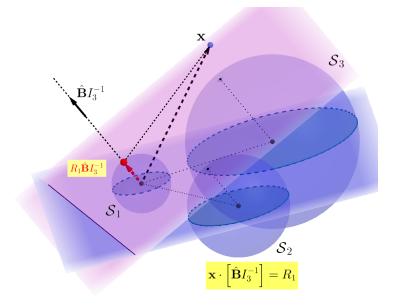# Use GA (Geometric Algebra) to Identify the Planes that are Tangent to Three Given Spheres

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James Smith

Abstract

To help fill the need for examples of introductory-level problems that have been solved via Geometric Algebra (GA), we derive the equation for a plane that is tangent to three given planes. The approach that we use determines the unit bivector of the tangent plane from the interior and exterior products of the vectors that connect the centers of the given spheres. A more-general version of this approach is presented in an appendix.

Contents

1 Introduction

To specify a plane, all we need to know is the unit bivector $\left(\hat{\mathbf{B}}\right)$ parallel to the plane, and one point ($\mathbf{p}$) within the plane. We can then write the equation of the tangent plane as

$$(\mathbf{x} - \mathbf{p}) \wedge \hat{\mathbf{B}} = 0. \tag{1.1}$$

Our challenge is to identify the required points and unit bivectors for the two planes that are tangent to a given set of three spheres.

2 Some of the Ideas that We Will Find Useful

1. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\|\mathbf{a} \wedge \mathbf{b}\|^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2$.
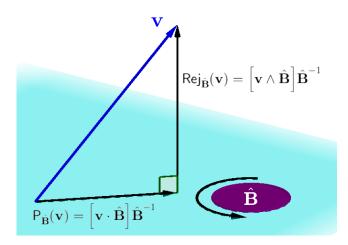
Figure 1: The relationship between the vector $\mathbf{v}$ and its projection $\left[\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{v}) \right]$ and "rejection" $\left[\mathsf{Rej}_{\hat{\mathbf{B}}}(\mathbf{v}) \right]$ with respect to the bivector $\hat{\mathbf{B}}$.

2. The relationship between a vector $\mathbf{v}$ and its projection and "rejection" with respect to a bivector $\hat{\mathbf{B}}$ is as shown in Fig. 1.

3. The unit vector from the center of a sphere to a point of tangency is the dual of the unit bivector of the tangent plane. Note that we need to choose the correct sense of rotation of the bivector, according to the right-hand rule, so that $\hat{\mathbf{B}}I_3^{-1}$ is directed away from the center (Fig. 2). See also [1] , pp. 105-108.

4. Let $\mathbf{v}$ be a vector, and α, β be scalars. Then, the multiplicative inverse of the multivector $\alpha I_3 + \beta \mathbf{v}$ is $\dfrac{-\alpha I_3 + \beta \mathbf{v}}{\alpha^2 + \beta^2} = \dfrac{\alpha I_3^{-1} + \beta \mathbf{v}}{\alpha^2 + \beta^2}$.

5. Whenever an expression contains the outer product of two perpendicular vectors (call them $\mathbf{a}$ and $\mathbf{b}$), we should consider substituting the geometric product $\mathbf{ab}$ for $\mathbf{a} \wedge \mathbf{b}$.

3 Formulation of the Problem in GA Terms

For any given set of three spheres, there are two planes that are tangent to all three of them (Fig. 3). We will formulate the problem for only one of the two tangent planes (Fig. 4), leaving the formulation for the other plane as an exercise to the reader. We call the smallest sphere $\mathcal{S}_1$, and its radius R_1. The radii of $\mathcal{S}_2$ and $\mathcal{S}_3$ are (respectively) R_2 and R_3.

We begin by reducing the problem from "three spheres" to "one point and two spheres", as in the solution of the classic Apollonius problem . The center

Reducing the problem to one of "two spheres and one point" is not necessary, but does help us to identify the essential elements of the problem.

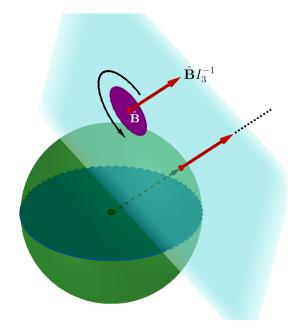Figure 2: The unit vector from the center of a sphere to a point of tangency is the dual of the unit bivector of the tangent plane. (Note that we need to choose the correct sense of rotation of the bivector, according to the right-hand rule, so that $\hat{\mathbf{B}}I_3^{-1}$ is directed away from the center.)
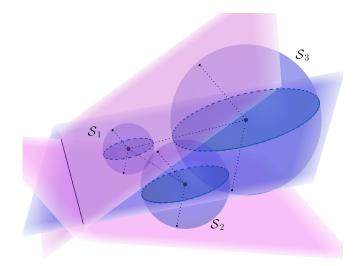

Figure 3: For any given set of three spheres, there are two planes that are tangent to three all of them.
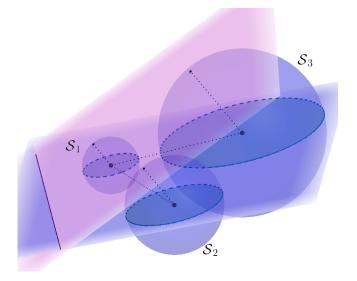
Figure 4: We will identify the tangent plane that is shown here. We denote the radii of the three spheres as (respectively) R_1, R_2, and R_3, with R_1 being the smallest.

of the "reduced" sphere $\mathcal{S}_2'$ is the same as for $\mathcal{S}_2$; $\mathcal{S}_2'$'s radius is $r_2 = R_2 - R_1$ (Fig. 5). Similarly, center of the "reduced" sphere $\mathcal{S}_3'$ is the same as for $\mathcal{S}_3$; and $\mathcal{S}_3'$'s radius is $r_3 = R_3 - R_1$.

To complete the formulation, we use the center of $\mathcal{S}_1$ as the origin, and define the vectors as in Fig. 6. According to this formulation,

$$\begin{aligned}
\mathbf{s}_2 &= \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) + \left(\mathbf{s_2}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1} \\
&= \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\hat{\mathbf{B}}I_3^{-1} \\
&= \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) + r_2\hat{\mathbf{B}}I_3 \ .
\end{aligned}$$
(3.1)

$$\begin{aligned}
\mathbf{s}_3 &= \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + \left(\mathbf{s_3}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1} \\
&= \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) - r_2\hat{\mathbf{B}}I_3^{-1} \\
&= \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + r_2\hat{\mathbf{B}}I_3 \ .
\end{aligned}$$
(3.2)

We will see later that we might have been better off to leave the results as $\mathbf{s}_2 = \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\hat{\mathbf{B}}I_3^{-1}$ and $\mathbf{s}_3 = \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) - r_3\hat{\mathbf{B}}I_3^{-1}$.

4 Our Strategy

We need to identify the unit bivector $\hat{\mathbf{B}}$, and one point within the tangent plane. We will identify $\hat{\mathbf{B}}$ via the outer product of the projections of $\mathbf{s}_2$ and $\mathbf{s}_3$. As the required point, we will use $R_1\hat{\mathbf{B}}I_3^{-1}$ (the point of tangency with $\mathcal{S}_1$). Then, we will write the equation of the plane as

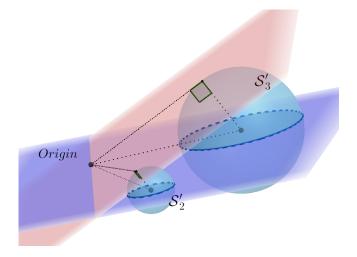$$\left(\mathbf{x} - R_1\hat{\mathbf{B}}I_3^{-1}\right) \wedge \hat{\mathbf{B}} = 0.$$
(4.1)

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Figure 5: Reduction of the problem to "two spheres and one point". The origin is at the center of $\mathcal{S}_1$. The centers and radii of radius center of spheres $\mathcal{S}_2'$ and $\mathcal{S}_3'$ are as follows:

The center of $\mathcal{S}_2'$ is the same as for $\mathcal{S}_2$; $\mathcal{S}_2'$'s radius is $r_2 = R_2 - R_1$.

The center of $\mathcal{S}_3'$ is the same as for $\mathcal{S}_3$; $\mathcal{S}_3'$'s radius is $r_3 = R_3 - R_1$.

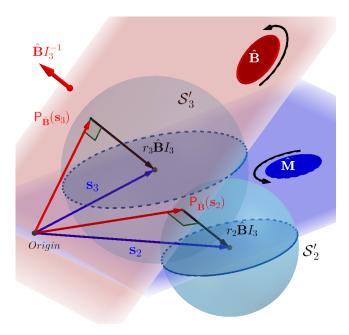Figure 6: Our formulation of the problem in GA terms. Note that the unit vector of the direction from the ends of the projections to the centers of the spheres is $-\hat{\mathbf{B}}I_3^{-1}$, $= \hat{\mathbf{B}}I_3$. $\hat{\mathbf{M}}$ is the unit bivector of the plane that contains the vectors $\mathbf{s}_2$ and $\mathbf{s}_3$. $\hat{\mathbf{B}}$ (the bivector that we wish to identify) contains their projections.

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5   Solution

5.1   Preliminary Work

Here, we will write a few results that we will need later. First, $\hat{\mathbf{B}}$ is the unit
bivector of the plane that contains (or to be more correct, is parallel to) the
bivector $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)$. Therefore,

$$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) = \|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|\hat{\mathbf{B}}. \tag{5.1}$$

From Item 1 in Section 2, we might infer that to find $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|$,
we will need to know the lengths of the projections of $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right)$ and $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)$.
Therefore, we will write expressions for those lengths now:

$$\begin{aligned}
\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right)\| &= \sqrt{s_2^2 - r_2^2}, \text{ and} \\
\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\| &= \sqrt{s_3^2 - r_3^2}.
\end{aligned} \tag{5.2}$$

5.2   Identify $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|$

We will deal with $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|$ first, so that when we have obtained an
expression for $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)$, we may replace that expression with $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge$
$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|\hat{\mathbf{B}}$.

Item 1 in Section 2 noted that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\|\mathbf{a} \wedge \mathbf{b}\|^2 = a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2$. Thus,

$$\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|^2 = \|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right)\|^2\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|^2 - \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right]^2.$$

We have already found expressions for $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right)\|$ and $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|$ (Eqs. (5.2)). To
find $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)$, we start from the relations (Eqs. (3.1) and (3.2)) between
each "$\mathbf{s}$" vector and its projection. Specifically,

$$\underbrace{\left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) + r_2\hat{\mathbf{B}}I_3\right]}_{= \, \mathbf{s}_2} \cdot \underbrace{\left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + r_3\hat{\mathbf{B}}I_3\right]}_{= \mathbf{s}_3} = \mathbf{s}_2 \cdot \mathbf{s}_3$$

$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right)$ and $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)$ are perpendicular to $\hat{\mathbf{B}}I_3$; therefore, the left-hand side
reduces to

$$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + r_2r_3\left(\hat{\mathbf{B}}I_3\right)^2 = \mathbf{s}_2 \cdot \mathbf{s}_3.$$

$\hat{\mathbf{B}}I_3$ is a unit vector, so $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + r_2r_3 = \mathbf{s}_2 \cdot \mathbf{s}_3$, from which

$$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) = \mathbf{s}_2 \cdot \mathbf{s}_3 - r_2r_3 \ .$$

Putting the previous ideas together,

$$\begin{aligned}
\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\| &= \sqrt{\left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right)\right]^2 \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right]^2 - \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right]^2} \\
&= \sqrt{\left(s_2^2 - r_2^2\right)\left(s_3^2 - r_3^2\right) - \left[\mathbf{s}_2 \cdot \mathbf{s}_3 - r_2r_3\right]^2}.
\end{aligned}$$

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Expanding the products and simplifying,

$$\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\| = \sqrt{s_2^2 s_3^2 - \left(\mathbf{s}_2 \cdot \mathbf{s}_3\right)^2 - \left(r_3^2 s_2^2 - 2r_3 r_2 \mathbf{s}_2 \cdot \mathbf{s}_3 + r_2^2 s_3^2\right)}$$

$$= \sqrt{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|^2 - \|r_3 \mathbf{s}_2 - r_2 \mathbf{s}_3\|^2}. \qquad (5.3)$$

5.3 Obtain an expression for $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)$

We begin by writing

$$\underbrace{\left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) + r_2\hat{\mathbf{B}}I_3\right]}_{=\mathbf{s}_2} \wedge \underbrace{\left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + r_3\hat{\mathbf{B}}I_3\right]}_{=\mathbf{s}_3} = \mathbf{s_2} \wedge \mathbf{s_3}.$$

Because $\left[\hat{\mathbf{B}}I_3\right] \wedge \left[\hat{\mathbf{B}}I_3\right] = 0$,

$$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right)\right] \wedge \left[\hat{\mathbf{B}}I_3^{-1}\right] + \left[r_2\hat{\mathbf{B}}I_3\right] \wedge \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right] = \mathbf{s_2} \wedge \mathbf{s_3} ,$$

and

$$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right) + \left[r_3\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right] \wedge \left[\hat{\mathbf{B}}I_3\right] = \mathbf{s_2} \wedge \mathbf{s_3}.$$

We wish to find $\hat{\mathbf{B}}$, so let's introduce it in the first term on the left-hand side, via the substitution indicated by Eq. (5.1). Our idea is that we may be able to transform the left-hand side into a product of $\hat{\mathbf{B}}$ and some multivector that has an inverse.

$$\underbrace{\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|\hat{\mathbf{B}}}_{=\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_2) \wedge \mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_3)} + \left[r_3\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right] \wedge \left[\hat{\mathbf{B}}I_3\right] = \mathbf{s_2} \wedge \mathbf{s_3}.$$

With the same idea in mind, we note that we can write

$$\left[r_3\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right] \wedge \left[\hat{\mathbf{B}}I_3\right]$$

as

$$\left[r_3\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right] \left[\hat{\mathbf{B}}I_3\right].$$

Thus,

$$\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|\hat{\mathbf{B}} + \left[r_3\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right] \left[\hat{\mathbf{B}}I_3\right] = \mathbf{s_2} \wedge \mathbf{s_3}. \qquad (5.4)$$

Although that result could be transformed readily into the form of a product of $\hat{\mathbf{B}}$ and some multivector, we would still not be able to solve for $\hat{\mathbf{B}}$, because we cannot as yet identify the vector $\left[r_3\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right]$. To get an idea, let's "see" Eq. (5.4) as an addition of bivectors, examine a geometric representation of the addition of bivectors in general (Fig. 7). The key element in that representation is the "hinge" vector that is parallel to both of the addends. Could $\left[r_3\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) - r_2\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\right]$ be that "hinge" in our case? More specifically,
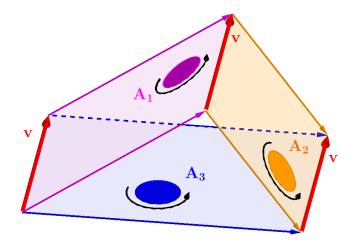
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Figure 7: Geometric representation of the sum of bivectors, $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{A}_3$. The "hinge" vector $\mathbf{v}$ is a factor of both addends, and therefore of the result ($\mathbf{A}_3$) as well.

we showed (above) that that vector is parallel to $\hat{\mathbf{B}}$. Is that vector also parallel to $\mathbf{s}_2 \wedge \mathbf{s}_3$? Yes, it is. We can show this by writing $\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_2)$ as $\mathbf{s}_2 - r_2\hat{\mathbf{B}}I_3$, and $\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_3)$ as $\mathbf{s}_3 - r_3\hat{\mathbf{B}}I_3$:

$$r_3 \underbrace{\left[\mathbf{s}_2 - r_2\hat{\mathbf{B}}I_3\right]}_{=\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_2)} - r_2 \underbrace{\left[\mathbf{s}_3 - r_3\hat{\mathbf{B}}I_3\right]}_{=\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_3)} = r_3\mathbf{s}_2 - r_2\mathbf{s}_3.$$

Therefore, Eq. (5.4) becomes

$$\|\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_2) \wedge \mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_3)\|\hat{\mathbf{B}} + [r_3\mathbf{s}_2 - r_2\mathbf{s}_3]\left[\hat{\mathbf{B}}I_3\right] = \mathbf{s_2}\wedge\mathbf{s_3}. \tag{5.5}$$

Two comments are in order. First, a study of Eq. (5.5) shows us why Eq. (5.3) makes sense: the bivector $\mathbf{s}_2 \wedge \mathbf{s}_3$ is the sum of the bivectors $\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_2) \wedge \mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{s}_3)$ and $[r_3\mathbf{s}_2 - r_2\mathbf{s}_3]\left[\hat{\mathbf{B}}I_3\right]$, which are perpendicular to each other. Therefore, the relation between the norms of the three vectors has the form of the Pythagorean Theorem.

The second comment is that although we could now proceed with our idea of identifying $\hat{\mathbf{B}}$, we can also change our goal, with benefit.

5.4   Changing Our Goal to Take Advantage of an Unforeseen Opportunity

Our goal has been to identify $\hat{\mathbf{B}}$, after which we would write the equation of the tangent plane as $\left(\mathbf{x} - R_1\hat{\mathbf{B}}I_3^{-1}\right) \wedge \hat{\mathbf{B}} = 0$. (See Eq. (4.1).) However, an examination of Eq. (5.5) suggests that we could instead find the vector $\hat{\mathbf{B}}I_3^{-1}$ directly. One advantage of this change of plans is that we can then write the

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equation of the plane as the more user-friendly $\left(\mathbf{x} - R_1\hat{\mathbf{B}}I_3^{-1}\right) \cdot \left[\hat{\mathbf{B}}I_3^{-1}\right] = 0$, and thus as

$$\mathbf{x} \cdot \left[\hat{\mathbf{B}}I_3^{-1}\right] = R_1. \tag{5.6}$$

Another advantage (as we shall see) is that the expression for $\hat{\mathbf{B}}I_3^{-1}$ will be both readily interpretable, and easy to check via programs such as GeoGebra.

5.5   Continuing in Our New Direction: Find $\hat{\mathbf{B}}I_3^{-1}$ Rather than $\hat{\mathbf{B}}$

To make the necessary transformation of Eq. (5.5), we will use the identities $I_3I_3^{-1} = 1$ and $I_3^{-1} = {}^-I_3$. We begin by writing Eq. (5.5) as

$$\left[\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|\right]\left[I_3I_3^{-1}\hat{\mathbf{B}}\right] - [r_3\mathbf{s}_2 - r_2\mathbf{s}_3]\left[\hat{\mathbf{B}}I_3^{-1}\right] = \mathbf{s_2}\wedge\mathbf{s_3}$$

Now, we use $I_3^{-1}\hat{\mathbf{B}} = \hat{\mathbf{B}}I_3^{-1}$, and the fact that GA products possess the associative property, to write

$$\left[\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3\right]\left[\hat{\mathbf{B}}I_3^{-1}\right] - [r_3\mathbf{s}_2 - r_2\mathbf{s}_3]\left[\hat{\mathbf{B}}I_3^{-1}\right] = \mathbf{s_2}\wedge\mathbf{s_3}$$

Factoring, we obtain

$$\left\{\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3 - (r_3\mathbf{s}_2 - r_2\mathbf{s}_3)\right\}\left[\hat{\mathbf{B}}I_3^{-1}\right] = \mathbf{s_2}\wedge\mathbf{s_3}$$

Left-multiplying both sides by the inverse of the factor in "{}s",

$$\hat{\mathbf{B}}I_3^{-1} = \left[\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3 - (r_3\mathbf{s}_2 - r_2\mathbf{s}_3)\right]^{-1}[\mathbf{s}_2 \wedge \mathbf{s}_3].$$

To find that inverse, we use Item 4 from Section 2

$$\left[\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3 - (r_3\mathbf{s}_2 - r_2\mathbf{s}_3)\right]^{-1}$$

$$= \frac{-\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3 - (r_3\mathbf{s}_2 - r_2\mathbf{s}_3)}{\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|^2 + \|r_3\mathbf{s}_2 - r_2\mathbf{s}_3\|^2}. \tag{5.7}$$

From Eq. (5.3), $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|^2 = \|\mathbf{s}_2 \wedge \mathbf{s}_3\|^2 - \|r_3\mathbf{s}_2 - r_2\mathbf{s}_3\|^2$. Making this substitution in the denominator of Eq. (5.7), then simplifying (and recalling that $I_3^{-1} = {}^-I_3$), we obtain

$$\left[\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3 - (r_3\mathbf{s}_2 - r_2\mathbf{s}_3)\right]^{-1}$$

$$= \frac{-\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3 - (r_3\mathbf{s}_2 - r_2\mathbf{s}_3)}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|^2}$$

$$= \frac{\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|I_3^{-1} + (r_2\mathbf{s}_3 - r_3\mathbf{s}_2)}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|^2}. \tag{5.8}$$

Therefore,

$$\hat{\mathbf{B}}I_3^{-1} = \left[\frac{\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\| I_3^{-1} + \left(r_2\mathbf{s}_3 - r_3\mathbf{s}_2\right)}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|^2} \right] \left[\mathbf{s}_2 \wedge \mathbf{s}_3\right]$$

$$= \left[\frac{\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\| I_3^{-1} + \left(r_2\mathbf{s}_3 - r_3\mathbf{s}_2\right)}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|} \right] \hat{\mathbf{M}} , \qquad (5.9)$$

where $\hat{\mathbf{M}} = \left[\mathbf{s_2} \wedge \mathbf{s}_3\right] / \|\mathbf{s_2} \wedge \mathbf{s}_3\|$.

We can put that result in a more readily-interpreted form as follows:

$$\hat{\mathbf{B}}I_3^{-1} = \left[\frac{\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_2\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{s}_3\right)\|}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|} \right] \left[I_3^{-1}\hat{\mathbf{M}}\right] + \left[\frac{1}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|} \right] \left[\left(r_2\mathbf{s}_3 - r_3\mathbf{s}_2\right)\hat{\mathbf{M}}\right]$$

$$= \left[\frac{\sqrt{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|^2 - \|r_3\mathbf{s}_2 - r_2\mathbf{s}_3\|^2}}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|} \right] \left[\hat{\mathbf{M}}I_3^{-1}\right]$$
$$+ \left[\frac{1}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|} \right] \left[\left(r_2\mathbf{s}_3 - r_3\mathbf{s}_2\right)\hat{\mathbf{M}}\right]$$

$$= \left\{ \sqrt{1 - \left[\frac{\|r_3\mathbf{s}_2 - r_2\mathbf{s}_3\|}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|} \right]^2} \right\} \left[\hat{\mathbf{M}}I_3^{-1}\right]$$
$$+ \left[\frac{1}{\|\mathbf{s}_2 \wedge \mathbf{s}_3\|} \right] \left[\left(r_2\mathbf{s}_3 - r_3\mathbf{s}_2\right)\hat{\mathbf{M}}\right] . \qquad (5.10)$$

The first term on the right-hand side is a scalar multiple of the unit normal vector (in the right-hand sense) of the bivector $\mathbf{s}_2 \wedge \mathbf{s}_3$ —and thus of the plane that contains the centers of the three given spheres. The second term is a scalar multiple of the 90 deg rotation (in the right-hand sense) of the vector $\left(r_2\mathbf{s}_3 - r_3\mathbf{s}_2\right)$ in the plane of $\mathbf{s}_2 \wedge \mathbf{s}_3$. The result is shown in Fig. 8.

6 For the Reader

In what way does this procedure need to be changed in order to identify the unit bivector of the second tangent plane?

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).
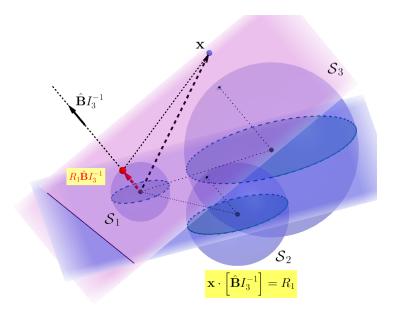
Figure 8: The result: All vectors $\mathbf{x}$ whose endpoints lie in the tangent plane fulfill the condition $\mathbf{x} \cdot \left[\hat{\mathbf{B}} I_3^{-1} \right] = R_1$.

A General Case

Here, we will develop an equation analogous to Eq. (5.10), but for the general case in which

$$\mathbf{v}_1 = \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) + \lambda_1 \hat{\mathbf{B}} I_3^{-1} \tag{A.1}$$

and

$$\mathbf{v}_2 = \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right) + \lambda_2 \hat{\mathbf{B}} I_3^{-1} \;, \tag{A.2}$$

where either both of the λ's are positive, or both are negative. As in the main text, we find $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)\|$ by first finding $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right)\|$, $\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)\|$, and $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)$, so that we may use the relation

$$\|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) \wedge \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)\|^2 = \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right)\right]^2 \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)\right]^2 - \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)\right]^2 . \tag{A.3}$$

From Eqs. (A.1) and (A.2),

$$\begin{aligned} \|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{x}_1\right)\|^2 &= v_1^2 - \lambda_1^2, \;\; \text{and} \\ \|\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)\|^2 &= v_2^2 - \lambda_2^2. \end{aligned} \tag{A.4}$$

To obtain an expression for $\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right)$, we write

$$\left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) + \lambda_1 \hat{\mathbf{B}} I_3^{-1}\right] \cdot \left[\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right) + \lambda_2 \hat{\mathbf{B}} I_3^{-1}\right] = \mathbf{v}_1 \cdot \mathbf{v}_2,$$

from which

$$\mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_1\right) \cdot \mathsf{P}_{\hat{\mathbf{B}}}\left(\mathbf{v}_2\right) = \mathbf{v}_1 \cdot \mathbf{v}_2 - \lambda_1 \lambda_2. \tag{A.5}$$

Therefore, Eq. (A.3) becomes

$$\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|^2 = \left[v_1^2 - \lambda_1^2\right]\left[v_2^2 - \lambda_2^2\right] - \left[\mathbf{v}_1 \cdot \mathbf{v}_2 - \lambda_1\lambda_2\right]^2$$

$$= v_1^2 v_2^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \left[(\lambda_2\mathbf{v}_1)^2 - 2\lambda_1\lambda_2\mathbf{v}_1 \cdot \mathbf{v}_2 + (\lambda_1\mathbf{v}_2)^2\right]$$

$$= \|\mathbf{v}_1 \wedge \mathbf{v}_2\|^2 - \|\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2\|^2. \tag{A.6}$$

Next, we form the outer product from Eq. (A.1) and (A.2):

$$\left[P_{\hat{\mathbf{B}}}(\mathbf{v}_1) + \lambda_1\hat{\mathbf{B}}I_3^{-1}\right] \wedge \left[P_{\hat{\mathbf{B}}}(\mathbf{v}_2) + \lambda_2\hat{\mathbf{B}}I_3^{-1}\right] = \mathbf{v}_1 \wedge \mathbf{v}_2,$$

from which

$$P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2) + \left[\lambda_2 P_{\hat{\mathbf{B}}}(\mathbf{v}_1) - \lambda_1 P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\right] \wedge \left[\hat{\mathbf{B}}I_3^{-1}\right] = \mathbf{v}_1 \wedge \mathbf{v}_2,$$

$$\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|\hat{\mathbf{B}} + \left[\lambda_2 P_{\hat{\mathbf{B}}}(\mathbf{v}_1) - \lambda_1 P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\right] \wedge \left[\hat{\mathbf{B}}I_3^{-1}\right] = \mathbf{v}_1 \wedge \mathbf{v}_2,$$

and (because $\hat{\mathbf{B}}I_3^{-1}$ is perpendicular to the vector $\left[\lambda_2 P_{\hat{\mathbf{B}}}(\mathbf{v}_1) - \lambda_1 P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\right]$)

$$\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|\hat{\mathbf{B}} + \left[\lambda_2 P_{\hat{\mathbf{B}}}(\mathbf{v}_1) - \lambda_1 P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\right]\left[\hat{\mathbf{B}}I_3^{-1}\right] = \|\mathbf{v}_1 \wedge \mathbf{v}_2\|\hat{\mathbf{M}},$$

where $\hat{\mathbf{M}} = [\mathbf{v}_1 \wedge \mathbf{v}_2]/\|\mathbf{v}_1 \wedge \mathbf{v}_2\|$. Writing $P_{\hat{\mathbf{B}}}(\mathbf{v}_1)$ as $\mathbf{v}_1 - \lambda_1\hat{\mathbf{B}}I_3^{-1}$ and $P_{\hat{\mathbf{B}}}(\mathbf{v}_2)$ as $\mathbf{v}_2 - \lambda_2\hat{\mathbf{B}}I_3^{-1}$, then simplifying, the factor $\left[\lambda_2 P_{\hat{\mathbf{B}}}(\mathbf{v}_1) - \lambda_1 P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\right]$ becomes $\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2$. Thus,

$$\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|\hat{\mathbf{B}} + (\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)\left[\hat{\mathbf{B}}I_3^{-1}\right] = \|\mathbf{v}_1 \wedge \mathbf{v}_2\|\hat{\mathbf{M}}.$$

Continuing as in the main text, we use $I_3^{-1}\hat{\mathbf{B}} = \hat{\mathbf{B}}I_3^{-1}$, and the fact that GA products possess the associative property, to obtain

$$\left[\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|I_3\right]\left[\hat{\mathbf{B}}I_3^{-1}\right] + (\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)\left[\hat{\mathbf{B}}I_3^{-1}\right] = \|\mathbf{v}_1 \wedge \mathbf{v}_2\|\hat{\mathbf{M}},$$

$$\left[\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|I_3 + (\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)\right]\left[\hat{\mathbf{B}}I_3^{-1}\right] = \|\mathbf{v}_1 \wedge \mathbf{v}_2\|\hat{\mathbf{M}},$$

and

$$\hat{\mathbf{B}}I_3^{-1} = \left\{\left[\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|I_3 + (\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)\right]\right\}^{-1}\|\mathbf{v}_1 \wedge \mathbf{v}_1\|\hat{\mathbf{M}}. \tag{A.7}$$

The inverse on the right-hand side is

$$\frac{\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|I_3^{-1} + (\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)}{\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|^2 + (\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)^2}.$$

Recalling that $\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|^2 = \|\mathbf{v}_1 \wedge \mathbf{v}_2\|^2 - \|\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2\|^2$, the denominator reduces to $\|\mathbf{v}_1 \wedge \mathbf{v}_2\|^2$, and Eq. (A.7) becomes

$$\hat{\mathbf{B}}I_3^{-1} = \left\{\frac{\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|I_3^{-1} + (\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)}{\|\mathbf{v}_1 \wedge \mathbf{v}_2\|}\right\}\hat{\mathbf{M}}$$

$$= \left[\frac{\|P_{\hat{\mathbf{B}}}(\mathbf{v}_1) \wedge P_{\hat{\mathbf{B}}}(\mathbf{v}_2)\|}{\|\mathbf{v}_1 \wedge \mathbf{v}_2\|}\right]\left[I_3^{-1}\hat{\mathbf{M}}\right] + \frac{1}{\|\mathbf{v}_1 \wedge \mathbf{v}_2\|}\left[(\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)\hat{\mathbf{M}}\right].$$

Now, we use $\|\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{v}_1)\wedge\mathsf{P}_{\hat{\mathbf{B}}}(\mathbf{v}_2)\| = \sqrt{\|\mathbf{v}_1\wedge\mathbf{v}_2\|^2 - \|\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2\|^2}$ and $I_3^{-1}\hat{\mathbf{M}} = \hat{\mathbf{M}}I_3^{-1}$:

$$\hat{\mathbf{B}}I_3^{-1} = \left[\frac{\sqrt{\|\mathbf{v}_1\wedge\mathbf{v}_2\|^2 - \|\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2\|^2}}{\|\mathbf{v}_1\wedge\mathbf{v}_2\|}\right]\left[\hat{\mathbf{M}}I_3^{-1}\right]$$
$$+ \frac{1}{\|\mathbf{v}_1\wedge\mathbf{v}_2\|}\left[(\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)\hat{\mathbf{M}}\right].$$

Simplifying,

$$\hat{\mathbf{B}}I_3^{-1} = \left\{\sqrt{1 - \left[\frac{\|\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2\|}{\|\mathbf{v}_1\wedge\mathbf{v}_2\|}\right]^2}\right\}\left[\hat{\mathbf{M}}I_3^{-1}\right]$$
$$+ \frac{1}{\|\mathbf{v}_1\wedge\mathbf{v}_2\|}\left[(\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)\hat{\mathbf{M}}\right]. \tag{A.8}$$

The first term on the right-hand side is a scalar multiple of the unit normal vector (in the right-hand sense) of the bivector $\mathbf{v}_1\wedge\mathbf{v}_2$. The second term is a scalar multiple of the 90 deg rotation (in the right-hand sense) of the vector $(\lambda_2\mathbf{v}_1 - \lambda_1\mathbf{v}_2)$ in the plane of $\mathbf{v}_1\wedge\mathbf{v}_2$.