton-neutron isotopic pair [31]. Some quantum numbers of the new ECS quarks are given in Table 1.

Table 1: Quantum numbers of the proposed ECS quarks.

(ECS)-quarks	Q: electric charge	I_{sz}: ECS isospin component	B_s: Baryonic number
u˜,c˜,t˜	1/3	-½	-1/3
d˜,s˜,b˜	-2/3	½	-1/3

Similar with hypothetical non-regular quarks, hypothetical non regular leptons are, a) a zero-charged version of the electron, $\tilde{e}^0$, and, b) a positively charged version of the electron neutrino, $\tilde{\nu}_e^+$. Non-regular leptons can, therefore, be obtained from the swap of electric charge between electrons and electron neutrinos in the internal space. We call these proposed non-regular leptons, electric-charge-swap (ECS) leptons [31]. The quantum numbers of the new ECS leptons are given in Table 2 [31].

Table 2. Quantum numbers (weak ECS isospin I_s, charge Q, ECS hypercharge Y_s, ECS lepton number L_S) of the ECS leptons $\tilde{e}_L^0$, $\tilde{\nu}_e^+$.

(ECS)-lepton	I	Is-z	Q	Y_s L_s
$\tilde{\nu}_e^+$	½	½	1	1 -1
$\tilde{e}_L^0$	½	-½	0	1 -1

The simplest way to realize the global $SO(3)_{ECS}$ group provided by Equation (11) is by adding the ECS electron $\tilde{e}_L^0$ and grouping it together with the ECS electron neutrino $\tilde{\nu}_e^+$, electron antineutrino $\bar{\nu}_e$, and electron e^- into a triplet [1]. Similar for quarks ,we have to add an 1/3-electrically charged version of the up (α) quark types (ECS-α quarks) and group them together with the -2/3-electrically-charged version of the down (κ) quark types (ECS-κ quarks), and the anti-up $(\bar{\alpha})$ and down (κ) quark types into a triplet.

The representative matrix of a general element of the $SO(3)_{ECS}$ group can be written as:

$$O_{ECS} = \mathrm{e}^{-i\sum_{\alpha}\theta_s^{\alpha} X_{\alpha}}, \ \alpha=(1,2,3), \tag{5}$$

with $\theta^{\alpha} = (\theta^1,\theta^2,\theta^3)$ being arbitrary real group parameters independent of space-time coordinates, and O_{ECS} being an orthogonal 3 × 3 matrix:

$$O_{ECS}^T O_{ECS} = I = O_{ECS} O_{ECS}^T . \tag{6}$$

The three basic ECS rotation matrices that rotate fermions by an angle θ_s^{α} (α = 1, 2, 3) about the x-, y-, or z-axis in three internal dimensions can be explicitly written as follows:

$$O_{ECS} = \mathrm{e}^{-i\theta_s^{\alpha} X_{\alpha}} \ . \tag{7}$$

In this representative space, the representative matrices of the generators of the $SO(3)_{ECS}$ group are denoted by X_α (α = 1, 2, 3):

$$X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

These matrices satisfy the following condition:

$$[\mathrm{X}_{\alpha}, \mathrm{X}_{\beta}] = iC_{\alpha\beta\gamma} X_{\gamma} \ , \tag{9}$$

where $C_{\alpha\beta\gamma}$ are structure constants of the $SO(3)_{ECS}$ group. The generator X_α is Hermitian and traceless:

$$X_{\alpha}^{\dagger} = X_{\alpha} \ \ , \ \mathrm{Tr}\mathrm{X}_{\alpha} = 0 \, . \tag{10}$$

To calculate the trace, we use the sum of the diagonal elements of the rotation matrix given by Equation (7):

$$Tr(O_{ECS}) = 2\cos\|\theta_s\| + 1 \, . \tag{11}$$

From Equation (11), it follows that the arbitrary absolute value of the ECS angle is:

$$\begin{gathered} \theta_s = \|\theta_s\| = \arccos\left(\frac{Tr(O_{ECS}) - 1}{2}\right) = \frac{\pi}{2} - \arcsin\left(\frac{Tr(O_{ECS}) - 1}{2}\right), \\ F(\theta_s) \equiv \sin\|\theta_s\| = \frac{Tr(O_{ECS})}{2} \end{gathered} \ . \tag{12}$$

From the mathematical point of view, in R-Category—a category theory with invertible morphisms [40]—the geometric structures we consider here are always associated with local Lie brackets [,] on sections of some vector bundles (Lie algebroids). Here, we define the ECS-Standard Model (ECSM) algebra $\tilde{\aleph} = \tilde{u}(2) \cong \mathbb{R}^4$ from the SM algebroid A_{SM} over the product $M = S^3 \times S^1$ manifold, which satisfies the conditions:

$$[\rho(\mathrm{I}_2), \rho(\tau_\alpha)] = \rho([\mathrm{I}_2, \tau_\alpha]) \tag{13}$$

$$[\mathrm{I}_2, \varphi\tau_\alpha] = \varphi[\mathrm{I}_2, \tau_\alpha] + (\rho(\mathrm{I}_2)\cdot\varphi)\tau_\alpha \qquad , \alpha=(1,2,3) \tag{14}$$

$$[\rho(\tau_\alpha), \rho(\tau_\beta)] = \rho([\tau_\alpha, \tau_\beta]) \tag{15}$$

$$[\tau_\alpha, \varphi\tau_\beta] = \varphi[\tau_\alpha, \tau_\beta] + (\rho(\tau_\alpha)\cdot\varphi)\tau_\beta \ , \quad \beta=(1,2,3) \tag{16}$$

(with τ_α being smooth sections of A_{SM}, and ϕ being a smooth function on $M = S^3 \times S^1$), when the anchor's smooth multiplication factor of the SM-algebroid, A_{SM}, is given by Equation (12), and the ECS generators are derived from the SM generators by the anchor map:

$$\tilde{I}_2 = \rho(I_2) = \mathrm{F}(\theta_s)I_2 = I_2 \sin\theta_s \in \Gamma(TM)\,, \tag{17}$$

$$\tilde{\tau}_2 = \rho(\tau_2) = \mathrm{F}(\theta_s)\tau_2 = \tau_2 \sin\theta_s \in \Gamma(TM)\,. \tag{18}$$

In the above equations, $\Gamma(TM)$ indicates the sections of the tangent bundle TM.

$$F = \mathrm{F}(\theta_s) = \sin\theta_s \tag{19}$$

indicates the smooth function on $S^3 \times S^1$, and θ_s are the arbitrary ECS-angles if we parametrise the unit 3-sphere by hyperspherical coordinates (x_0, x_1, x_2, x_3) and use $(\psi, \theta_s, \varphi)$. By restricting the domain of Equation (12), we obtain:

$$\begin{aligned} &\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1], \\ &\theta_s \mapsto \sin\theta_s \end{aligned} . \tag{20}$$

Function (20) is both one-one and onto; therefore, it has an inverse function:

$$\begin{aligned} &\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ &\theta_s \mapsto \sin^{-1}\theta_s \end{aligned} . \tag{21}$$

By Equations 20 and 21, the anchor of the SM-algebroid, A_{SM}, is thus both an one-one and onto map between the SM and ECS generators. Equations (13) and (15) define the ECSM algebra $\tilde{\aleph} = \tilde{u}(2) \cong \mathbb{R}^4$ in terms of the SM algebra $\aleph = u(2) \cong \mathbb{R}^4$ as follows:

$$[\tilde{I}_2, \tilde{\tau}_\alpha] = \mathrm{F}(\theta_s)[I_2, \tau_\alpha] = \sin\theta_s [I_2, \tau_\alpha] = 0 \;\;, \tag{22}$$

$$[\tilde{\tau}_\alpha, \tilde{\tau}_\beta] = \mathrm{F}(\theta_s)[\tau_\alpha, \tau_\beta] = \sin\theta_s [\tau_\alpha, \tau_\beta]\,. \tag{23}$$

Following Marius Crainic and Rui Loja Fernandes (2003) [41] and previous article [1], we now deduce the known inerrability. We observe that the $\tilde{U}(2), \mathrm{S\tilde{U}}(2)$, and $\tilde{U}(1)$ gauge groupoids are the $\mathrm{F}(\theta_s)$-valued U(2), SU(2), and U(1) gauge groups [50–51] in θs.

The ECS-angle θ_s in the anchor map (Equations 17-18) is strictly a global parameter, and may originate from a different group: either from the global $SO(3)_{ECS}$ group or from the finite subgroups of the ECS Möbius transformations (see [1]). Furthermore, as Weinstein illustrates [42], there is no assumption that a gauge transformation actually extends to the entire object U(2)*:* it may be that the gauge symmetry does not extend globally but affects only a part of U(2), while the ECS symmetry extends globally.

It turns out that the concept of spontaneous symmetry breaking plays an important role in the proposed theory of ECS electroweak interaction. The broken large groupoid symmetry

$$S\tilde{U}(2)_L \times \tilde{U}(1)_Y \to \tilde{U}(1)_{EM} \tag{24}$$

gives the massive ECS particles $\tilde{W}$ and $\tilde{Z}$ bosons [1]. In Equations (24), we observe that the spontaneous breaking of the large groupoid symmetry can only occur when the symmetry of electroweak interaction breaks spontaneously [1]:

$$SU(2)_L \times U(1)_Y \to U(1)_{EM} . \tag{25}$$

Equation (25) predicts the massive particles W and Z bosons, whose correct mass has already been known since 1983 [52–54]. In Equation 24, $\tilde{U}(1)_{EM}$ is the ECS gauge groupoid of electromagnetism, which is the $F(\theta_s)$-valued $U(1)_{EM}$ common electromagnetism in θs. Therefore, we may view the ECS electromagnetism sectors of the $\tilde{U}(1)_{EM}$ gauge groupoid as the mirror sectors [55-57] that are suppressed for small ECS angles θ_s (a proposition explored in a forthcoming work [58]). Following [1], the ECSM gauge boson masses are given as follows:

$$M_{\tilde{A}}(\theta_s) = M_A = 0, M_{\tilde{W}}(\theta_s) = M_W \, F(\theta_s), M_{\tilde{Z}}(\theta_s) = M_Z \, F(\theta_s) , \tag{26}$$

where

$$M_A = 0, M_W = \frac{1}{2} g\upsilon, M_Z = \frac{1}{2} \upsilon \sqrt{g^2 + g'^2} \tag{27}$$

are the gauge boson masses of the SM vector [52–54].Here, $\tilde{W}^{\pm}$ and $\tilde{Z}^0$ are the gauge bosons that mediate the ECS-exchanging electroweak interaction between the families of fermions (for details see [31]). The masses and mixing of the ECSM quarks and leptons have a common origin, as suggested in the SM [52–54]. Following [1], the ECSM quark masses given as follows:

$$m_{ij}^{\tilde{d}}(\theta_s) = \sum_n Y_{ij}^{\tilde{d}} \left\langle \tilde{\phi}^0(\theta_s) \right\rangle = m_{ij}^d \, F(\theta_s) \ , \ m_{ij}^{\tilde{u}}(\theta_s) = \sum_n Y_{ij}^{\tilde{u}} \left\langle \tilde{\phi}^0(\theta_s) \right\rangle = m_{ij}^u \, F(\theta_s) \ . \tag{28}$$

where

$$m_{ij}^d = \sum_n Y_{ij}^d \left\langle \phi^0 \right\rangle \ , \ m_{ij}^u = \sum_n Y_{ij}^u \left\langle \phi^0 \right\rangle \tag{29}$$

are the SM quark masses [52–54]. The masses and mixing of the ECSM quarks and leptons have a common origin, as suggested in the SM [52–54]. The ECSM quark masses depend on the arbitrary couplings and cannot be predicted. Furthermore, since ECSM quarks are not observed in isolation, their masses are not precisely defined.

Similarly, for the charges ECSM-leptons $\tilde{\nu}_e^+$, we have:

$$m_{ij}^{\tilde{l}}(\theta_s) = \sum_n Y_{ij}^{\tilde{l}} \left\langle \tilde{\phi}^0(\theta_s) \right\rangle = m_{ij}^l \, F(\theta_s) \ , \tag{30}$$

where

$$m_{ij}^{l}=\sum_{n}Y_{ij}^{l}\left\langle \phi^{0}\right\rangle \tag{31}$$

are the SM lepton masses [52–54]. The masses of the neutral ECSM-leptons ($\tilde{e}^{0}$) are given by the effective interaction:

$$\sum_{ij} f_{ij}^{(\tilde{e}^{0})}\bar{\phi}^{0}\left(\theta_{s}\right)^{2}\bar{\tilde{e}}_{i}^{0}\tilde{e}_{j}^{0}=m_{\tilde{e}_{ij}^{0}}\bar{\tilde{e}}_{i}^{0}\tilde{e}_{j}^{0}, \tag{32}$$

where

$$m_{\tilde{e}_{ij}^{0}}\left(\theta_{s}\right)=\sum_{ij} f_{ij}^{(\tilde{e}^{0})}\bar{\phi}^{0}\left(\theta_{s}\right)^{2}=F(\theta_{s})^{2}\sum_{ij} f_{ij}^{(\nu)}(\bar{\phi}^{0})^{2}=F(\theta_{s})^{2}m_{\nu_{ij}}, \tag{33}$$

$$f_{ij}^{(\tilde{e}^{0})}=f_{ij}^{(\nu)}\approx\frac{1}{M}, \tag{34}$$

$f_{ij}^{(\tilde{e}^{0})}$, $f_{ij}^{(\nu)}$ are the effective couplings for neutral ECSM-leptons $\tilde{e}^{0}$ and neutrinos ν respectively, and $m_{\nu_{ij}}=\sum_{ij} f_{ij}^{(\nu)}(\bar{\phi}^{0})^{2}$ is the neutrinos masses [59]. We expect M to be of order $(10^{15}-10^{18})GeV$.

3. The averages of the F(θs) function over the multi-triangulation of a convex-particle P-gon

We observe that the $\tilde{U}(2),\mathrm{S\tilde{U}}(2)$, and the $\tilde{U}(1)$ ECS gauge groupoids are the $\mathrm{F}(\theta_{s})$-valued U(2), SU(2), and U(1) gauge groups [52–54] in *θs*. The ECS angle θ_{s} in Equation (19) is strictly an arbitrary global parameter, and may originate from a different group—either the $SO(3)_{ECS}$ group or the finite subgroups of the ECS Möbius transformations, as we explain in this section. Let Γ be an ECS subgroup of $PSL_2(C)_{ECS}$, consisting of elliptic elements together with the identity. Then Γ_{ECS} is conjugate in $PSL_2(C)_{ECS}$ to a subgroup of $PSU_2(C)_{ECS}$ [60-61,62-65].

Now, by the group isomorphism [31], for every finite subgroup of ECS rotations (i.e., a subgroup of $PSU_2(C)_{ECS}$), Γ_{ECS}, of ECS rotations in C^{∞} (Equation (4)), one of the following holds [60-61,62-65]:

1 .*Γ is ECS-cyclic;*

2 .*Γ is ECS-dihedral;*

3. *Γ is the ECS symmetry group of a regular ECS tetrahedron (A_4), ECS octahedron (S_4), or ECS icosahedron (A_5).*

One can show that two finite ECS subgroups in $PSL_2(C)_{ECS}$ are conjugate if and only if they are isomorphic.

Here, D_F (with F being the number of fermions) refers to the symmetries of the Fermionic polygon (F-gon:having *F* fermionicsides)—a group of order 2*F*. In abstract algebra, D_{2F} refers to this same ECS dihedral group (for details of the Dirac equation on the polygon regions see [66-72, 1]). D_F is a subgroup of $O(2)_{ECS}\cong U(1)_{ECS}$, i.e., the group of ECS rotations (about the origin) and ECS reflections (across axes through the origin) of the plane. However, the notation ‘D_F’ is also used for a subgroup of $SO(3)_{ECS} \cong SU(2)_{ECS}$, which is also an abstract group: the proper symmetry group of an F-gon embedded in three-dimensional internal space (if the number of fermions is F ≥ 3).

The sum of the interior ECS angles of a simple *F*-gon is $(F-2)\pi$ radians. This is because any simple *F*-gon can be considered to be made up of $(F-2)$ triangles, each of which has an angle sum of π radians:

$$\theta_s(F) = \left(1 - \frac{2}{F}\right)\pi , \tag{35}$$

where

$$F = F_{ordinary} + F_{ECS} . \tag{36}$$

F is the total number of fermions, given by the sum of ordinary ($F_{ordinary}$) and ECS fermions (F_{ECS}). Following [73], we find that the number (C_F) of triangulations of a convex fermionic *F*-gon in the internal space satisfies the recursive formula:

$$C_F = C_0 C_{F-1} + C_1 C_{F-2} + ... + C_{F-1} C_0 \ , \tag{37}$$

where $C_0 = 1$ [74-76]. The numbers C_F are now called fermionic Catalan numbers. From (37), it follows that $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, and so on. Using generating functions and Segner's formula, an explicit formula for C_F can be developed, as follows [76]:

$$C_F = \frac{(2F)!}{(F+1)!F!}, \tag{38}$$

with F being the total number of fermions.

After the triangulation of a convex fermionic *F*-gon in three-dimensional internal space, Equation (19) becomes:

$$\sin\left(\theta_s(\mathrm{F})\right)_{(\Delta_F)} \equiv \frac{\pi \sin\left(\theta_s(\mathrm{F})\right)_{averge}}{C_F} = \frac{2}{C_F} \text{ , on } [0, \pi/2] \text{ ([1]),} \tag{39}$$

where

$$\sin\left(\theta_s(\mathrm{F})\right)_{averge} = \frac{2}{\pi} \text{ , on } [0, \pi/2]. \tag{40}$$

This is the normalised average sin of the ECS angle θ_s over the triangulation of a convex fermionic *F*-gon by the fermionic Catalan numbers C_F. Following, J. Jonsson 2003 [77], and Vincent Pilaud and Francisco Santos 2009 [78], the number of multi-triangulations of the convex particle-gon is equal to the determinant:

$$\det(C_p) = \left\| \begin{pmatrix} C_{F\pm B} & 0 & 0 \\ 0 & C_F & 0 \\ 0 & 0 & C_{B|} \end{pmatrix} \right\| = (C_{\mathrm{F}\pm B})(C_F)(C_B) \ , \tag{41}$$

where

$$C_p = \begin{pmatrix} C_{\mathrm{F}\pm B} & 0 & 0 \\ 0 & C_F & 0 \\ 0 & 0 & C_B \end{pmatrix}. \tag{42}$$

C_p ($p = \{F\pm B, F, B\}$) denotes the p-th particle Catalan number, and C_B, C_F, and C_{F+B} are the fermionic, bosonic, and combination Catalan numbers, respectively. $F = F_{Ord} + F_{ECS}$ [1], and $B = B_{Ord} + B_{ECS}$ are the total number of fermions and bosons, respectively. After the multi-triangulation of a convex particle *P*-gon in three-dimensional internal space, Equation (19) becomes:

$$\sin(\theta_s(\mathrm{P}))_{(\Delta_P)} = \frac{\pi \sin(\theta_s(\mathrm{P}))_{averge}}{\det(C_P)} = \frac{2}{(C_{F\pm B})(C_F)(C_B)} \text{ , on } [0, \pi/2], \tag{43}$$

where

$$\sin\left(\theta_s(P)\right)_{averge} = \frac{2}{\pi} \text{ , on } [0, \pi/2] \text{ .} \tag{44}$$

This is the normalised average sin of the ECS angle θ_s over the multi-triangulation of a convex P-gon by the particle Catalan numbers C_P.

4. Calculation of the cosmological constant

Now that the averages of the F(θs) function have been determined by the particle Catalan numbers C_P, we consider two possible scenarios of ECS contribution to the SM:

Loop level: The ECS Physics at loop-level differs from the SM physics; the ECSM mass is not identical to the SM mass. Therefore,

$$\sin(\theta_s(\mathrm{P}))_{(\Delta_P)} = \frac{2}{(C_{F\pm B})(C_F)(C_B)} = \frac{M_{p-\Delta_n}}{M_{p-SM}} \neq 1 \text{, on } [0, \pi/2] \tag{45}$$

implies

$$M_{p-\Delta_n} = \sin(\theta_s(\mathrm{P}))_{(\Delta_P)} M_{p-SM} = \frac{2M_{p-SM}}{(C_{F\pm B})(C_F)(C_B)} \text{, on } [0, \pi/2], \tag{46}$$

where $M_{p-\Delta_n}$ are the multitriangulation masses of the ECS particles (from Equations (26),(28),(30),(33)), M_{p-SM} are the corresponding masses of the SM particles, and C_p are the particle Catalan numbers for the multitriangulations of the particle P-gon.

Tree level: The ECS Physics at tree level is the same as the SM physics, and the ECSM masses are identical to the SM masses. Therefore,

$$\sin(\theta_s(\mathrm{P}))_{(\Delta_P)} = \frac{2}{(C_2)(C_1)(C_1)} = \frac{M_{p-\Delta_2}}{M_{p-SM}} = 1 \text{, on } [0, \pi/2] \tag{47}$$

implies

$$M_{p-\Delta_2} = M_{p-SM} \text{ ,} \tag{48}$$

where $M_{p-\Delta_2}$ are the ECS particle multitriangulation masses of the fermionic 2-gon, bosonic 1-gon, and the combination between fermions and bosons in the 1-gon are the corresponding SM particles masses. $C_2 = 2$ is the fermionic Catalan number for the triangulation of the fermionic 2-gon, $C_1 = 1$ is the bosonic Catalan number for the triangulation of the bosonic 1-gon, and $C_{2\text{-}1} = 1$ is the combination between the fermion and boson Catalan numbers for the triangulation of the 2-gon.

In the proposed ECS model ((46),(47)), the tree-level cosmological term, Λ_0, denotes Einstein's own 'bare' cosmological constant which in itself leads to a curvature of empty space when there is no matter or radiation present. We thus only consider the effect of the new ECS physics at loop level. It is usually assumed that the vacuum energy density ($<\rho>$) is equivalent to a contribution to the 'effective' cosmological constant in Einstein's Equations (1):

$$\Lambda_{eff} = \Lambda_0 + \frac{8\pi G}{c^4}\langle\rho\rangle_{vac} \,. \tag{49}$$

Summing over the zero-point energies of all normal modes for both ECS and ordinary fields yields the vacuum energy density:

$$\begin{aligned}\langle\rho\rangle_{vac} &= \int_0^\infty \frac{4\pi k^2_{\Delta_{(\tilde{w},\tilde{z})}}dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2_{\Delta_{(\tilde{w},\tilde{z})}} + m^2_{\Delta_{(\tilde{w},\tilde{z})}}} + \int_0^\infty \frac{4\pi k^2_{\Delta_{(\tilde{L}\tilde{\nu}_L)}}dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2_{\Delta_{(\tilde{L},\tilde{\nu}_L)}} + m^2_{\Delta_{(\tilde{L},\tilde{\nu}_L)}}} \\ &+ \int_0^\infty \frac{4\pi k^2_{\Delta_{(\tilde{\alpha},\tilde{\kappa})}}dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2_{\Delta_{(\tilde{\alpha},\tilde{\kappa})}} + m^2_{\Delta_{(\tilde{\alpha},\tilde{\kappa})}}} + \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2 + m^2_{(\mathrm{W,Z})}} \\ &+ \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2 + m^2_{(\mathrm{L},\nu_L)}} + \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2 + m^2_{(\alpha,\kappa)}} + \int_0^\infty \frac{k_\gamma^3 dk}{(2\pi)^2} + \int_0^\infty \frac{k_g^3 dk}{(2\pi)^2}\end{aligned} \tag{50}$$

For massive theories with a particle mass M_p, it is easy to translate this arbitrary dimensionless scale η to an arbitrary "unit of wave number" $k = \eta M_p$. For massless theories, one can similarly identify an arbitrary unit of wave number $k = \eta\Lambda$. For the proposed massive ECS theory with a multitriangulation particle mass $M_{\Delta p}$, we obtain the multitriangulation wave number:

$$k_{\Delta_P} = \eta M_{\Delta_{\tilde{P}}} = \frac{2\eta M_p}{\det(C_p)} = \frac{2k}{(C_{F+B})(C_F)(C_B)} \,, \tag{51}$$

where the "unit of wave number" is $k = \eta M_p$, and $M_{\Delta p}$ is given by the Equation (46). Substituting Equation (51) to (50) and assuming a cutoff up to the Planck energy scale and a cutoff up to the QCD energy scale Λ_{QCD} for gluons fields, we have:

$$\langle\rho\rangle_{vac} = \frac{\Lambda^4_{Planck}}{\pi^2\left(C_{F+B}C_F C_B\right)^4} + \frac{\Lambda^4_{Planck}}{16\pi^2} + \frac{\Lambda^4_{QCD}}{16\pi^2} = \langle\rho\rangle^{ECS}_{vac} + \frac{\Lambda^4_{Planck}}{16\pi^2} + \frac{\Lambda^4_{QCD}}{16\pi^2} \,, \tag{52}$$

where

$$\left\langle \rho \right\rangle_{vac}^{ECS} = \int_0^{\Lambda_{Planck}} \frac{4\pi k_{\Delta_p}^2 dk}{(2\pi)^3} \frac{1}{2} \sqrt{k_{\Delta_p}^2 + m_{\Delta_p}^2} = \frac{1}{16\pi^2} \left[k_{\Delta_p}^4 \right]_0^{\Lambda_{Planck}}$$

$$= \frac{1}{16\pi^2} \left[\frac{(2)^4 k^4}{\left((C_{F+B} C_F C_B)^4 \right)} \right]_0^{\Lambda_{Planck}} = \frac{\Lambda_{Planck}^4}{\pi^2 \left(C_{F+B} C_F C_B \right)^4} \tag{53}$$

is the ECS-vacuum energy density,

$$\int_0^{\Lambda_{Planck}} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m_{SM}^2} = \int_0^{\Lambda_{Planck}} \frac{k_\gamma^3 dk}{(2\pi)^2} \simeq \frac{\Lambda_{Planck}^4}{16\pi^2} \simeq \mathrm{O}(10^{76} GeV^4)\,, \tag{54}$$

is the SM particle vacuum energy density, and

$$\int_0^{\Lambda_{QCD}} \frac{k_g^3 dk}{(2\pi)^2} \simeq \frac{\Lambda_{QCD}^4}{16\pi^2} \simeq O(10^{-6} GeV^4) \tag{55}$$

is the gluon vacuum energy density. Substituting Equations ((53)-(55)) to (49), we have:

$$\Lambda_{eff} = \Lambda_0 + \frac{8\pi G}{c^4} \left\langle \rho \right\rangle_{vac} = \Lambda_0 + \frac{8\pi G}{c^4} \left\langle \rho \right\rangle_{vac}^{ECS} + \frac{8\pi G}{c^4} \left\langle \rho \right\rangle_{vac}^{ordynary} = \Lambda_{eff}^{ECS} + \frac{8\pi G}{c^4} \left\langle \rho \right\rangle_{vac}^{ordinary}$$
$$\approx O(\Lambda_{Planck}^4) \,, \tag{56}$$

where

$$\Lambda_{eff}^{ECS} = \Lambda_0 + \frac{8\pi G}{c^4} \left\langle \rho \right\rangle_{vac}^{ECS} \,, \tag{57}$$

is the ECS ‘effective’ cosmological constant contributing in Einstein’s Equations (1). Λ_0, denotes Einstein’s ‘bare’ cosmological constant, and $< \rho >^{ECS}$ is the ECS vacuum energy density. For an equal number of ordinary and ECS fermions [1], and an equal number of ordinary and ECS bosons, the calculated one-loop radiative corrections to the ‘bare’ cosmological constant Λ_0 are given in Table 3.

Number of Ordinary Fermions ($F_{Ordinary}$)	*12*
Number of ECS Fermions (F_{ECS})	*12*
Total Number of Fermions (F) Eq. (36)	*24*
Fermionic Catalan Number (C_F) Eq (38)	*2.28×10^{12}*
Number of Ordinary Bosons ($B_{Ordinary}$)	*3*
Number of ECS Bosons (B_{ECS})	*3*
Total Number of Bosons (B)	*6*
Bosonic Catalan Number (C_B)	*1.32×10^{2}*
Combination of Fermions and Bosons	*30*
Combination Catalan Number (C_{F+B})	*2.621×10^{16}*
Multitriangulation of P-gon det(C_p) Eq.(41)	*4.12×10^{30}*
ECS vacuum energy density $< \rho >^{ECS}$ Eq.(53)	*$2.4\times10^{-47} GeV^4$*

Table 3. One-loop radiative corrections to the 'bare' cosmological constant Λ_0, as calculated from Equation (53), for the reciprocal 4th power of particle Catalan number $1/(\det C_p)^4 = O(10^{-123})$.

For 24 fermions and 6 bosons, we calculate a vacuum energy density of $10^{-47}GeV^4$ which is very close to the experimental value [16]. Our approach, therefore, may solve the fine-tuning problem, since it provides an all-orders solution which is suppressed by the particle Catalan number C_p.

5. Conclusion

Following a previous article [1], we investigate a version of the SM algebroid with its anchor map depending on the ECS angle, θs. We find that many SM algebras depend on the ECS angle, θs. We call these ECSM algebras. Furthermore, the SM algebroid is integrable to the SM groupoid; our results, therefore, potentially extend well beyond the investigated case. We then investigate how the breaking of the symmetry of the SM groupoid gives the massive ECS particle. We find that the ECS particle mass is related to the SM particle mass through the ECS angle θs. We investigate the finite subgroups of the ECS Möbius transformations. In this case, the ECS-angle θs could originate from the ECS dihedral group that refers to the symmetry of the particle polygon (P-gon). The ECS angle θs can then be determined through the multitriangulation of a convex particle P-gon. Finally, we find that, at loop-level, the ECS Physics is different from the SM physics, and the ECSM mass is suppressed by the particle Catalan numbers C_P. For 24 fermions [1], and 6-vector gauge bosons, the calculated one-loop radiative correction to the 'bare' cosmological constant Λ_0 is $10^{-47}\,GeV^4$—very close to the experimental value.

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