Lagrange multipliers and adiabatic limits II

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In this part II to [FW] we finish, assuming local properness only, the proof of the one-to-one correspondence of gradient flow lines of index difference one between the restricted functional and the Lagrange multiplier functional for deformation parameters of the metric close to the singular one. In particular, we prove that, although the metric becomes singular, we have uniform bounds for the Lagrange multiplier of finite energy solutions and all its derivatives. The uniform bound is the crucial ingredient for a compactness theorem for gradient flow lines of arbitrary deformation parameter.

If the functionals are Morse we further prove uniform exponential decay. We finally show combined with the linear theory in part I that if the metric is Morse-Smale the adiabatic limit map is bijective.

1. Introduction

We build on part I [FW] and use the notation conventions introduced there. So M is a finite dimensional smooth manifold and $F, H\colon M \to \mathbb{R}$ are two smooth functions. We assume that zero is a regular value of H and H is locally proper around zero, so that, in particular, the level set $\Sigma := H^{-1}(0) \stackrel{\iota}{\hookrightarrow} M$, called the **base** or the **constraint**, is a closed codimension one submanifold of M.

The **Lagrange multiplier functional** is defined by

$$F_H\colon M \times \mathbb{R} \to \mathbb{R}, \quad (x, \tau) \mapsto F(x) + \tau H(x).$$

Since zero is a regular value of H, it holds that (x, τ) is a critical point of the Lagrange multiplier functional F_H if and only if x is a critical point of the restriction $f := \iota^* F = F|_\Sigma$ and $\tau = \chi \circ x$ for the smooth function χ given by

$$\chi := -\frac{\langle \overline{\nabla} H, \overline{\nabla} F \rangle}{|\overline{\nabla} H|^2}, \quad \text{along } M_{\mathrm{reg}} := \{p \in M \mid dH(p) \neq 0\} \supset \Sigma. \tag{1.1}$$

Here $\overline{\nabla}$ is the gradient with respect to a Riemannian metric G on M. The graph map of χ is called the **canonical embedding** and denoted by

$$i\colon M_{\mathrm{reg}} \to M_{\mathrm{reg}} \times \mathbb{R}, \quad p \mapsto (p, \chi(p))\,. \tag{1.2}$$

The graph map extends to path spaces with values in M_{reg}, same notation i.

For $\varepsilon > 0$ define a family of Riemannian metrics $h^\varepsilon := G \oplus \varepsilon^2$ on $M \times \mathbb{R}$. Gradient flow trajectories of F_H with respect to the metric h^ε, called **$\boldsymbol{\varepsilon}$-trajectories**, are smooth solutions $z = (u, \tau)\colon \mathbb{R} \to M \times \mathbb{R}$ of the ODE

$$\begin{aligned} \partial_s u &= -\overline{\nabla} F(u) - \tau \overline{\nabla} H(u), \\ \tau' &= -\varepsilon^{-2} H(u). \end{aligned} \tag{1.3}$$

If one lets ε formally go to zero in this equation, one obtains $H(u) = 0$ and $\tau = \chi(u)$. So u takes values in Σ and is a gradient flow line of $f = F|_\Sigma$ with respect to the restricted metric $g := \iota^* G = G|_\Sigma$, called a **base trajectory** or a **0-solution**, namely a smooth solution $q\colon \mathbb{R} \to \Sigma \subset M$ to the ODE

$$\partial_s q = -\nabla f(q), \qquad \nabla f(q) = \overline{\nabla} F(q) + \chi(q) \overline{\nabla} H(q), \tag{1.4}$$

where ∇ is the gradient with respect to g. The energies of base trajectories q, respectively ε-trajectories (u, τ), are given by the L^2 norms

$$E^0(q) = \|\partial_s q\|^2, \qquad E^\varepsilon(u,\tau) = \|\partial_s u\|^2 + \varepsilon^2 \|\tau'\|^2.$$

While the energy of every base trajectory is finite, in fact even uniformly finite $E^0(q) \le \max f - \min f$, for ε-trajectories finite energy

$$E^\varepsilon(u,\tau) < \infty, \qquad \Rightarrow \qquad E^\varepsilon(u,\tau) \le \max f - \min f =: \operatorname{osc} f < \infty \tag{1.5}$$

not only implies uniformly finite in $\varepsilon > 0$, see [FW], but also boundedness of (u, τ) and derivatives, uniformly in $\varepsilon \in (0, 1]$, see Theorem 3.1.

Theorem A (Compactness). *Let $\varepsilon_\nu > 0$ be a sequence converging to zero and (u_ν, τ_ν) a sequence of finite energy ε_ν-trajectories of (1.3). Then (u_ν, τ_ν) has a converging subsequence, on compact sets and with all derivatives, to a downward gradient trajectory of $f\colon \Sigma \to \mathbb{R}$ with respect to the metric g.*

Theorem B (Exponential decay). *Suppose that F_H, equivalently f, are Morse. Then finite energy gradient flow lines of (1.3) for $\varepsilon \in (0, 1]$ decay exponentially with all their derivatives, uniformly for all ε.*

Two critical points $x^\mp$ of $f\colon \Sigma \to \mathbb{R}$ are called **asymptotic boundary conditions** of a smooth map $q\colon \mathbb{R} \to \Sigma$ if $\lim_{s\to\mp\infty} q(s) = x^\mp$ and of a pair of smooth maps $(u, \tau)\colon \mathbb{R} \to M \times \mathbb{R}$ if

$$\lim_{s\to\mp\infty} (u(s), \tau(s)) = \big(x^\mp, \chi(x^\mp)\big) = i(x^\mp), \tag{1.6}$$

we also say that the map q, respectively (u, τ), **connects** x^- and x^+. We write $\mathcal{M}^0_{x^-,x^+}$ and $\mathcal{M}^\varepsilon_{x^-,x^+}$ for the sets of connecting base, respectively ε-, trajectories. With asymptotic boundary conditions in place there are the **energy identities** $E^0(q) = f(x^-) - f(x^+)$ whenever $q \in \mathcal{M}^0_{x^-,x^+}$ and

$$E^\varepsilon(u,\tau) = \|\partial_s u\|^2 + \varepsilon^2\|\tau'\|^2 = f(x^-) - f(x^+) =: c^* \tag{1.7}$$

whenever $(u, \tau) \in \mathcal{M}^\varepsilon_{x^-,x^+}$. Here and throughout $\|\cdot\|$ denotes L^2 norms.

Theorem C (Surjectivity). *Let (f, g) be Morse-Smale. Let $x^\mp \in \operatorname{Crit} f$ be critical points of Morse index difference one. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the injective maps $\mathcal{T}^\varepsilon\colon \mathcal{M}^0_{x^-,x^+} \to \mathcal{M}^\varepsilon_{x^-,x^+}$ between moduli spaces of connecting gradient flow trajectories, see [FW], are surjective.*

Idea of proof: Suppose the theorem is not true. Then there are sequences $\varepsilon_i \to 0$ and $(u_i, \tau_i) \in \mathcal{M}^{\varepsilon_i}_{x^-,x^+}$ not in the image of $\mathcal{T}^{\varepsilon_i}$. By transversality $\overline{\nabla} H \pitchfork \Sigma$ we project the values of $u_i \colon \mathbb{R} \to M$ to Σ and get maps $\mathfrak{q}_i \colon \mathbb{R} \to \Sigma$ almost solving the base equation (1.4). The base implicit function Theorem A (IFT II) then provides true solutions $q_i \in \mathcal{M}^0_{x^-,x^+}$ nearby. Now the ambient implicit function Theorem (IFT I), the main result of part I [FW], shows that $(u_i, \tau_i) = \mathcal{T}^{\varepsilon_i}(q_i(\cdot + \sigma))$ for a suitable time shift σ. Contradiction.

Note that Theorem C combined with injectivity established in part I proves bijectivity, i.e. the Main Theorem as stated in part I.

Notation. a) Conventions in [FW, Sec. 1]. b) Injectivity radius $\iota_\Sigma = \iota(\Sigma, g) > 0$ and $\iota(T_\Sigma M, G) > 0$ is the one of (M, G) along $T_\Sigma M \to \Sigma$.

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2. Outline

In this section we give a detailed overview of this article. We also provide formulas and enlist techniques that are used later on. Certain properties of the ε-equations (1.3) hold true along the unit parameter interval $\varepsilon \in (0, 1]$, while others only work along a smaller interval $(0, \varepsilon_\kappa]$.

Hypothesis 2.1.

(i) Zero is a regular value of H.
(ii) **Local properness.** $\exists \kappa > 0$ such that $H^{-1}[-\kappa, \kappa] \subset M$ is compact.
(iii) The metric G on M is geodesically complete.

Remark 2.2. (i) and (ii) assert that $\Sigma := H^{-1}(0)$ is a smooth compact hypersurface in M. Part (iii) guarantees that closed and bounded is equivalent to compact (Theorem of Hopf-Rinow; see e.g. [O'N83, Ch. 5 Thm. 21]).

The equations

The ε-equation for smooth maps $(u,\tau)\colon \mathbb{R}\to M\times\mathbb{R}$ and the first three derivatives, for latter reference named $u1, u2, \dots, \tau 1, \tau 2, \dots$, are given by

(2.8)

$$\begin{aligned}
\partial_s u &\stackrel{u1}{=} -\overline{\nabla} F(u) - \tau\overline{\nabla} H(u)\\
\overline{\nabla}_s\partial_s u &\stackrel{u2}{=} -\overline{\nabla}_s\overline{\nabla} F(u) - \tau\overline{\nabla}_s\overline{\nabla} H(u) - \tau'\overline{\nabla} H(u)\\
\overline{\nabla}_s\overline{\nabla}_s\partial_s u &\stackrel{u3}{=} -\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} F(u) - \tau\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} H(u) - 2\tau'\overline{\nabla}_s\overline{\nabla} H(u) - \tau''\overline{\nabla} H(u)\\
\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\partial_s u &\stackrel{u4}{=} -\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} F(u) - \tau\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} H(u) - 3\tau'\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} H(u)\\
&\quad - 3\tau''\overline{\nabla}_s\overline{\nabla} H(u) - \tau'''\overline{\nabla} H(u)
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2\tau' &\stackrel{\tau 1}{=} -H(u)\\
\varepsilon^2\tau'' &\stackrel{\hat\tau 2}{=} -\left\langle\overline{\nabla} H(u),\partial_s u\right\rangle\\
&\stackrel{\tau 2}{=} \left\langle\overline{\nabla} H(u),\overline{\nabla} F(u)\right\rangle + \tau\left|\overline{\nabla} H(u)\right|^2\\
\varepsilon^2(\tau')'' &\stackrel{\hat\tau 3}{=} -\left\langle\overline{\nabla}_s\overline{\nabla} H(u),\partial_s u\right\rangle - \left\langle\overline{\nabla} H(u),\overline{\nabla}_s\partial_s u\right\rangle\\
&\stackrel{\tau 3}{=} \left\langle\overline{\nabla}_s\overline{\nabla} H(u),\overline{\nabla} F(u)+\tau\overline{\nabla} H(u)\right\rangle\\
&\quad + \left\langle\overline{\nabla} H(u),\overline{\nabla}_s\overline{\nabla} F(u)+\tau\overline{\nabla}_s\overline{\nabla} H(u)\right\rangle + \tau'\left|\overline{\nabla} H(u)\right|^2\\
\varepsilon^2(\tau'')'' &= -\left\langle\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} H(u),\partial_s u\right\rangle - 2\left\langle\overline{\nabla}_s\overline{\nabla} H(u),\overline{\nabla}_s\partial_s u\right\rangle\\
&\quad - \left\langle\overline{\nabla} H(u),\overline{\nabla}_s\overline{\nabla}_s\partial_s u\right\rangle\\
&\stackrel{\tau 4}{=} -\left\langle\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} H(u),\partial_s u\right\rangle - 2\left\langle\overline{\nabla}_s\overline{\nabla} H(u),\overline{\nabla}_s\partial_s u\right\rangle\\
&\quad + \left\langle\overline{\nabla} H(u),\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} F(u)+\tau\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla} H(u)+2\tau'\overline{\nabla}_s\overline{\nabla} H(u)\right\rangle\\
&\quad + \tau''\left|\overline{\nabla} H(u)\right|^2
\end{aligned}$$

pointwise at $s\in\mathbb{R}$ where $\langle\cdot,\cdot\rangle := G(\cdot,\cdot)$ with induced norm $|\cdot|$. The relation between the Levi-Civita connection $\overline{\nabla}$ of (M,G) and the Levi-Civita connection ∇ of $(\Sigma,g) := (H^{-1}(0),\iota^* G)$ is explained in part I [FW].

The first set of equations, those for the derivatives of u, translate the higher, say k-fold, derivatives of u into products of $\partial_s u$ and individual τ derivatives of order up to $k-1$. The second set of equations, those for the derivatives of τ, say the k-fold derivative, also produce products of $\partial_s u$ with τ derivatives up to order $k-2$.

Remark 2.3 (Small parameters). The presence of the factor ε^2 makes the second set of equations not very useful to obtain estimates for derivatives of τ. It is crucial to observe that τ also appears in the first ε-equation, even without any ε factor, but unfortunately accompanied by a factor $\overline{\nabla} H(u)$. However, since zero is a regular value of H, the factor $\overline{\nabla} H(u)$ is nonzero when u takes values near Σ – which is true whenever $\varepsilon > 0$ is small for the following reason. We shall see that τ' is L^∞ bounded by a constant C, uniformly in $\varepsilon \in (0,1]$, thus equation $\tau 1$ provides the estimate

$$|H(u)| \leq \varepsilon^2 C.$$

Thus the images $u(\mathbb{R})$ of ε-solutions (u, τ) shrink to $\Sigma = H^{-1}(0)$, as $\varepsilon \to 0$. So, whenever $\varepsilon > 0$ is small, one can resolve $u1$ for τ, see (2.10) below.

On the other hand, even along the whole unit parameter interval $(0,1] \ni \varepsilon$ the ε-equations (1.3) exhibit a number of important properties such as uniform a priori bounds for u and τ and their derivatives, uniform gradient bounds for the u component in terms of L^2 intervals, which then leads to uniform exponential decay of the derivatives of the u component.
There are corresponding results (gradient bounds and exponential decay) for the τ component, but one has to consider separately the cases of small $(0, \varepsilon_\kappa)$ and large $[\varepsilon_\kappa, 1]$ parameters. Moreover, these τ component results are not relevant for the main goal of this article, to prove surjectivity of $\mathcal{T}^\varepsilon$ in Theorem C. Hence in the following outline we focus on the u component and the unit interval $(0,1]$. Only in the end we briefly mention the τ component.

Unit parameter interval $(0, 1]$

Section 3 "Uniform a priori bounds". By Hypothesis 2.1 fix a constant $\kappa > 0$ such that the pre-image $H^{-1}[-\kappa, \kappa]$ is compact. In Theorem 3.1, part (i), we show that any finite energy solution (u, τ) of the ε-equations (1.3) has the property that the component $u \colon \mathbb{R} \to M$ takes values in the compact

$$K = K_\kappa := \{p \in M \mid \mathrm{dist}_G(p, H^{-1}[-\kappa, \kappa]) \leq \tfrac{\mathrm{osc} f}{\kappa}\}$$

and τ and all derivatives of u and τ are L^∞ bounded, uniformly in $\varepsilon \in (0, 1]$.

The uniform C^2 bounds already imply local convergence and broken trajectory compactness of fixed asymptotics solution sequences; see Section 3.2.

Section 4 "Gradient bounds by energy intervals". We derive an L^∞ bound C for the derivative of u, uniformly in $\varepsilon \in (0, 1]$, in terms of L^2

norm intervals about any given point $s \in \mathbb{R}$, in symbols

$$|\partial_s u(s)| \leq C\|\partial_s u\|_{L^2[s-1,s+1]} \leq C E^\varepsilon_{[s-1,s+1]}(u,\tau).$$

By $E^\varepsilon_I(u,\tau)$ we denote the energy of (u,τ) along an interval $I \subset \mathbb{R}$. We use the technique of proof from [SW06, Sec. 6] to establish inequalities of the form $\mathfrak{f}'' - \mathfrak{f}' \geq -\mathfrak{f}$ for suitable C^2 functions $\mathfrak{f}\colon \mathbb{R} \to [0,\infty)$. These imply bounds for $|f(s)|$ in terms of the integral of $\mathfrak{f}$ over suitable intervals about s.

Motivated by [SW06, Sec. 6] one expects to get uniform pointwise bounds for the sum $|\partial_s u(s)|^2 + \tau'(s)^2$ in terms of ε-energy intervals by choosing for $\mathfrak{f}$ the ε-energy density $\frac{1}{2}|\partial_s u|^2 + \frac{1}{2}\varepsilon^2\tau'^2$. This does not work at all. In the case at hand one must not consider $\partial_s u$ and τ' simultaneously. Instead we use $\mathfrak{f}_0 := \frac{1}{2}|\partial_s u|^2$ to estimate $|\partial_s u(s)|$ and $\mathfrak{f}_1 := \frac{1}{2}|\overline{\nabla}_s \partial_s u|^2$ to estimate $|\overline{\nabla}_s \partial_s u(s)|$. By-products are L^2 estimates of $\varepsilon\tau''$, respectively $\varepsilon\tau'''$, in terms of $\|\partial_s u\|$. To estimate $|\tau'(s)|$ decompose the interval in two pieces $(0,1] = (0,\varepsilon_\kappa) \cup [\varepsilon_\kappa, 1]$. On $(0,\varepsilon_\kappa)$ just use (2.11). As $[\varepsilon_\kappa, 1]$ is compact and bounded away from $\varepsilon = 0$, the expected standard approach via the ε-energy density works.

Section 5 "Uniform exponential decay". Suppose f is a Morse function. We derive exponential decay, uniformly in $\varepsilon \in (0,1]$, of the derivative $\partial_s u$ of any solution (u,τ) of the ε-equations (1.3) subject to given asymptotic boundary conditions $x^\mp \in \operatorname{Crit} f$ of Morse index difference 1; see (1.6). Here we use again the density $\mathfrak{f}_0 := \frac{1}{2}|\partial_s u|^2$ and not $\frac{1}{2}|\partial_s u|^2 + \frac{1}{2}\varepsilon^2\tau'^2$. To deal with $|\tau'(s)|$ we decompose again $(0,1] = (0,\varepsilon_\kappa) \cup [\varepsilon_\kappa, 1]$. As in Section 4 the case $(0,\varepsilon_\kappa)$ is a free meal. Although $[\varepsilon_\kappa, 1]$ is compact and bounded away from the singularity $\varepsilon = 0$, this time the expected standard approach via the ε-energy density does <u>not</u> work, in sharp contrast to Section 4. Unexpected salvation arrives in the form of equation $\tau 1$ in (2.8), see (5.46).

To get exponential decay there are two methods, see Appendix A.2, the energy method (Section 5) and the action-energy inequality (Appendix A.2).

Small parameter interval $(0, \varepsilon_\kappa)$

For a number of results, for instance local and global surjectivity of the map $\mathcal{T}^\varepsilon$, we will need bounds for τ' and τ'' which do not come with a factor ε^{-2} as in the $\tau 1$ and $\tau 2$ equations in (2.8), see (6.60) and (7.71). The way out is to note that τ, τ', τ'' do also appear in the $u1$-$u3$ equations in which there is no ε at all. However, each of them comes in a product with $\overline{\nabla} H(u)$ which, if vanishing, would obstruct resolving for the τ derivative term. Since zero is a regular value of H and $\Sigma = H^{-1}(0)$ is compact, the gradient is bounded away from zero along Σ. Theorem 3.1 shows that this property persists, more precisely, given c_*, there are $c_\kappa, \varepsilon_\kappa > 0$ with

$$|\overline{\nabla} H(u)| \ge c_\kappa, \quad \forall\ \varepsilon\text{-solution with } E^\varepsilon(u,\tau) < \infty \text{ and } \varepsilon \in (0, \varepsilon_\kappa), \tag{2.9}$$

As mentioned earlier the component u takes values in a compact $K \subset M$.

Now we can resolve equation $u1$ for τ, namely, scalar multiply by $\overline{\nabla} H$, then divide by $|\overline{\nabla} H|^2$ and recall definition (1.1) of χ to get that

$$\tau \overset{u1}{=} \chi(u) - \frac{\left\langle \partial_s u, \overline{\nabla} H(u) \right\rangle}{|\overline{\nabla} H(u)|^2} \tag{2.10}$$

pointwise at $s \in \mathbb{R}$. Similarly, abbreviating $H = H(u)$ and $F = F(u)$ we get

$$\tau' \overset{u2}{=} -\frac{\left\langle \overline{\nabla} H, \overline{\nabla}_s \partial_s u + \overline{\nabla}_s \overline{\nabla} F + \tau \overline{\nabla}_s \overline{\nabla} H \right\rangle}{|\overline{\nabla} H|^2}, \qquad |\tau'| \le \tfrac{c_{F,H}}{c_\kappa} \left(|\overline{\nabla}_s \partial_s u| + |\partial_s u| \right), \tag{2.11}$$

pointwise at $s \in \mathbb{R}$ and where $c_{F,H}$ depends on the $C^3(K)$ norms[1] of F and H and the uniform bound of τ. Furthermore, we get analogously the identity

$$\tau'' \overset{u3}{=} -\frac{\left\langle \overline{\nabla} H, \overline{\nabla}_s \overline{\nabla}_s \partial_s u + \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} F + \tau \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} H + 2\tau' \overline{\nabla}_s \overline{\nabla} H \right\rangle}{|\overline{\nabla} H|^2} \tag{2.12}$$

and the estimate

$$|\tau''| \le \tfrac{d_{F,H}}{c_\kappa} \left(|\overline{\nabla}_s \overline{\nabla}_s \partial_s u| + |\partial_s u|^2 + |\partial_s u| \cdot |\tau'| \right) \tag{2.13}$$

both pointwise at $s \in \mathbb{R}$ and where $d_{F,H}$ depends on the $C^3(K)$ norms of F and H and the uniform bound of τ.

Section 6 "Local Surjectivity of $\mathcal{T}^\varepsilon$– time shift". Under the Morse-Smale condition there is an interval $(0, \varepsilon_0] \subset (0, \varepsilon_\kappa)$ such that the following holds. Given an ε-solution $(u^\varepsilon, \tau^\varepsilon)$ between critical points of index difference one, then any sufficiently close connecting base trajectory q has a time shift $q_\sigma := q(\sigma + \cdot)$ which gets mapped under $\mathcal{T}^\varepsilon$ to the given ε-solution. What determines the value of σ is that the difference vector from $i(q_\sigma)$ to $(u^\varepsilon, \tau^\varepsilon)$ must be in the image of the adjoint linearized operator. One translates the image property into being a zero $\theta^\varepsilon(\sigma) = 0$ of a suitably defined function.

Section 7 "Surjectivity of $\mathcal{T}^\varepsilon$". The program to prove surjectivity is described right after Theorem C.

Appendix A "Implicit function theorem II – detect base solutions". This section is independent of the others and provides the analysis

[1] here C^2 is Ok, but for later reference (proof of Theorem 5.1) we already ask C^3

for solutions of the base equation (1.4), the well known case of the downward gradient flow on a compact manifold. We prove the main tool to prove surjectivity of the map $\mathcal{T}^\varepsilon$, the base implicit function theorem (IFT II), Theorem A.1. The theorem requires the Morse-Smale condition for the base data. In this case the restriction function $f = F|_\Sigma$ to the base $\Sigma = H^{-1}(0)$ is automatically Morse. Theorem A.1 asserts that roughly speaking, if a map $q\colon \mathbb{R} \to \Sigma$ with asymptotic limits $x^\mp \in \mathrm{Crit} f$ almost solves the base equation and $|\partial_s q(s)|$ decays like $\frac{1}{1+s^2}$, then there is a 0-solution nearby. Exponential decay methods are discussed in Section A.2.

3. Uniform a priori bounds

The following theorem is fundamental, not only for the present article II, but also part I [FW] is built on (i).

Theorem 3.1 (Uniform a priori bounds). *Assume Hypothesis 2.1 with constant κ. Then there are, a compact subset $K \subset M$, and constants $c_0, c_1, c_2, c_3, c_\kappa > 0$ and $\varepsilon_\kappa \in (0,1]$, such that the following holds. Suppose that (u,τ) solves the ε-equations (1.3) and is of finite energy $E^\varepsilon(u,\tau) < \infty$.*

(i) *If $\varepsilon \in (0,1]$, then u takes values in K and there are bounds*

$$|\tau(s)| \le c_0, \quad |\partial_s u(s)| + \left|\tau'(s)\right| \le c_1, \quad \left|\overline{\nabla}_s \partial_s u(s)\right| + \left|\tau''(s)\right| \le c_2,$$

and $\left|\overline{\nabla}_s \overline{\nabla}_s \partial_s u(s)\right| \le c_3$, at every instant $s \in \mathbb{R}$.

(ii) *If $\varepsilon \in (0, \varepsilon_\kappa]$, then $\left|\overline{\nabla} H(u(s))\right| \ge c_\kappa$ at every instant $s \in \mathbb{R}$.*

3.1. Proof of Theorem 3.1

Let $\varepsilon \in (0,1]$ and let $z = (u,\tau)$ be a solution of the ε-equations (1.3) of finite energy, thus $E^\varepsilon(u,\tau) \le \mathrm{osc} f$ by (1.5). Abbreviate $z_s := z(s)$, similarly for u_s, τ_s. Assume Hypothesis 2.1 with constant $\kappa > 0$.

Step 1. *There exists a compact subset $K = K_\kappa \subset M$ such that if $\varepsilon \in (0,1]$ and $z = (u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ is a finite energy solution of the ε-equations (1.3), then u takes values in K, in symbols $u_s \in K$ for every $s \in \mathbb{R}$.*

To prove this we abbreviate $\nabla^\varepsilon = \nabla^{h^\varepsilon}$ and $|\cdot|_\varepsilon = |\cdot|_{h^\varepsilon}$. For $\alpha \in (0,1]$ consider the set of times s where the gradient is very large or wild

$$S_\kappa := \{s \in \mathbb{R}\colon\ |\nabla^\varepsilon F_H(z_s)|_{h^\varepsilon_{z_s}} \ge \tfrac{\kappa}{\varepsilon^\alpha}\}. \tag{3.14}$$

Given a wild time $s \in S_\kappa$, we denote the last entry time into the wild set S_κ before time s by

$$s_0 = s_0(s) := \sup\{s' < s \mid s' \notin S_\kappa\}. \tag{3.15}$$

Note that $[s_0, s] \subset S_\kappa$ and

$$|\nabla^\varepsilon F_H(z_{s_0})|_{h^\varepsilon_{z_{s_0}}} = \frac{\kappa}{\varepsilon^\alpha}. \tag{3.16}$$

Now we estimate the distance in terms of the energy, namely

$$\begin{aligned}
\sqrt{\operatorname{dist}_G(u_s, u_{s_0})^2 + \varepsilon^2 |\tau_s - \tau_{s_0}|^2} &= \operatorname{dist}_{h^\varepsilon}(z_s, z_{s_0}) \\
&\le \int_{s_0}^s |z'_\sigma|_\varepsilon \, d\sigma \\
&= \int_{s_0}^s \frac{|\nabla^\varepsilon F_H(z_\sigma)|_\varepsilon^2}{|\nabla^\varepsilon F_H(z_\sigma)|_\varepsilon} \, d\sigma \\
&\le \frac{\varepsilon^\alpha}{\kappa} \int_{s_0}^s |\nabla^\varepsilon F_H(z_\sigma)|_\varepsilon^2 \, d\sigma \\
&\le \frac{\varepsilon^\alpha}{\kappa} \int_{-\infty}^\infty \underbrace{|\nabla^\varepsilon F_H(z_\sigma)|_\varepsilon^2}_{|\partial_s u_\sigma|^2 + \varepsilon^2 {\tau'_\sigma}^2} \, d\sigma \\
&= \tfrac{\varepsilon^\alpha}{\kappa} E^\varepsilon(z) \\
&\overset{(1.5)}{\le} \tfrac{\operatorname{osc} f}{\kappa} \varepsilon^\alpha.
\end{aligned} \tag{3.17}$$

where step three is by (1.3) in the form $z'_\sigma = -\nabla^\varepsilon F_H(z_\sigma)$. This shows that

$$\operatorname{dist}_G(u_s, u_{s_0}) \le \tfrac{\operatorname{osc} f}{\kappa} \varepsilon^\alpha \overset{\alpha = \frac{1}{2}}{=} \tfrac{\operatorname{osc} f}{\kappa} \sqrt{\varepsilon}. \tag{3.18}$$

On the other hand at time s_0 there are, by (3.16) and (1.3), the two identities

$$\tfrac{\kappa^2}{\varepsilon^{2\alpha}} = |\nabla^\varepsilon F_H(w_{s_0})|_\varepsilon^2 = \left|\overline{\nabla} F(u_{s_0}) + \tau_{s_0} \overline{\nabla} H(u_{s_0})\right|^2 + \varepsilon^{-2} H(u_{s_0})^2. \tag{3.19}$$

This shows that

$$|H(u_{s_0})| \le \kappa \varepsilon^{1-\alpha} \overset{\alpha = \frac{1}{2}}{=} \kappa \sqrt{\varepsilon}. \tag{3.20}$$

Estimates (3.18) and (3.20) suggest to choose $\alpha = \frac{1}{2}$. Since $\varepsilon \in (0, 1]$ we get

$$|H(u_{s_0})| \le \kappa, \qquad \operatorname{dist}_G(u_s, u_{s_0}) \le \tfrac{\operatorname{osc} f}{\kappa}. \tag{3.21}$$

Now we define the set

$$K = K_\kappa := \{p \in M \mid \mathrm{dist}_G(p, H^{-1}[-\kappa,\kappa]) \le \tfrac{\mathrm{osc} f}{\kappa}\}.$$

By (3.21) we see that $u_{s_0} \in H^{-1}[-\kappa,\kappa]$ and $u_s \in K$. Hence $u(S_\kappa) \subset K$.

For a non-wild time $s \notin S_\kappa$ we have the inequality

$$\tfrac{\kappa^2}{\varepsilon^{2\alpha}} > |\nabla^\varepsilon F_H(w_s)|_\varepsilon^2 = \left|\overline{\nabla} F(u_s) + \tau_s \overline{\nabla} H(u_s)\right|^2 + \varepsilon^{-2} H(u_s)^2. \tag{3.22}$$

Thus $|H(u_s)| < \kappa\sqrt{\varepsilon} \le \kappa$, hence $u_s \in H^{-1}[-\kappa,\kappa]$, and therefore $u_s \in K$.

The set K is closed and bounded. Thus, since G is geodesically complete, the set K is compact by the Theorem of Hopf-Rinow; see e.g. [O'N83, Ch. 5 Thm. 21]. This proves Step 1.

Step 2 (ε_κ)**.** *There exists $\varepsilon_\kappa \in (0,1]$ and $c_\kappa > 0$ such that $\left|\overline{\nabla} H(u_s)\right| \ge c_\kappa$ whenever $s \in \mathbb{R}$ and $z = (u,\tau)$ is a finite energy ε-solution with $\varepsilon \in (0,\varepsilon_\kappa]$.*

We prove Step 2. Since 0 is a regular value of H and $\Sigma = H^{-1}(0)$ is compact there exists a constant $c_\kappa > 0$ such that $\left|\overline{\nabla} H(x)\right| \ge 2c_\kappa$ for every $x \in \Sigma$.

Thus $\exists \mu > 0$ such that if $\mathrm{dist}_G(x,\Sigma) \le \mu$ with $x \in M$, then $\left|\overline{\nabla} H(x)\right| \ge c_\kappa$.

Claim. *Since $H^{-1}[-\kappa,\kappa]$ is compact there exists $\varepsilon_\kappa \in (0,1]$ such that if a point $x \in M$ satisfies $|H(x)| \le \kappa\sqrt{\varepsilon_\kappa}$, then* $\mathrm{dist}_G(x,\Sigma) \le \frac{\mu}{2}$.

To prove the claim suppose by contradiction that there are sequences $\varepsilon_\nu \to 0$ and $x_\nu \in H^{-1}[-\kappa\sqrt{\varepsilon_\nu}, \kappa\sqrt{\varepsilon_\nu}]$, but $\mathrm{dist}_G(x_\nu,\Sigma) > \frac{\mu}{2}$. By compactness of $H^{-1}[-\kappa,\kappa]$ there is a subsequence x_{ν_j} and a point $x \in H^{-1}(0)$ such that $x_{\nu_j} \to x$, as $j \to \infty$. By the corresponding property of each point x_ν the limit point x also satisfies $\mathrm{dist}_G(x,\Sigma) \ge \mu/2$. But $x \in H^{-1}(0) = \Sigma$. Contradiction.

With the notation from Step 1 we see from (3.20) that $|H(u_{s_0})| \le \kappa\sqrt{\varepsilon_\kappa}$, thus by the claim $\mathrm{dist}_G(u_{s_0},\Sigma) \le \frac{\mu}{2}$. Choose ε_κ smaller, if necessary, such that $\frac{\mu}{2} \ge \frac{\mathrm{osc} f}{\kappa}\sqrt{\varepsilon_\kappa}$ where $\frac{\mathrm{osc} f}{\kappa}\sqrt{\varepsilon_\kappa} \ge \mathrm{dist}_G(u_s, u_{s_0})$ by (3.18). Hence we have

$$\mathrm{dist}_G(u_s,\Sigma) \le \mathrm{dist}_G(u_s,u_{s_0}) + \mathrm{dist}_G(u_{s_0},\Sigma) \le \tfrac{\mu}{2} + \tfrac{\mu}{2} = \mu.$$

Hence by construction of μ prior to the claim it follows that $\left|\overline{\nabla} H(u_s)\right| \ge c_\kappa$. This proves Step 2.

Recall that the map u takes values in the compact set K, by Step 1, and that $\left|\overline{\nabla} H \circ u\right| \ge c_\kappa$ whenever $\varepsilon \in (0,\varepsilon_\kappa]$, by Step 2.

Step 3 (ε_κ)**.** *Every ε-solution (u,τ) with $\varepsilon\in(0,\varepsilon_\kappa]$ admits the a priori bound*

$$|\tau(s)|\le\tfrac{1}{c_\kappa^2}\max\nolimits_K\left|\left\langle\overline{\nabla}F,\overline{\nabla}H\right\rangle\right|$$

for every $s\in\mathbb{R}$.

We prove Step 3. Case $\tau(s)>0$: By identity $\tau2$ in (2.8) we get the estimate

$$\varepsilon\tau''(s)\overset{\tau2}{\ge}-\max_K|b|+c_\kappa^2\tau(s),\qquad b(p):=\left\langle\overline{\nabla}H(p),\overline{\nabla}F(p)\right\rangle$$

for every $s\in\mathbb{R}$. If $\tau(s)>\frac{1}{c_\kappa^2}\max_K|b|$, then $\varepsilon\tau''(s)>0$, hence in this range no s can be a maximum.

As $z=(u,\tau)$ has finite energy, it has a non-empty ω-limit set consisting of critical points of F_H, i.e. pairs $(x,\tau_x)\in\Sigma\times\mathbb{R}$ satisfying the critical point equation $\overline{\nabla}F(x)+\tau_x\overline{\nabla}H(x)=0$. Take the inner product with $\overline{\nabla}H$ to get

$$|\tau_x|=\frac{|\langle\overline{\nabla}F(x),\overline{\nabla}H(x)\rangle|}{|\overline{\nabla}H(x)|^2}\le\frac{\max_\Sigma|b|}{c_\kappa^2}.$$

Thus it follows that

$$\tau(s)\le\tfrac{1}{c_\kappa^2}\max\nolimits_K|b|$$

for every $s\in\mathbb{R}$; otherwise $\tau(s)$ would have to return to its asymptotic limits and therefore there would be a maximum in the above range, contradiction.

Case $\tau(s)<0$: Similarly, if $\tau(s)<0$ is negative, then we get the estimate

$$\varepsilon\tau''(s)\overset{\tau2}{\le}\max_K|b|+c_\kappa^2\tau(s),\qquad b(p):=\left\langle\overline{\nabla}H(p),\overline{\nabla}F(p)\right\rangle$$

for every $s\in\mathbb{R}$. If $\tau(s)<-\frac{1}{c_\kappa^2}\max_K|b|$, then $\varepsilon\tau''(s)<0$, hence in this range no s can be a minimum. Thus

$$\tau(s)\ge-\tfrac{1}{c_\kappa^2}\max\nolimits_K|b|$$

for every $s\in\mathbb{R}$; otherwise there is a minimum, contradiction.

Step 4. *There is a constant $C_0>0$ such that any ε-solution (u,τ) is bounded*

$$|\tau(s)|\le C_0,\qquad|\partial_s u(s)|\le\max_K\left|\overline{\nabla}F\right|+C_0\max_K\left|\overline{\nabla}H\right|=:D_1$$

for all $s\in\mathbb{R}$ and $\varepsilon\in(0,1]$.

We prove Step 4. For τ the case $\varepsilon\in(0,\varepsilon_\kappa]$ was already proved in Step 3. Thus let $\varepsilon\in(\varepsilon_\kappa,1]$. Recall that κ is the constant from Hypothesis 2.1 on local properness of H. For $\kappa_0 \in (0, \kappa]$ we define the wild set S_{κ_0} by (3.14). For $s \in S_{\kappa_0}$ let s_0 be the last entry time into the wild set S_{κ_0} before s; see (3.15). From (3.17) with $\alpha = \frac{1}{2}$ we get the estimate $\varepsilon^2 \left|\tau_s - \tau_{s_0}\right|^2 \leq \varepsilon (\mathrm{osc} f)^2/\kappa_0^2$, so

$$|\tau_s - \tau_{s_0}| \leq \frac{\mathrm{osc} f}{\sqrt{\varepsilon}\kappa_0} \leq \frac{\mathrm{osc} f}{\sqrt{\varepsilon_\kappa}\kappa_0}. \tag{3.23}$$

Equation (3.19) for κ_0 is of the form

$$\frac{\kappa_0^2}{\varepsilon} = \left|\overline{\nabla} F(u_{s_0}) + \tau_{s_0} \overline{\nabla} H(u_{s_0})\right|^2 + \varepsilon^{-2} H(u_{s_0})^2. \tag{3.24}$$

Thus $H(u_{s_0})^2 \leq \kappa_0^2 \varepsilon \leq \kappa_0^2$, hence $|H(u_{s_0})| \leq \kappa_0$. As in the claim in the proof of Step 2 choose $\kappa_0 > 0$ so small that if $x \in M$ satisfies $|H(x)| \leq \kappa_0$ then $\left|\overline{\nabla} H(x)\right| \geq c_\kappa$. By this choice of κ_0 it follows that $\left|\overline{\nabla} H(u_{s_0})\right| \geq c_\kappa$. Firstly, by (3.24) we have $\left|\overline{\nabla} F(u_{s_0}) + \tau_{s_0} \overline{\nabla} H(u_{s_0})\right| \leq \kappa_0/\sqrt{\varepsilon} \leq \kappa_0/\sqrt{\varepsilon_\kappa}$. Secondly

$$\left|\overline{\nabla} F(u_{s_0}) + \tau_{s_0} \overline{\nabla} H(u_{s_0})\right| \geq \left|\tau_{s_0} \overline{\nabla} H(u_{s_0})\right| - \left|\overline{\nabla} F(u_{s_0})\right| \geq c_\kappa \left|\tau_{s_0}\right| - \max_K \left|\overline{\nabla} F\right|.$$

Combining these two facts we obtain

$$|\tau_{s_0}| \leq \frac{1}{c_\kappa} \left(\frac{\kappa_0}{\sqrt{\varepsilon_\kappa}} + \max_K \left|\overline{\nabla} F\right| \right).$$

Therefore, by (3.23), we obtain the bound

$$|\tau_s| \leq |\tau_s - \tau_{s_0}| + |\tau_{s_0}| \leq \frac{1}{c_\kappa} \left(\frac{\kappa_0}{\sqrt{\varepsilon_\kappa}} + \max_K \left|\overline{\nabla} F\right| \right) + \frac{\mathrm{osc} f}{\sqrt{\varepsilon_\kappa}\kappa_0} =: c_0$$

for all $s \in \mathbb{R}$ and $\varepsilon \in (\varepsilon_\kappa, 1]$. Hence, by Step 3, we obtain the uniform bound

$$|\tau| \leq \max\{C_{\mathrm{Step3}}, c_0\} =: C_0$$

whenever $\varepsilon \in (0, 1]$. Thus by (2.8) we get

$$|\partial_s u| \overset{u1}{\leq} \max_K \left|\overline{\nabla} F\right| + C_0 \max_K \left|\overline{\nabla} H\right|$$

whenever $\varepsilon \in (0, 1]$. This proves Step 4.

Step 5 (ε_κ)**.** *There is a constant $\tilde{C}_1 > 0$ such that every ε-solution (u, τ) with $\varepsilon \in (0, \varepsilon_\kappa]$ admits the a priori bound*

$$\left|\tau'(s)\right| \leq \tilde{C}_1$$

for every $s \in \mathbb{R}$.

We prove Step 5. The proof follows the same scheme as Step 3 just for τ' instead of τ and using in (2.8) the identity $\tau 3$. E.g. one gets the estimate

$$\varepsilon(\tau')''(s) \overset{\tau 3}{\geq} -\max_K |\tilde{b}| + c_\kappa^2 \tau'(s)$$

for every $s \in \mathbb{R}$ where, however, the definition of $\tilde{b}$ incorporates more terms as in Step 3 and the estimate for $\tilde{b}$ uses the a priori bounds $|\tau|$ and $|\partial_s u|$ obtained in Step 4. Now proceed as in Step 3. This proves Step 5.

Step 6. *There are constants C_1, D_2 such that any ε-solution (u, τ) is bounded*

$$\left|\tau'(s)\right| \leq C_1, \qquad \left|\overline{\nabla}_s \partial_s u(s)\right| \leq D_2$$

for all $s \in \mathbb{R}$ and $\varepsilon \in (0, 1]$.

We prove Step 6. Recall from (2.8) the gradient flow equation $\tau 1$, namely

$$\tau'(s) = -\frac{1}{\varepsilon^2} H(u(s)).$$

Therefore $|\tau'| \leq \frac{1}{\varepsilon^2} \max_K |H| \leq \frac{1}{\varepsilon_\kappa^2} \max_K |H|$ in case $\varepsilon \in [\varepsilon_\kappa, 1]$; here we used that u takes values in K by Step 1. Therefore the a priori bound for τ', uniformly for $\varepsilon \in (0, 1]$, follows together with Step 5.

The uniform a priori bound for $\overline{\nabla}_s \partial_s u$ follows from the gradient flow equation $u2$ in (2.8) together with the uniform bounds for $\tau, \tau', \partial_s u$ which were already established. This proves Step 6.

Step 7. *There are constants C_2, D_3 such that any ε-solution (u, τ) is bounded*

$$\left|\tau''(s)\right| \leq C_2, \qquad \left|\overline{\nabla}_s \overline{\nabla}_s \partial_s u(s)\right| \leq D_3$$

for all $s \in \mathbb{R}$ and $\varepsilon \in (0, 1]$.

The proof of Step 7 follows the same scheme as the proof of Step 6 using equation $\tau 4$ in (2.8), together with the estimates established in Step 6. This proves Step 7. The proof of Theorem 3.1 is complete.

Remark 3.2 (Higher order bounds in Theorem 3.1). Bootstrapping further one gets bounds for all derivatives of τ and u, uniformly in $\varepsilon \in (0, 1]$.

3.2. Compactness

Local convergence of a sequence (u_i, τ_i) of ε_i-connecting trajectories, given a sequence $\varepsilon_i \to 0$, follows already from the a priori L^∞ bounds of (u_i, τ_i)

and its first and second derivatives, as provided by Theorem 3.1. Here not even the Morse condition is needed.

Corollary 3.3 (Local convergence). *Let* $(\varepsilon_\nu)_{\nu\in\mathbb{N}} \subset (0,1]$ *be a sequence with limit zero* $\lim_{\nu\to\infty}\varepsilon_\nu = 0$*. Suppose* $(u_\nu,\tau_\nu)\colon \mathbb{R}\to M\times\mathbb{R}$ *is a sequence of finite energy solutions of the* ε_ν*-equation (1.3). Then there is a subsequence* ν_j *and a solution* $q\colon \mathbb{R}\to\Sigma$ *of the* 0*-equation (1.4) such that* u_{ν_j} *converges in* C^∞_{loc} *to* q *and* τ_{ν_j} *in* C^∞_{loc} *to* $\chi\circ q$*.*

Proof. Iterating the arguments in the proof of Theorem 3.1, using recursively (2.8), we obtain that τ_ν and u_ν, with all derivatives, are uniformly bounded for all ν and all times. Therefore by the Theorem of Arzelà-Ascoli there exists a subsequence ν_j such that (u_{ν_j},τ_{ν_j}) converges in the C^∞_{loc} topology. Since (u_{ν_j},τ_{ν_j}) is a solution of the ε_{ν_j}-equation the limit $q := \lim_{j\to\infty} u_{\nu_j}$ has to be a finite energy solution of the 0-equation. Since for the 0-equation the Lagrange multiplier τ is uniquely determined by q, namely as composition $\chi\circ q$, we further have $\lim_{j\to\infty}\tau_{\nu_j} = \chi\circ q$. □

If one prescribes *asymptotics* of index difference *one*, then the *Morse-Smale* condition leads to *compactness*, in C^∞, of the space of connecting trajectories.

Lemma 3.4 (Compactness for fixed index difference 1 asymptotics). *Assume* (f,g) *is Morse-Smale. Let* $x^\mp \in \operatorname{Crit} f$ *be critical points of Morse index difference one. Let* $(u_\nu,\tau_\nu)\colon\mathbb{R}\to M\times\mathbb{R}$ *be a sequence of solutions of the* ε_ν*-equation with fixed asymptotics* $\lim_{s\to\mp\infty}(u_\nu(s),\tau_\nu(s)) = (x^\mp,\chi\circ x^\mp))$ *and such that* $\varepsilon_\nu\to 0$*, as* $\nu\to\infty$*. Then there exists a subsequence* u_{ν_j} *and a solution* q *of the* 0*-equation with asymptotics* $x^\mp$ *such that* u_{ν_j} *converges in* C^∞ *to* q *and* τ_{ν_j} *converges in* C^∞ *to* $\chi\circ q$*.*

Proof. It is well known, see e.g. [Sal90, Sch93] or [FN20, §4.7], that in view of the local convergence from Corollary 3.3 the sequence u_ν has a subsequence u_{ν_j} which converges in the sense of Floer-Gromov to a broken gradient flow line $q = (q^1,\dots,q^m)$ from x^- to x^+. Since the index difference of x^- and x^+ is 1 and the metric g is Morse-Smale the limit q is unbroken, that is $m = 1$, i.e. $q = q^1$ is a gradient flow line from x^- to x^+. This proves Lemma 3.4. □

Compactness under the Morse condition. In Section 7 the proof of Theorem C, which asserts surjectivity of the map $\mathcal{T}^\varepsilon$, uses the following two refined compactness results.

Lemma 3.5 (Local convergence). *Assume f is Morse. Let $x^\mp \in \mathrm{Crit} f$ and $(u_i, \tau_i) \in \mathcal{M}^{\varepsilon_i}_{x^-,x^+}$ where $\varepsilon_i > 0$ is a real sequence converging to zero. Then there is a pair of critical points $x_0, x_1 \in \mathrm{Crit}\, f$, a connecting base trajectory $q \in \mathcal{M}^0_{x_0,x_1}$, and a subsequence, still denoted by (u_i, τ_i), such that*

(i) *(u_i, τ_i) converges to $i(q) = (q, \chi(q))$ strongly in C^1 and weakly in $W^{2,2}$ on every compact subset of $\mathbb{R}$. Moreover, the difference $\tau_i - \chi(u_i)$ converges to zero in the C^1 norm on every compact subset of $\mathbb{R}$;*

(ii) *whenever $s \in \mathbb{R}$ and $T > 0$ the following limits exist and are given by*

$$f(q(s)) = \lim_{i\to\infty} F_H(u_i(s), \tau_i(s)), \qquad E^0_{[-T.T]}(q) = \lim_{i\to\infty} E^{\varepsilon_i}_{[-T.T]}(u_i, \tau_i). \tag{3.25}$$

Proof. (i) By the a priori Theorem 3.1 there is a constant $c > 0$ such that

$$\|\tau_i\|_\infty + \|\partial_s u_i\|_\infty + \|\tau_i'\|_\infty + \|\overline{\nabla}_s \partial_s u_i\|_\infty + \|\tau_i''\|_\infty \le c \tag{3.26}$$

for every $i \in \mathbb{N}$. Since $|H \circ u_i| = \varepsilon_i^2 |\tau_i'| \le \varepsilon_i^2 c$ the sequence u_i is bounded in C^0; compare Hypothesis 2.1. Thus, by (3.26), the sequence (u_i, τ_i) is bounded in C^2 and hence in $W^{2,2}([-T,T])$ for each $T > 0$. Thus, by the Arzelà-Ascoli theorem and the Banach-Alaoglu theorem, there is a subsequence, still denoted by (u_i, τ_i), that converges strongly in C^1 and weakly in $W^{2,2}$ on every compact subset of $\mathbb{R}$ to some $W^{2,2}_{\mathrm{loc}}$-function $(q, \tau)\colon \mathbb{R} \to M \times \mathbb{R}$. As $\|\tau_i'\|_\infty \le c$, by (3.26), the sequence $H \circ u_i = -\varepsilon_i^2 \tau_i'$ converges to zero in the C^0 norm. Hence $H(q) \equiv 0$ which shows that $q\colon \mathbb{R} \to \Sigma$ actually takes values in $\Sigma = H^{-1}(0)$. We get that

$$\tau_i \overset{(2.10)}{=} \chi(u_i) - \frac{\langle \partial_s u_i, \overline{\nabla} H(u_i)\rangle}{|\overline{\nabla} H(u_i)|^2} \overset{i\to\infty}{\longrightarrow} \chi(q) - \frac{\langle \partial_s q, \overline{\nabla} H(q)\rangle}{|\overline{\nabla} H(q)|^2} = \chi(q) \tag{3.27}$$

where the limit is pointwise at $s \in \mathbb{R}$ and where we used that $\overline{\nabla} H(q)$ is pointwise orthogonal to $\partial_s q \in T_q H^{-1}(0)$. This proves that $\tau = \chi(q)$. Observe that both differences, namely $\tau_i - \chi(u_i)$ and also

$$\tau_i' - (\chi(u_i))' = -\frac{d}{ds} \frac{\langle \partial_s u_i, \overline{\nabla} H(u_i)\rangle}{|\overline{\nabla} H(u_i)|^2},$$

converge to zero in C^0_{loc}. The latter holds by weak $W^{1,2}_{\mathrm{loc}}$ convergence $\partial_s u_i \rightharpoonup \partial_s q$. As $u_i \to q$ and $\tau_i \to \chi(q)$, both in C^1 on compact subsets of $\mathbb{R}$, we get

$$0 \overset{(1.3)}{=} \partial_s u_i + \overline{\nabla} F|_{u_i} + \tau_i \overline{\nabla} F|_{u_i} \overset{i\to\infty}{\longrightarrow} \partial_s q + \overline{\nabla} F|_q + \chi|_q \overline{\nabla} F|_q \overset{(1.4)}{=} \partial_s q + \nabla f|_q$$

on any compact subset of $\mathbb{R}$. So the limit $q\colon \mathbb{R}\to\Sigma$ satisfies the base equation (1.4). Thus q is smooth, by regularity in part I [FW], and so is $\tau=\chi\circ q$.

(ii) The fact that $u_i\to q$ in C^1_{loc}, as $i\to\infty$, proves the first equality in

$$\begin{aligned} E^0_{[-T,T]}(q) &= \lim_{i\to\infty}\int_{-T}^{T}\lvert\partial_s u_i\rvert^2\,ds \\ &= \lim_{i\to\infty}\int_{-T}^{T}\left(\lvert\partial_s u_i\rvert^2+\varepsilon_i^2(\tau_i')^2\right)ds \\ &= \lim_{i\to\infty} E^{\varepsilon_i}_{[-T,T]}(u_i,\tau_i) \end{aligned} \tag{3.28}$$

and this is true for any $T\ge 0$. Equality two holds since $\lVert\tau_i'\rVert_\infty\le c$ is uniformly bounded and $\varepsilon_i\to 0$. Equality three is by definition of ε_i-energy. Hence the limit q has finite energy $c^*:=f(x^-)-f(x^+)$ by the ε_i-energy identity (1.7), and so, by Proposition A.5, belongs to a moduli space $\mathcal{M}^0_{x_0,x_1}$ for some $x_0,x_1\in\mathrm{Crit}\,f$.

Pick $s\in\mathbb{R}$, then since $u_i\to q$ in C^0_{loc} we get pointwise convergence

$$f(q(s))=F(q(s))=\lim_{i\to\infty}\left(F(u_i(s))+\tau_i(s)H(u_i(s))\right)$$

due to $H\circ q\equiv 0$, since q takes values in $H^{-1}(0)$. This proves Lemma 3.5. $\square$

Lemma 3.6 (Convergence with fixed asymptotics – broken trajectory limit). *Let f be Morse. Let $x^\mp\in\mathrm{Crit} f$ and $(u_i,\tau_i)\in\mathcal{M}^{\varepsilon_i}_{x^-,x^+}$ where $\varepsilon_i>0$ is a real sequence converging to zero. In this case there are critical points $x^-=x^0,x^1,\dots,x^\ell=x^+\in\mathrm{Crit}\,f$, connecting base trajectories $q^k\in\mathcal{M}^0_{x^{k-1},x^k}$ for $k\in\{1,\dots,\ell\}$, a subsequence, still denoted by (u_i,τ_i), and time shift sequences $s_i^k\in\mathbb{R}$, $k\in\{1,\dots,\ell\}$, such that the following holds.*

- (i) *For each $k\in\{1,\dots,\ell\}$ the sequence $s\mapsto\left(u_i(s_i^k+s),\tau_i(s_i^k+s)\right)$ converges to $(q^k,\chi(q^k))$ as in the local convergence Lemma 3.5.*
- (ii) *For each $k\in\{1,\dots,\ell\}$ it holds $\partial_s q^k\not\equiv 0$, hence $x^{k-1}\ne x^k$, and, furthermore, each difference sequence $s_i^k-s_i^{k-1}$ diverges to infinity.*
- (ii) *For any $k\in\{1,\dots,\ell\}$ and any $\rho>0$ there $\exists T>0$ such that, for all i and $s\in\mathbb{R}$, it holds $u_i([s_i^k+T,s_i^{k+1}-T])\subset B_\rho(x^k)$, equivalently*

$$s_i^k+T\le s\le s_i^{k+1}-T\quad\Rightarrow\quad \mathrm{dist}(u_i(s),x^k)<\rho.$$

Here $s_i^0:=-\infty$ and $s_i^{\ell+1}:=\infty$ and dist *is the Riemannian distance.*

Note: while Morse is essential in the lemma, Morse-Smale is not required.

Proof. The proof in [SW06, Le. 10.3] carries over literally. Let us just add the argument that the difference $s_i^2 - s_i^1$ diverges to infinity: Abbreviate $z_i := (u_i, \tau_i)$. By (i), the sequence $(s_i^1)_* z_i$ converges to $(q^1, \chi(q^1))$, as in the local convergence Lemma 3.5, and $(s_i^2)_* z_i$ converges to $(q^2, \chi(q^2))$. Assume by contradiction that the sequence $s_i^2 - s_i^1$ is bounded. Then there is a subsequence, same notation, which converges to some time $T \in \mathbb{R}$. Since $(s_i^2)_* z_i = (s_i^2 - s_i^1)_* (s_i^1)_* z_i$, we get

$$(q^2, \chi(q^2)) = \lim_{i\to\infty} (s_i^2)_* z_i = \lim_{i\to\infty} (s_i^2 - s_i^1)_* (s_i^1)_* z_i = T_*(q^1, \chi(q^1)).$$

So q^1 is just a reparametrization of q^2. Thus their upper asymptotics, namely. x^0 for q^1 and x^1 for q^2, must be equal. But this contradicts $x^0 \neq x^1$. □

4. Uniform gradient bounds by energy intervals

The following result refines the a priori Theorem 3.1 for $|\partial_s u(s)|$ in the sense that on the right hand side it brings in energy intervals about $s \in \mathbb{R}$. The latter decay exponentially in s, uniformly in $\varepsilon \in (0, 1]$, as we shall prove later on in Theorem 5.1 under the Morse condition for f.

Theorem 4.1 (Gradient bounds for u). *There exists a constant $C > 0$ such that if $\varepsilon \in (0, 1]$ and (u, τ) is a finite energy ε-trajectory, then*

$$|\partial_s u(s)|^2 \le C \|\partial_s u\|^2_{L^2([s-1,s+1])} \tag{4.29}$$

and

$$\|\overline{\nabla}_s \partial_s u_s\|_{L^2([s-\frac14,s+\frac14])} + \varepsilon \|\tau''\|_{L^2([s-\frac14,s+\frac14])} \le C \|\partial_s u\|_{L^2([s-30,s+30])} \tag{4.30}$$

for every $s \in \mathbb{R}$. Moreover, it holds that

$$|\overline{\nabla}_s \partial_s u(s)|^2 \le C \|\partial_s u\|^2_{L^2([s-30,s+30])} \tag{4.31}$$

and

$$\|\overline{\nabla}_s \overline{\nabla}_s \partial_s u_s\|_{L^2[s-\frac14,s+\frac14]} + \varepsilon \|\tau'''\|_{L^2[s-\frac14,s+\frac14]} \le C \|\partial_s u\|_{L^2[s-60,s+60]}$$

for every $s \in \mathbb{R}$.

Where are the gradient bounds used? We use the L^∞ gradient bounds to prove exponential decay of $\partial_s u$ and $\overline{\nabla}_s \partial_s u$, see (5.43) and Corollary 5.2. We use the $L^2(\mathbb{R})$ bound for $\varepsilon\tau''$ in the global surjectivity Theorem C, see (7.71).

Corollary 4.2 (Gradient bounds for τ). *Under the assumptions of Theorem 4.1 it holds that*

$$|\tau'(s)|^2 \le \begin{cases} C\|\partial_s u\|^2_{L^2([s-30,s+30])}, & \varepsilon\in(0,\varepsilon_\kappa], \\ CE^\varepsilon_{[s-1,s+1]}(u,\tau), & \varepsilon\in[\varepsilon_\kappa,1], \end{cases} \tag{4.32}$$

for every $s\in\mathbb{R}$ where $E^\varepsilon_I(u,\tau)$ is the energy of (u,τ) along an interval $I\subset\mathbb{R}$.

On $(0,\varepsilon_\kappa]$ obtain the corollary by inserting (4.29) and (4.31) into (2.11). On $[\varepsilon_\kappa,1]$ the factor ε^2 in $\hat\tau 3$ in (2.8) gets irrelevant. A simple version of the technique in the proof of the theorem will do, so we do the proof in the end.

To get pointwise bounds we use the following mean value type inequality whose proof in [SW06] relies on Gruber's parabolic mean value inequality.

Lemma 4.3 (Le. B.3[SW06]). *There is a universal constant $d>0$ such that the following is true. Fix two constants $r\in(0,1]$ and $\delta\ge 0$. If $\mathfrak{f}\colon[-r^2-\delta r,\delta r]\to\mathbb{R}$ is a C^2 function, C^1 suffices in case $\delta=0$, that satisfies*

$$\delta^2\mathfrak{f}''-\mathfrak{f}'\ge-\mu\mathfrak{f},\qquad \mathfrak{f}\ge 0,$$

for some constant $\mu\ge 0$, then

$$\mathfrak{f}(0)\le\frac{4de^{\mu r^2}}{r^2}\int_{-r^2-\delta r}^{\delta r}\mathfrak{f}(s)\,ds.$$

Concerning L^2 estimates there is the following tool.

Lemma 4.4 (Le. B.6[SW06]). *Fix constants $r,R,\delta>0$ and functions $\mathfrak{f},\mathfrak{g},\mathfrak{u}\colon[-(R+r)^2-\delta(R+r),\delta(R+r)]\to\mathbb{R}$ with $\mathfrak{f}\in C^2$ and $\mathfrak{g},\mathfrak{u}\in C^0$. If*

$$\delta^2\mathfrak{f}''-\mathfrak{f}'\ge\mathfrak{g}-\mathfrak{u},\qquad \mathfrak{f}\ge0,\quad \mathfrak{g}\ge0,\quad \mathfrak{u}\ge0,$$

then

$$\begin{aligned}\int_{-(\frac{R}{2})^2+\delta\frac{R}{2}}^{-\delta\frac{R}{2}}\mathfrak{g}(s)\,ds &\le 4\left(1+\tfrac{r}{R}\right)^2\int_{-(R+r)^2-\delta(R+r)}^{\delta(R+r)}\mathfrak{u}(s)\,ds\\ &\quad+4\left(1+\tfrac{r}{R}\right)^2\left(\tfrac{4}{r^2}+\tfrac{1}{Rr}\right)\int_{-(R+r)^2-\delta(R+r)}^{\delta(R+r)}\mathfrak{f}(s)\,ds.\end{aligned}$$

Proof of Theorem 4.1. Let $\varepsilon \in (0,1]$. Let (u,τ) be a finite energy solution of the ε-equations (1.3). Define $2\mathfrak{f}_0 := |\partial_s u|^2$ and $2\mathfrak{g}_0 := |\overline{\nabla}_s \partial_s u|^2$. We have

$$\mathfrak{f}_0'' - \mathfrak{f}_0' = 2\mathfrak{g}_0 + \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \partial_s u - \overline{\nabla}_s \partial_s u \right\rangle . \tag{4.33}$$

To estimate the inner product use $\underline{u3}$ and $\hat{\tau}2$ in (2.8) to get

$$\begin{aligned}\left\langle \partial_s u, \underline{\overline{\nabla}_s \overline{\nabla}_s \partial_s u} - \overline{\nabla}_s \partial_s u \right\rangle &= \left\langle \partial_s u, -\overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} F - \tau \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} H - 2\tau' \overline{\nabla}_s \overline{\nabla} H \right\rangle \\ &\quad + \varepsilon^2 \tau''^2 - \left\langle \partial_s u, \overline{\nabla}_s \partial_s u \right\rangle .\end{aligned}$$

Theorem 3.1 requires the finite energy hypothesis and provides uniform L^∞ bounds for τ, τ', $\partial_s u$ and tells that u takes values in a compact $K \subset M$. By Young's inequality and (4.33) we get the inequality $\mathfrak{f}_0'' - \mathfrak{f}_0' \geq 2\mathfrak{g}_0 + \varepsilon^2 \tau''^2 - \mu \mathfrak{f}_0 - \mathfrak{g}_0$ pointwise at $s \in \mathbb{R}$ for a suitable constant $\mu_1 > 0$, so [2]

$$\mathfrak{f}_0'' - \mathfrak{f}_0' \geq \mathfrak{g}_0 + \varepsilon^2 \tau''^2 - \mu_1 \mathfrak{f}_0 \geq -\mu_1 \mathfrak{f}_0. \tag{4.34}$$

Now pick a number $s_0 \in \mathbb{R}$ and apply Lemma 4.3 with $\delta = 1$ and $r = \frac{1}{3}$ to the function $w(s) := \mathfrak{f}_0(s_0 + s)$ to obtain

$$\tfrac{1}{2} |\partial_s u(s_0)|^2 = w(0) \leq \frac{4d e^{\frac{\mu_1}{9}}}{9} \int_{-\frac{4}{9}}^{\frac{1}{3}} \mathfrak{f}_0(s_0 + s)\, ds \leq d e^{\frac{\mu_1}{9}} \|\partial_s u\|^2_{L^2([s_0 - \frac{1}{2}, s_0 + \frac{1}{2}])} .$$

This proves the pointwise estimate (4.29). By (4.34), Lemma 4.4 with $\delta = 1$, $R = 2 + \sqrt{6}$, $r = -3 - \sqrt{6} + \sqrt{30}$ applies to $s \mapsto w(s) := \mathfrak{f}_0(s_0 + s)$, thus

$$\int_{s_0 - \frac{3}{2} - \sqrt{\frac{3}{2}}}^{s_0 - 1 - \sqrt{\frac{3}{2}}} \left(|\overline{\nabla}_s \partial_s u|^2 + \varepsilon^2 \tau''^2 \right) ds \leq \tilde{C}_1 \int_{s_0 + \sqrt{30} - 30}^{s_0 + \sqrt{30} - 1} |\partial_s u|^2\, ds \tag{4.35}$$

for a constant $\tilde{C}_1 = \tilde{C}_1(R, r) > 0$.

Define $2\mathfrak{f}_1 := |\partial_s u|^2 + |\overline{\nabla}_s \partial_s u|^2$ and $2\mathfrak{g}_1 := |\overline{\nabla}_s \partial_s u|^2 + |\overline{\nabla}_s \overline{\nabla}_s \partial_s u|^2$. We prove estimate (4.31) and its successor. By Young's inequality we obtain

$$\begin{aligned}&\mathfrak{f}_1'' - \mathfrak{f}_1' \\ &= 2\mathfrak{g}_1 + \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \partial_s u - \overline{\nabla}_s \partial_s u \right\rangle + \left\langle \overline{\nabla}_s \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla}_s \partial_s u - \overline{\nabla}_s \overline{\nabla}_s \partial_s u \right\rangle \\ &\geq 2\mathfrak{g}_1 + \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \partial_s u \right\rangle + \left\langle \overline{\nabla}_s \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla}_s \partial_s u \right\rangle - \mathfrak{f}_1 - \mathfrak{g}_1 .\end{aligned} \tag{4.36}$$

[2] To get the key estimate (4.34) it is crucial, and in sharp contrast to [SW06, Sec. 6], not to use $\partial_s u$ together with τ' in the function $\mathfrak{f}_0$.

To estimate the first inner product use $\underline{u3}$ and $\hat{\tau}2$ in (2.8) to get

$$\left\langle \partial_s u, \underline{\overline{\nabla}_s\overline{\nabla}_s\partial_s u}\right\rangle = \left\langle \partial_s u, -\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}F - \tau\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}H - 2\tau'\overline{\nabla}_s\overline{\nabla}H\right\rangle + \varepsilon^2\tau''^2.$$

For inner product two in (4.36) use $\underline{u4}$ in (2.8) and $\underline{\underline{\hat{\tau}3}}$ in (2.8) to get

$$\begin{aligned}
&\left\langle \overline{\nabla}_s\partial_s u, \underline{\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\partial_s u}\right\rangle \\
&= -\left\langle \overline{\nabla}_s\partial_s u, \overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}F + \tau\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}H + 3\tau'\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}H + 3\tau''\overline{\nabla}_s\overline{\nabla}H\right\rangle \\
&\quad \underline{\underline{-\left\langle \overline{\nabla}_s\partial_s u, \overline{\nabla}H\right\rangle\tau'''}} \\
&= -\left\langle \overline{\nabla}_s\partial_s u, \overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}F + \tau\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}H + 3\tau'\overline{\nabla}_s\overline{\nabla}_s\overline{\nabla}H + 3\tau''\overline{\nabla}_s\overline{\nabla}H\right\rangle \\
&\quad + \varepsilon^2\tau'''^2 + \left\langle \overline{\nabla}_s\overline{\nabla}H, \partial_s u\right\rangle\tau'''.
\end{aligned}$$

Theorem 3.1 uses the finite energy hypothesis and provides uniform L^∞ bounds for τ, τ', τ'', τ''', $\partial_s u$, $\overline{\nabla}_s\partial_s u$ and tells that u takes values in a compact $K \subset M$. By Young's inequality and (4.36) we get the inequality

$$\mathfrak{f}_1'' - \mathfrak{f}_1' \ge \mathfrak{g}_1 + \varepsilon^2\tau''^2 + \varepsilon^2\tau'''^2 - \mu_2\mathfrak{f}_1 \ge -\mu_2\mathfrak{f}_1 \tag{4.37}$$

pointwise at $s \in \mathbb{R}$ and for a suitable constant $\mu_2 > 0$. Now proceed as for $\mathfrak{f}$ and $\mathfrak{g}$ in (4.34) to obtain

$$|\overline{\nabla}_s\partial_s u(s)| \le C_2\left(\|\partial_s u\|_{L^2([s-\frac{1}{2},s+\frac{1}{2}])} + \|\overline{\nabla}_s\partial_s u\|_{L^2([s-\frac{1}{2},s+\frac{1}{2}])}\right)$$

and

$$\begin{aligned}
\|\overline{\nabla}_s\overline{\nabla}_s\partial_s u_s\|_{L^2[s-\frac{1}{4},s+\frac{1}{4}]} &+ \varepsilon\|\tau'''\|_{L^2[s-\frac{1}{4},s+\frac{1}{4}]} \\
&\le C_3\left(\|\partial_s u\|_{L^2[s-30,s+30]} + \|\overline{\nabla}_s\partial_s u\|_{L^2[s-30,s+30]}\right)
\end{aligned}$$

for every $s \in \mathbb{R}$. But $\|\overline{\nabla}_s\partial_s u\|_{L^2[s-\frac{1}{4},s+\frac{1}{4}]} \le C_4\|\partial_s u\|_{L^2[s-30,s+30]}$, by (4.30), and therefore $\|\overline{\nabla}_s\partial_s u\|_{L^2[s-30,s+30]} \le C_5\|\partial_s u\|_{L^2[s-60,s+60]}$, by additivity of the integral. This proves Theorem 4.1. □

Proof of Corollary 4.2. Pick $\varepsilon \in [\varepsilon_\kappa, 1]$. Define $2\mathfrak{f}_2 := |\partial_s u|^2 + \varepsilon^2\tau'^2$ and $2\mathfrak{g}_2 := |\overline{\nabla}_s\partial_s u|^2 + \varepsilon^2\tau''^2$. Straightforward calculation (step 1) and using $\underline{u3}$,

$\underline{\hat{\tau}3}$ in (2.8) (step 2) gives

$$
\begin{aligned}
\mathfrak{f}_2'' - \mathfrak{f}_2' &= 2\mathfrak{g}_2 + \left\langle \partial_s u, \underline{\overline{\nabla}_s \overline{\nabla}_s \partial_s u} - \overline{\nabla}_s \partial_s u \right\rangle + \tau' \underline{\varepsilon^2 \tau'''} - \varepsilon^2 \tau' \tau'' \\
&\geq \mathfrak{g}_2 - \mathfrak{f}_2 - \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} F + \tau \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} H \right\rangle - 2\tau' \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla} H \right\rangle \\
&\quad + \varepsilon^2 \tau''^2 - \tau' \left\langle \overline{\nabla}_s \overline{\nabla} H, \partial_s u \right\rangle - \tau' \left\langle \overline{\nabla} H, \overline{\nabla}_s \partial_s u \right\rangle - \varepsilon \tau' \varepsilon \tau'' \\
&\geq \mathfrak{g}_2 + \tfrac{1}{2} \varepsilon^2 \tau''^2 - \tfrac{1}{2} |\overline{\nabla}_s \partial_s u|^2 - \mu_1 \mathfrak{f}_2 - \tfrac{1}{2} \|\overline{\nabla} H\|^2_{L^\infty(K)} {\tau'}^2 \tfrac{\varepsilon^2}{\varepsilon_\kappa^2} \\
&\geq \varepsilon^2 \tau''^2 - \mu \mathfrak{f}_2 .
\end{aligned}
\tag{4.38}
$$

In step 3 we used the a priori bounds for $\tau, \tau', \partial_s u$ from Theorem 3.1 and the fact that u takes values in a compact subset K of M, so the gradients of H and F are uniformly L^∞ bounded. All bounds go into a suitable constant $\mu_1 > 0$. We multiplied a negative term with $\varepsilon^2 \varepsilon_\kappa^2 \leq 1$. Pick $s_0 \in \mathbb{R}$ and apply Lemma 4.3 with $\delta = \varepsilon$ and $r = \frac{1}{3}$ to the function $w(s) := \mathfrak{f}_2(s_0 + s)$ to get

$$
\tfrac{\varepsilon^2}{2} \tau'(s_0)^2 \leq w(0) \leq \frac{4de^{\frac{\mu_1}{9}}}{9} \int_{-\frac{1}{9}-\frac{\varepsilon}{3}}^{\frac{\varepsilon}{3}} \mathfrak{f}_2(s_0 + s)\, ds \leq de^{\frac{\mu_1}{9}} E^{\varepsilon}_{[s_0 - \frac{1}{2}, s_0 + \frac{1}{2}]} .
$$

Divide by ε^2 and use that $\frac{1}{\varepsilon^2} \leq \frac{1}{\varepsilon_\kappa^2}$. □

5. Uniform exponential decay

To prove the global surjectivity Theorem C, in (7.70), we need to verify for approximate base solutions $\mathfrak{q} \colon \mathbb{R} \to \Sigma$ – associated to ε-solutions $(u, \tau) \colon \mathbb{R} \to M \times \mathbb{R}$ via suitable projection to Σ – the decay assumption (A.1) that appears in the base implicit function theorem. It is for this purpose that we now establish exponential decay (5.40) for $\partial_s u$ and τ', uniformly in $\varepsilon \in (0, 1]$.

Theorem 5.1 (Exponential decay – $\partial_s u$). *Let f be Morse and $x^{\mp} \in$* Crit$f$. *Then there are constants $\delta, c, \tilde{c}, \rho > 0$ such that the following is true. Let $\varepsilon \in (0, 1]$ and $(u, \tau) \in \mathcal{M}^{\varepsilon}_{x^-, x^+}$. If $T_0 > 0$ has δ-small asymptotic energy*

$$
E^{\varepsilon}_{\mathbb{R} \setminus [-T_0, T_0]}(u, \tau) < \delta, \tag{5.39}
$$

then there is pointwise and slicewise exponential decay in the sense that

$$
\begin{aligned}
|\partial_s u(s)| &\leq c e^{-\rho(|s| - T_0 - 1)}, \\
\|\partial_s u\|^2_{L^2(\mathbb{R} \setminus [-T, T])} &\leq c e^{-2\rho(|T| - T_0)} \|\partial_s u\|^2_{L^2(\mathbb{R} \setminus [-T_0, T_0])},
\end{aligned}
\tag{5.40}
$$

whenever $|s|, |T| \geq T_0 + 1$.

L^2 norms are uniformly bounded $\|\partial_s u\|^2 \le E^\varepsilon(u,\tau) = f(x^-) - f(x^+) =: c^*$.

Corollary 5.2 (Exponential decay). *Under the assumptions of Theorem 5.1 it holds that*

$$|\overline{\nabla}_s \partial_s u(s)|^2 \le d e^{-2\rho|s|} \|\partial_s u\|^2_{L^2(\mathbb{R}\setminus[-T_0,T_0])} \tag{5.41}$$

for every $|s| \ge T_0 + 31$ *and*

$$\tau'(s)^2 \le \begin{cases} d e^{-2\rho|s|} \|\partial_s u\|^2_{L^2(\mathbb{R}\setminus[-T_0,T_0])}, & \varepsilon \in (0, \varepsilon_\kappa], \\ \tilde{c} e^{-2\rho(|s|-T_0-1)}, & \varepsilon \in [\varepsilon_\kappa, 1], \end{cases} \tag{5.42}$$

for every $|s| \ge T_0 + 2$. *Here* $d := cCe^{60+2T_0}$ *and* $C > 0$ *is from Theorem 4.1.*

Proof. Given Theorem 5.1 and a real $|s| \ge T_0 + 31$, combine estimate (4.31) for $|\overline{\nabla}_s \partial_s u(s)|^2$ with the L^2 norm exponential decay estimate (5.40) for $T := s - 30$ to get (5.41). Given (4.32), the same argument proves the estimate for $\tau'(s)^2$ along the small parameter interval $(0, \varepsilon_\kappa]$. The proof for the large parameter interval $[\varepsilon_\kappa, 1]$ will be given at the end of the section. □

Tools: Stability and Critical point detection. The proof of Theorem 5.1 relies on the following two lemmas. Note that they are about points and vectors, not maps and vector fields, despite the same notation.

Lemma 5.3 (Critical point detection). *Let f be Morse and $\Sigma \subset K \subset M$ compact. Then, for any $\delta_0 > 0$, there exists $\delta_1 > 0$ with the following significance. Let $(u,\tau) \in K \times \mathbb{R}$ be an almost critical point of F_H, namely*

$$|\overline{\nabla} F(u) + \tau \overline{\nabla} H(u)| + |H(u)| < \delta_1.$$

Then there exists $x_0 \in \operatorname{Crit} f$ and a δ_0-short difference vector towards $i(x_0) = (x_0, \chi(x_0))$, more precisely, there is a pair $(X_0, \ell_0) \in T_u M \times \mathbb{R}$ such that

$$u = \operatorname{Exp}_{x_0} X_0, \qquad \tau = \chi(x_0) + \ell_0, \qquad |X_0| + |\ell_0| \le \delta_0.$$

Proof. Suppose by contradiction that the assertion is wrong. Then there is a constant $\delta_0 > 0$ and sequences $\delta_{1,\nu} \to 0$ and $(u_\nu, \tau_\nu) \in M \times \mathbb{R}$ satisfying

$$|\overline{\nabla} F(u_\nu) + \tau_\nu \overline{\nabla} H(u_\nu)| + |H(u_\nu)| < \delta_{1,\nu} \longrightarrow 0, \text{ as } \nu \to \infty,$$

but not the conclusion of the lemma for the given constant δ_0. By compactness of K there is a subsequence of u_ν, still denoted by u_ν, such that

$u_\nu \to q \in M$, as $\nu \to \infty$. But $H(q) = 0$, by the second summand in the limit, thus $q \in \Sigma$. By the first summand in the limit, using that $u_\nu \to q$ and $\overline{\nabla} H(q) \neq 0$, the sequence τ_ν has a unique limit $\tau \in \mathbb{R}$. Since the limit of the first summand is zero and by definition (1.1) of χ we have firstly that $\tau = \chi(q)$, thus $\ell_0 = 0$, and secondly that $0 = \overline{\nabla} F(q) + \chi(q)\overline{\nabla} H(q) = \nabla f(q)$; here identity two is (1.4). So $x_0 := q$ is a critical point of f. For each ν the pair (X_ν, ℓ_ν) is determined by

$$\mathrm{Exp}_q X_\nu = u_\nu \to q, \qquad \chi(q) + \ell_\nu = \tau_\nu \to \chi(q).$$

Thus $X_\nu \to 0$ and $\ell_\nu \to 0$ in contradiction to $|X_\nu| + |\ell_\nu| > \delta_0$ for every ν. □

Lemma 5.4 (Stability). *Let f be Morse. Then there are positive constants δ_0 and c such that the following is true. Given a base critical point $x_0 \in \mathrm{Crit}\, f$ and a point $(u, \tau) \in M \times \mathbb{R}$ δ_0-close to $i(x_0) = (x_0, \chi(x_0))$ in the sense that*

$$u = \mathrm{Exp}_{x_0} X_0, \qquad \tau = \chi(q) + \ell_0, \qquad |X_0| + |\ell_0| \leq \delta_0,$$

then

$$|X| + |\ell| \leq c\big(|\overline{\nabla}_X \overline{\nabla} F(u) + \tau \overline{\nabla}_X \overline{\nabla} H(u) + \ell \overline{\nabla} H(u)| + |dH(u)X|\big)$$

for all $(X, \ell) \in T_u M \times \mathbb{R}$.

Proof. The covariant Hessian operator $A^1_{u,\tau}$ for $\varepsilon = 1$[3] acts on $T_u M \times \mathbb{R}$ by

$$A^1_{u,\tau} \begin{pmatrix} X \\ \ell \end{pmatrix} := \begin{pmatrix} \overline{\nabla}_X \overline{\nabla} F|_u + \tau \overline{\nabla}_X \overline{\nabla} H|_u + \ell \overline{\nabla} H|_u \\ dH|_u X \end{pmatrix}.$$

Pick $\delta_0 < \iota(T_\Sigma M, G)$. That $A^1_{u,\tau}$ is symmetric with respect to the metric $h^1 = G \oplus 1$ on $M \times \mathbb{R}$ boils down to the identity shown in part I [FW, Sec. 4.2.4] after the formula for the adjoint $(D^\varepsilon_{u,\tau})^*$. It is an instructive exercise to check that injectivity of $A^0_{x_0}$ (true since f is Morse), implies injectivity of $A^1_{x_0,\chi(x_0)}$, hence bijectivity (by symmetry). Thus $A^1_{x_0,\chi(x_0)}$ admits an inverse which, in finite dimension, is automatically bounded, say by a constant $C(x_0)$. But bijectivity is preserved under small perturbations (with respect to the operator norm). Hence the result for $A^1_{\mathrm{Exp}_{x_0} X, \chi(x_0)+\ell}$, say with constant $c = 2C(x_0)$, follows from continuous dependence on $|X|$ and $|\ell|$ and by choosing

[3] Why not use $A^\varepsilon_{u,\tau}$ and h^ε? As that way we would get an ε on the left hand side of (5.44).

δ_0 small enough. Since Σ is compact and f is Morse, the set $\operatorname{Crit} f$ is finite, so we may choose the same constants δ_0 and c for all $x_0 \in \operatorname{Crit} f$. □

The next lemma is standard and included for convenience of the reader, for a proof see e.g. [Web10, Le. 3.13].

Lemma 5.5. *Suppose $\mathfrak{f}\colon \mathbb{R} \to [0,\infty)$ is a bounded C^2 function and $\rho \geq 0$ and $T_0 > 0$ are constants. If $\mathfrak{f}''(s) \geq \rho^2 \mathfrak{f}(s)$ whenever $|s| \geq T_0$, then*

$$\mathfrak{f}(s) \leq e^{-\rho(s-T_0)}\mathfrak{f}(T_0),\ \forall s \geq T_0, \qquad \mathfrak{f}(s) \leq e^{\rho(s-T_0)}\mathfrak{f}(-T_0),\ \forall s \leq -T_0.$$

Proof of uniform exponential decay.

Proof of Theorem 5.1 (Exponential decay – $\partial_s u$). Let $x^{\mp} \in \operatorname{Crit} f$, $\varepsilon \in (0,1]$, and $(u,\tau) \in \mathcal{M}^{\varepsilon}_{x^-,x^+}$. Let K be the compact set provided by Theorem 3.1.

Let $C > 0$ be the constant of the gradient bound Theorem 4.1. Let $\delta_0 > 0$ and $c > 0$ be the constants of the stability Lemma 5.4 and δ_1 the constant associated to δ_0 by the critical point detection Lemma 5.3. Enlarge c to $c > 1$. Shrink $\delta > 0$ such that $2\sqrt{C\delta} \leq \delta_1$. In the following we shrink δ further.

As usual we will often write u_s for $u(s)$. For the vector norm in $T_{u_s}M$ and the one in $\mathbb{R}$ we use the same symbol $|\cdot|$.

Choose $T_0 > 0$ sufficiently small such that hypothesis (5.39) – the asymptotic energy is δ-small – is satisfied. (Such T_0 exists by finiteness of the total energy.) Then, by the gradient bounds (4.29,4.32) and since $\|\partial_s u\|^2 \leq E^{\varepsilon}(u,\tau)$, we have

$$|\partial_s u_s| + |\tau_s'| \leq 2\sqrt{C E^{\varepsilon}_{[s-30,s+30]}(u,\tau)} \overset{(5.39)}{<} 2\sqrt{C\delta} \leq \delta_1 \tag{5.43}$$

for $s \in \mathbb{R}$. Hence, whenever $|s| \geq T_0 + 1$, Lemma 5.3 for $(X_0,\ell_0) := (\partial_s u_s, \tau_s')$ detects a critical point $x^{(s)} \in \operatorname{Crit} f$ whose canonical embedding $(x^{(s)}, \chi(x^{(s)}))$ is of distance less than δ_0 to the time-s trajectory point (u_s,τ_s). Thus the stability Lemma 5.4 applies to the pair (u_s,τ_s) and the vector field $(X_0,\ell_0) := (\partial_s u_s, \tau_s')$. By (2.8), there are the identities

$$\overline{\nabla}_s \overline{\nabla} F|_{u_s} + \tau_s \overline{\nabla}_s \overline{\nabla} H|_{u_s} + \tau_s' \overline{\nabla} H|_{u_s} = -\overline{\nabla}_s \partial_s u_s, \qquad dH|_{u_s}\partial_s u_s = -\varepsilon^2 \tau_s'',$$

hence we obtain from the stability Lemma 5.4 the estimate

$$|\partial_s u_s|^2 + {\tau_s'}^2 \leq c\left(|\overline{\nabla}_s \partial_s u_s|^2 + \varepsilon^4 {\tau_s''}^2\right) \tag{5.44}$$

for every $|s| \geq T_0 + 1$.

Consider functions $\mathfrak{f}_0 := \frac{1}{2}|\partial_s u|^2$ and $\mathfrak{g}_0 := \frac{1}{2}|\overline{\nabla}_s \partial_s u|^2$. Next we show for $\mathfrak{f}_0$ an inequality of the form $\mathfrak{f}_0'' \geq \rho^2 \mathfrak{f}_0$, so Lemma 5.5 asserts exponential decay. Use the identities $\underline{u3}$ and $\underline{\underline{\hat{\tau}2}}$ in (2.8) to get (we abbreviate $H = H(u)$)

$$
\begin{aligned}
\mathfrak{f}_0'' &= 2\mathfrak{g}_0 + \left\langle \partial_s u, \underline{\overline{\nabla}_s \overline{\nabla}_s \partial_s u} \right\rangle \\
&= 2\mathfrak{g}_0 - \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} F + \tau \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} H \right\rangle \\
&\quad - 2\tau' \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla} H \right\rangle - \tau'' \underline{\underline{\left\langle \partial_s u, \overline{\nabla} H \right\rangle}} \\
&= |\overline{\nabla}_s \partial_s u|^2 + \varepsilon^2 \tau''^2 - \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} F + \tau \overline{\nabla}_s \overline{\nabla}_s \overline{\nabla} H \right\rangle \\
&\quad - 2\tau' \left\langle \partial_s u, \overline{\nabla}_s \overline{\nabla} H \right\rangle \\
&\geq |\overline{\nabla}_s \partial_s u|^2 + \varepsilon^2 \tau''^2 - \tilde{c} \left(|\partial_s u| + |\tau'| \right) |\partial_s u|^2 \\
&\geq \frac{1}{c} \left(|\partial_s u|^2 + {\tau'}^2 \right) - \frac{1}{2c} |\partial_s u|^2 \\
&\geq \frac{1}{c} \mathfrak{f}_0 = \rho^2 \mathfrak{f}_0, \quad \rho = \frac{1}{\sqrt{c}},
\end{aligned}
\tag{5.45}
$$

for every $|s| \geq T_0 + 1$; here T_0 is brought in by (5.44) in inequality three. Inequality one is by term by term inspection and Young's inequality, the constant $\tilde{c} > 0$ depends on the $C^3(K)$ norms of F and H and the uniform L^∞ bound of τ from Theorem 3.1. Inequality two uses, firstly, the stability estimate (5.44)[4] and, secondly, we chose $\delta > 0$ in (5.43) so small that

$$
\tilde{c} \left(|\partial_s u| + |\tau'| \right) \leq \tilde{c} 2 \sqrt{C\delta} \delta_1 \leq 1/2c < 1/2.
$$

By Theorem 3.1 the function $\mathfrak{f}_0$ is bounded by $c_1 > 0$, uniformly in $\varepsilon \in (0, 1]$. By Lemma 5.5 we have

$$
\mathfrak{f}_0(s) \leq e^{-\rho(|s| - T_0 - 1)} \left(\mathfrak{f}_0(T_0 + 1) + \mathfrak{f}_0(-T_0 - 1) \right),
$$

so

$$
|\partial_s u(s)| \leq 2\sqrt{c_1} e^{-\frac{|s| - (T_0 + 1)}{2\sqrt{c}}}
$$

whenever $|s| \geq T_0 + 1$ and $\varepsilon \in (0, 1]$. This proves assertion one of Theorem 5.1 where c and ρ do have new meanings though.

[4] here the term $\varepsilon^2 \tau''^2$ is by a factor ε^2 better than required.

As $2\mathfrak{f}_0(T_0+1) \le C\|\partial_s u\|^2_{L^2([T_0,T_0+2])}$ by (4.29), analogous for $2\mathfrak{f}_0(-T_0-1)$, together we conclude that for every $|s| \ge T_0+1$ holds the estimate

$$|\partial_s u(s)|^2 \le e^{-\rho(|s|-T_0-1)} C\|\partial_s u\|^2_{L^2(I_{T_0})}, \quad I_{T_0} := [-T_0-2, -T_0] \cup [T_0, T_0+2].$$

Fix $T \ge T_0+1$, integrate the inequality over $s \in (-\infty,-T] \cup [T,\infty)$ to get

$$\|\partial_s u\|^2_{L^2(\mathbb{R}\setminus[-T,T])} \le C\|\partial_s u\|^2_{L^2(I_{T_0})} \left(\int_{-\infty}^{-T} + \int_T^\infty\right) e^{-\rho(|s|-T_0-1)}\, ds.$$

Carrying out the integral to get assertion two in (5.40). This proves the exponential decay Theorem 5.1. □

To prove the yet missing exponential decay of $|\tau'(s)|$ along $[\varepsilon_\kappa, 1]$, see (5.42), one is inclined, given the successful strategy to prove (4.32), to repeat calculation (5.45) with $\mathfrak{f}_0$ substituted by the ε-energy density $\mathfrak{f}_2$. The new calculation starts exactly like (4.38) where one just needs to remove term $-\mathfrak{f}_2'$. Everything is fine except for the term $-\tau' \left\langle \overline{\nabla} H, \overline{\nabla}_s \partial_s u \right\rangle$ which neither is cubic, nor can it be incorporated via Young's inequality simultaneously into both positive terms $|\overline{\nabla}_s \partial_s u|^2$ and $\frac{1}{c}{\tau'}^2$ in (5.45). Interestingly enough, now it is the second ε-equation $\tau' = \varepsilon^{-2} H(u)$ which helps. The nasty factor ε^{-2} is tame along $[\varepsilon_\kappa, 1]$.

Proof of Corollary 5.2 for $[\varepsilon_\kappa, 1]$. Given $T_0 > 0$ such that asymptotic energy smallness (5.39) holds, fix $T_1 \ge T_0$ sufficiently large so we can write $u(s)$ as

$$u(s) = \mathrm{Exp}_{x^\mp} X^\mp(s)$$

for any $|s| \ge T_0$. We focus on X^-, the case X^+ is analogous. By $\tau 1$ we have

$$\begin{aligned} |\tau'(s)| &= \tfrac{1}{\varepsilon^2} |H \circ u(s) - H(x^-)| \\ &\le \tfrac{1}{\varepsilon_\kappa^2} |H \circ \mathrm{Exp}_{x^-} X^-(s) - H(x^-)| \\ &\stackrel{3}{=} |dH|_{\mathrm{Exp}_{x^-} tX^-(s)} X^-(s)| \\ &\le \|\overline{\nabla} H\|_{L^\infty(K)} \cdot |X^-(s)|. \end{aligned} \tag{5.46}$$

Step 3 uses the mean value theorem for a $t \in [0,1]$. The length of $X^-(s)$ is the Riemannian distance between $u(s)$ and x^-, see footnote to (6.52), in symbols

$$\begin{aligned}|X^-(s)| &= \mathrm{dist}_G\left(u(s), x^-\right)\\ &\le \mathrm{length}(u|_{(-\infty,s]})\\ &= \int_{-\infty}^s |\partial_s u(\sigma)|\, d\sigma\\ &\overset{4}{\le} ce^{\rho(T_0+1)}\int_{-\infty}^s e^{\rho\sigma}\, d\sigma\\ &= \tfrac{ce^{\rho(T_0+1)}}{\rho} e^{-\rho|s|}\end{aligned}$$

for every $s \le -T_0$ and where in step 4 we use (5.40). □

6. Local surjectivity of $\mathcal{T}^\varepsilon$ – time shift

Theorem 6.1 (Time shift). *Let (f,g) be Morse-Smale. Then there exist $c>0$ and $\Delta, \varepsilon_0 \in (0,1]$ such that the following holds. If $x^\mp \in \mathrm{Crit} f$ is a pair of index difference one connected by a base trajectory and an ε-trajectory*

$$q \in \mathcal{M}^0_{x^-,x^+}, \qquad (u^\varepsilon, \tau^\varepsilon) \in \mathcal{M}^\varepsilon_{x^-,x^+} \text{ with } \varepsilon \in (0, \varepsilon_0],$$

whose difference, firstly, is measured by a vector field $(X^\varepsilon, \ell^\varepsilon) \in C^\infty(\mathbb{R}, q^*TM \oplus \mathbb{R})$ determined by the identities

$$u^\varepsilon = \mathrm{Exp}_q X^\varepsilon, \qquad \tau^\varepsilon = \chi(q) + \ell^\varepsilon,$$

and, secondly, the difference is sufficiently L^∞ and L^2 small in the sense[5]

$$\|X^\varepsilon\|_\infty \le \Delta\varepsilon^{1/2}, \qquad \|X^\varepsilon\| \le \Delta\varepsilon^{1/2}, \tag{6.47}$$

then $\|\ell^\varepsilon\|_\infty + \|\ell^\varepsilon\| \le c$. Furthermore, there is a time-shift $\sigma \in \mathbb{R}$ such that

$$(u^\varepsilon, \tau^\varepsilon) = \mathcal{T}^\varepsilon\left(q(\sigma + \cdot)\right), \qquad |\sigma| < c\left(\|X^\varepsilon\| + \varepsilon^2\right).$$

Proof. Let $\kappa > 0$ be a constant as in Hypothesis 2.1. Let $c_\kappa > 0$ and $\varepsilon_\kappa \in (0,1]$ be the constants in Theorem 3.1. Fix $\delta_0 \in (0,1]$, $\varepsilon_0 \in (0, \varepsilon_\kappa)$, and $c > 1$ such that for all $x_0, x_1 \in \mathrm{Crit}\, f$ of index difference one the map $\mathcal{T}^\varepsilon \colon \mathcal{M}^0_{x_0,x_1} \to \mathcal{M}^\varepsilon_{x_0,x_1}$ is injective whenever $\varepsilon \in (0, \varepsilon_0]$, see [FW, Sec. 6].

[5] L^∞ condition enters (6.52) and Step 3 of proof, L^2 condition enters (6.60) and Step 3. Both conditions enter **sharply** in (6.62) in Step 5 to make uniqueness theorem applicable.

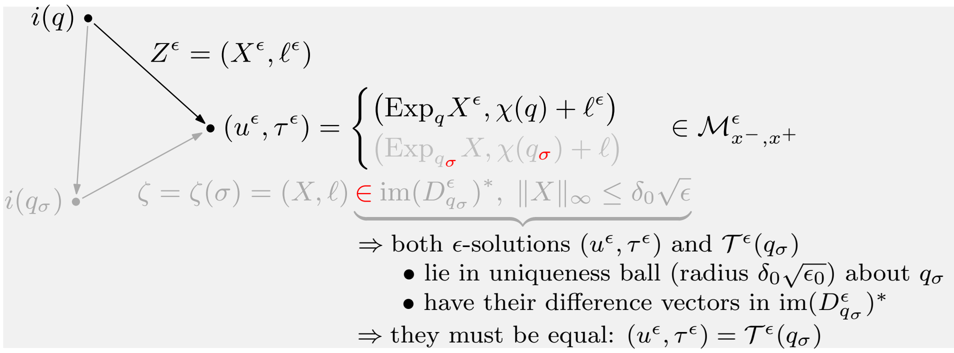

Figure 1: Detect time shift σ of q such that $\zeta(\sigma) \in \mathrm{im}\,(D^\varepsilon_{q_\sigma})^*$, $\|X\|_\infty \le \delta_0\sqrt{\varepsilon}$

It suffices to prove the result for a fixed pair $x^\mp \in \mathrm{Crit} f$ of index difference one and a fixed base trajectory $q \in \mathcal{M}^0_{x^-,x^+}$. The reason is that the assumptions and conclusions of the theorem are invariant under simultaneous time shift of q and $(u^\varepsilon, \tau^\varepsilon)$ and, up to time shift, there are only finitely many index one base trajectories q. Let $(u^\varepsilon, \tau^\varepsilon) \in \mathcal{M}^\varepsilon_{x^-,x^+}$ with $\varepsilon \in (0, \varepsilon_0]$. The constant defined by

$$c^* := f(x^-) - f(x^+) = \|\partial_s q\|^2 > 0 \overset{(1.7)}{=} \|\partial_s u^\varepsilon\|^2 + \varepsilon^2 \|(\tau^\varepsilon)'\|^2 \tag{6.48}$$

is positive as $\partial_s q \not\equiv 0$ by index difference 1. For $\sigma \in \mathbb{R}$ consider the time shift

$$q_\sigma(s) := q(s + \sigma), \qquad (q_\sigma)' := \tfrac{d}{ds} q_\sigma = \tfrac{d}{d\sigma} q_\sigma, \tag{6.49}$$

and the canonical embedding $i(q_\sigma) = (q_\sigma, \chi(q_\sigma))$, see (1.2). The difference between $(u^\varepsilon, \tau^\varepsilon)$ and $i(q_\sigma)$ is the pair $\zeta = \zeta(\sigma) = (X, \ell)$ determined by

$$u^\varepsilon = \mathrm{Exp}_{q_\sigma} X, \qquad \tau^\varepsilon = \chi(q_\sigma) + \ell. \tag{6.50}$$

Writing (X, ℓ) can be misleading, but $(X^{\varepsilon,\sigma}, \ell^{\varepsilon,\sigma})$ is unreadable. Note that

$$\zeta(0) = (X^\varepsilon, \ell^\varepsilon). \tag{6.51}$$

The pair $\zeta = (X, \ell)$, see Figure 1, is well defined whenever the distance[6]

$$\begin{aligned} \|X\|_\infty = \operatorname{dist}_\infty(q_\sigma, u^\varepsilon) &\le \operatorname{dist}_\infty(q_\sigma, q) + \operatorname{dist}_\infty(q, u^\varepsilon) \\ &\le |\sigma| \cdot \|\partial_s q\|_\infty + \|X^\varepsilon\|_\infty < \iota(T_\Sigma M) \end{aligned} \tag{6.52}$$

is smaller than the (by compactness of Σ positive) injectivity radius of the Riemannian vector bundle $(T_\Sigma M, G)$. To achieve such smallness

$$\text{fix } \sigma_0 \in (0,1]\colon\ \sigma_0 \|\partial_s q\|_\infty < \tfrac{\iota(T_\Sigma M)}{2}, \qquad \text{fix } \Delta \in (0, \delta_0]\colon\ \Delta\sqrt{\varepsilon_0} < \tfrac{\iota(T_\Sigma M)}{2}.$$

By the a priori bounds for $\partial_s q$ in (A.3) and for τ^ε in Theorem 3.1, by compactness of Σ, and by exponential decay (A.19), there is a constant $c_0 > 0$ such that

$$\|\partial_s q\|_\infty + \|\tau^\varepsilon\|_\infty + \|\chi\|_{L^\infty(\Sigma)} \le c_0 \tag{6.53}$$

and such that (used in (6.57))

$$\|\overline{\nabla}_s \partial_s q\| \le c_0. \tag{6.54}$$

We prove that for sufficiently small $\varepsilon > 0$ there is a time shift $\sigma \in [-\sigma_0, \sigma_0]$ such that our difference $\zeta = (X, \ell)$ satisfies the uniqueness hypotheses

$$\zeta(\sigma) \in \operatorname{im}(D^\varepsilon_{q_\sigma})^* \subset L^2, \qquad \|X(\sigma)\|_\infty \le \delta_0\sqrt{\varepsilon}, \tag{6.55}$$

of the implicit function theorem I [FW, Sec. 6, IFT I – Uniqueness]. Consequently there are two ε-solutions within the uniqueness range about q_σ, namely, not only $(u^\varepsilon, \tau^\varepsilon)$, by (6.50) and (6.55), but also $\mathcal{T}^\varepsilon(q_\sigma)$ as provided by the existence part of IFT I; see Figure 1. By uniqueness these are equal.

The proof of (6.55) is in five steps. Step 1 defines a function $\theta^\varepsilon(\sigma)$ which is zero iff $\zeta(\sigma) \in \operatorname{im}(D^\varepsilon_{q_\sigma})^*$. To find the zero one bounds $|\theta^\varepsilon(0)|$ from above (Step 2) and $\frac{d}{d\sigma}\theta^\varepsilon$ from below (Step 4); see Figure 2. Step 3 prepares estimates utilized in Step 4. The heart of the proof is Step 2 which hinges on the ε-equations (1.3) and ε-uniform exponential decay as prepared in Section 5.

[6] Let $\operatorname{dist}_\infty(q_\sigma, q) := \sup_{s\in\mathbb{R}} \operatorname{dist}(q_\sigma(s), q(s))$. The Riemann point distance $\operatorname{dist}(q_\sigma(s), q(s))$ is the infimum over all lengths of smooth curves between the two points. The length of the curve $q|_{[s,s+\sigma]}$ is $\le \|\partial_s q\|_{L^\infty([s,s+\sigma])}|\sigma|$, namely, maximal speed times length of time interval.

Step 1. *Shrinking $\varepsilon_0 > 0$, if necessary, then for every $\varepsilon \in (0, \varepsilon_0]$ and every $\sigma \in [-\sigma_0, \sigma_0]$ the following function is well defined, namely*

$$\theta^\varepsilon(\sigma) := -\langle \mathcal{Z}^\varepsilon_\sigma, \zeta \rangle_{0,2,\varepsilon} := -\langle \Xi^\varepsilon_\sigma, X \rangle - \varepsilon^2 \langle \lambda^\varepsilon_\sigma, \ell \rangle ,$$

where the pair $\zeta = \zeta(\sigma) = (X, \ell)$ is given by (6.50) and where

$$\mathcal{Z}^\varepsilon := \begin{pmatrix} \Xi^\varepsilon \\ \lambda^\varepsilon \end{pmatrix} := \begin{pmatrix} \partial_s q \\ d\chi|_q \partial_s q \end{pmatrix} - \begin{pmatrix} X^* \\ \ell^* \end{pmatrix} = I_q(\partial_s q) - R^\varepsilon_q D^\varepsilon_q I_q(\partial_s q)$$

$$\zeta^* := \begin{pmatrix} X^* \\ \ell^* \end{pmatrix} := \underbrace{(D^\varepsilon_q)^* \left(D^\varepsilon_q (D^\varepsilon_q)^* \right)^{-1}}_{=:R^\varepsilon_q} D^\varepsilon_q \begin{pmatrix} \partial_s q \\ d\chi|_q \partial_s q \end{pmatrix} = R^\varepsilon_q D^\varepsilon_q I_q(\partial_s q)$$

with $\mathcal{Z}^\varepsilon_\sigma$ denoting time shift. Here $I_q := di(q)$ is the linearized canonical embedding (1.2). The linear operator $D^\varepsilon_q := D^\varepsilon_{q,\chi(q)}$, the formal adjoint $(D^\varepsilon_q)^*$ with respect to the $(0, 2, \varepsilon)$ inner product, and the right inverse R^ε_q are introduced in [FW]. Most importantly, a zero σ of θ^ε is characterized by

$$\theta^\varepsilon(\sigma) = 0 \quad \Leftrightarrow \quad \zeta \in \operatorname{im} (D^\varepsilon_{q_\sigma})^*.$$

Shrink $\varepsilon_0 > 0$, if necessary, so the operator D^ε_q is surjective, by the key estimate [FW, Sec. 5.3.3], and, by [FW, Sec. 4.2.4], it has Fredholm index, hence kernel dimension, equal to one and we also have $(\ker D^\varepsilon_q)^\perp = \operatorname{im} (D^\varepsilon_q)^*$. We get $D^\varepsilon_q \mathcal{Z}^\varepsilon = 0$ since R^ε_q is a right inverse of D^ε_q. Below we will show that $\mathcal{Z}^\varepsilon \neq 0$ for every $\varepsilon > 0$ sufficiently small, so $\mathcal{Z}^\varepsilon$ spans $\ker D^\varepsilon_q$.

These properties also hold for the shifted trajectory q_σ. The to q_σ associated unshifted vector $\mathcal{Z}^\varepsilon(q_\sigma) = \mathcal{Z}^\varepsilon_\sigma$ is the shifted vector associated to q. So

$$\theta^\varepsilon(\sigma) = 0 \quad \Leftrightarrow \quad (\mathbb{R} \cdot \mathcal{Z}^\varepsilon_\sigma)^\perp = (\ker D^\varepsilon_{q_\sigma})^\perp = \operatorname{im} (D^\varepsilon_{q_\sigma})^*.$$

Here $\perp$ means orthogonal complement for the $(0, 2, \varepsilon)$ inner product.

It remains to show that $\mathcal{Z}^\varepsilon = I_q(\partial_s q) - \zeta^* \neq 0$ for any $\varepsilon > 0$ small: To see this note that $\partial_s q \neq 0$ since $x^- \neq x^+$, by Morse index difference one. So

$$\|I_q \partial_s q\|^2_{0,2,\varepsilon} = \|\partial_s q\|^2 + \varepsilon^2 \|d\chi|_q \partial_s q\|^2 \geq \|\partial_s q\|^2 \stackrel{(6.48)}{=} c^* > 0$$

is bounded away from zero. On the other hand, for ζ^* we have the identity

$$\zeta^* := R^\varepsilon_q D^\varepsilon_q \begin{pmatrix} \partial_s q \\ d\chi|_q \partial_s q \end{pmatrix} = R^\varepsilon_q \begin{pmatrix} 0 \\ (d\chi|_q \partial_s q)' \end{pmatrix} \tag{6.56}$$

which, by the key estimate [FW, Sec. 5.3.3] with constant C, leads to

$$
\begin{aligned}
&\|dH|_q X^*\| + \|X^*\| + \varepsilon\|\ell^*\| + \varepsilon\|\overline{\nabla}_s X^*\| + \varepsilon^2\|(\ell^*)'\| \\
&\le \varepsilon C\left(\varepsilon\|(0,(d\chi|_q\partial_s q)')\|_{0,2,\varepsilon} + \|\pi_\varepsilon(0,(d\chi|_q\partial_s q)')\|\right) \\
&\le \varepsilon^2 C(1+\mu_\infty)\|d^2\chi_q(\partial_s q,\partial_s q) + d\chi|_q\overline{\nabla}_s\partial_s q\| \\
&\le c_1\varepsilon^2 \quad \text{with } c_1 = C(1+\mu_\infty)\left(\|\chi\|_{C^2(\Sigma)}\|\partial_s q\|_\infty\|\partial_s q\| + \mu_\infty\|\overline{\nabla}_s\partial_s q\|\right)
\end{aligned} \tag{6.57}
$$

where $\mu_\infty := \max\{1, \|\nabla\chi\|_{L^\infty(\Sigma)}\} \in [1,\infty)$ and c_1 is finite by (6.48), (6.53), and (6.54). The projection π_ε is defined in [FW, Sec. 5.1.1] and works with $\beta = 2$ and any $\alpha \in [1,2]$. Thus $\|\zeta^*\|_{0,2,\varepsilon}^2 := \|X^*\|^2 + \varepsilon^2\|\ell^*\|^2 \le c_1^2\varepsilon^4$ converges to zero as ε tends to zero. Consequently $\mathcal{Z}^\varepsilon = I_q\partial_s q - \zeta^* \neq 0$ for $\varepsilon > 0$ sufficiently small and this proves Step 1.

Step 2. *There is a positive constant c_2 such that*

$$
|\theta^\varepsilon(0)| \le c_2\left(\|X^\varepsilon\| + \varepsilon^2\right)
$$

whenever $\varepsilon \in (0,\varepsilon_0]$.

Let $\varepsilon \in (0,\varepsilon_0]$. We have $\theta^\varepsilon(0) = -\langle \Xi^\varepsilon, X^\varepsilon\rangle - \varepsilon^2\langle\lambda^\varepsilon,\ell^\varepsilon\rangle$ by (6.51). First we prove that there is a positive constant c_3 such that there are L^2 estimates

$$
\|\Xi^\varepsilon\| + \|\lambda^\varepsilon\| \le c_3. \tag{6.58}
$$

For the first summand of $\Xi^\varepsilon = \partial_s q - X^*$ and also of $\lambda^\varepsilon = d\chi|_q\partial_s q - \ell^*$ we use that $\|\partial_s q\|^2 = c^*$ is finite by (6.48). For the second summands X^* and ℓ^* this is (6.57). This proves (6.58), consequently $|\theta^\varepsilon(0)| \le c_3\left(\|X^\varepsilon\| + \varepsilon^2\|\ell^\varepsilon\|\right)$.

So it remains to find an L^2 bound for $\ell^\varepsilon = \tau^\varepsilon - \chi(q)$. While we got L^∞-bounds for τ^ε and $\chi|_\Sigma$, by Theorem 3.1 and compactness of Σ, these do not help for L^2 due to non-compactness of the domain $\mathbb{R}$. As $\varepsilon < \varepsilon_\kappa$, we have

$$
\ell^\varepsilon = \tau^\varepsilon - \chi(q) \overset{(2.10)}{=} -\frac{\langle\partial_s u^\varepsilon, \overline{\nabla} H(u^\varepsilon)\rangle}{|\overline{\nabla} H(u^\varepsilon)|^2} + \chi(\mathrm{Exp}_q X^\varepsilon) - \chi(q). \tag{6.59}
$$

To deal with the difference $|\chi(\mathrm{Exp}_q X^\varepsilon) - \chi(q)|$, pointwise at $s \in \mathbb{R}$, use the lemma in [FW, Sec. 6.1] bringing in a factor $|X^\varepsilon(s)|$ whose L^2 norm $\|X^\varepsilon\| \le \Delta\sqrt{\varepsilon}$ is controlled by hypothesis (6.47). Since $\|\partial_s u^\varepsilon\| \le c^*$, by (1.7), we get

$$
\|\ell^\varepsilon\| = \|\tau^\varepsilon - \chi(q)\| \le \tfrac{c^*}{c_\kappa} + c_5\Delta\sqrt{\varepsilon}. \tag{6.60}
$$

where we also used the lower gradient bound c_κ in (6.48). This proves Step 2.

Step 3 (Shift q_σ)**.** *There exists a positive constant c_6 such that the vector field $X = X(s;\sigma)$ and the function $\ell = \ell(s;\sigma)$, defined by (6.50), satisfy*

$$\begin{aligned} \|\ell\|_\infty &\le c_0 & \|X\|_\infty &\le c_0|\sigma| + \Delta\varepsilon^{\frac{1}{2}} < \iota_M \\ \|\overline{\nabla}_s X\| &\le c_6 & \|\overline{\nabla}_\sigma X + \partial_s q_\sigma\| &\le c_6\big(|\sigma| + \Delta\varepsilon^{\frac{1}{2}}\big) \\ \|\ell\| &\le c_6 & \|X\| &\le c_6|\sigma| + \Delta\varepsilon^{\frac{1}{2}} \end{aligned}$$

whenever $\varepsilon \in (0, \varepsilon_0]$ and $|\sigma| \le \sigma_0$.

The first estimate of Step 3 holds true because $\|\ell\|_\infty = \|\tau^\varepsilon - \chi(q_\sigma)\|_\infty \le c_0$ by (6.50) and (6.53). The second estimate was proved in (6.52).

To prove the third and fourth estimates differentiate the identity

$$\mathrm{Exp}_{q_\sigma} X = u^\varepsilon$$

with respect to s and σ using the maps E_i from Thm. 6.7 in [FW] to get

$$\begin{aligned} E_1(q_\sigma, X)\partial_s q_\sigma + E_2(q_\sigma, X)\overline{\nabla}_s X &= \partial_s u^\varepsilon \\ E_1(q_\sigma, X)\partial_s q_\sigma + E_2(q_\sigma, X)\overline{\nabla}_\sigma X &= 0. \end{aligned}$$

The inverse $E_2(q_\sigma, X)^{-1}$ exists for small $\|X\|_\infty$ since $E_i(q_\sigma, 0) = \mathbb{1}$ by Thm. 6.7. Apply E_2^{-1} to both identities. The first identity then tells that

$$\|\overline{\nabla}_s X\| = \|E_2^{-1}\left(\partial_s u^\varepsilon - E_1 \partial_s q_\sigma\right)\| \le c_7\left(\|\partial_s u^\varepsilon\| + \|\partial_s q\|\right) = c_7 2\sqrt{c^*}$$

where the last step is by (6.48). Resolve the second identity for $\overline{\nabla}_\sigma X$ to get

$$\|\overline{\nabla}_\sigma X + \partial_s q_\sigma\| = \|(E_2^{-1}E_1 - \mathbb{1})\partial_s q_\sigma\| \le c_8 \|\partial_s q\| \|X\|_\infty \le c_9(\Delta\varepsilon^{1/2} + c_6|\sigma|).$$

Inequality 1 uses [FW, Le. 6.8 i)], inequality 2 (6.48) and the earlier $\|X\|_\infty$ estimate. Note that $\|\overline{\nabla}_\sigma X\| \le c_9(\Delta\sqrt{\varepsilon} + c_6\sigma) + \|\partial_s q_\sigma\| \le c_9(1 + c_6) + \sqrt{c^*}$.

Before doing the fifth estimate we do the sixth. Let $X \not\equiv 0$, otherwise we are done. Given $s \in \mathbb{R}$, by the fundamental theorem of calculus we have

$$\begin{aligned} \|X(\sigma)\| = \int_0^\sigma \tfrac{d}{d\tilde\sigma}\|X(\tilde\sigma)\|\, d\tilde\sigma + \|X(0)\| &= \int_0^\sigma \tfrac{\langle X(\tilde\sigma), \nabla_{\tilde\sigma} X(\tilde\sigma)\rangle}{\|X(\tilde\sigma)\|}\, d\tilde\sigma + \|X^\varepsilon\| \\ &\le |\sigma| \max_{\tilde\sigma \in [0,\sigma]} \|\nabla_{\tilde\sigma} X(\tilde\sigma)\| + \Delta\varepsilon^{\frac{1}{2}} \end{aligned}$$

where the inequality is by Cauchy-Schwarz and hypothesis (6.47). Now use that $\|\nabla_{\tilde\sigma} X(\tilde\sigma)\| \le c_9(1 + c_6) + \sqrt{c^*}$, as we already showed.

To prove estimate five note that $\|\frac{d}{d\sigma}\ell\| = \|\frac{d}{d\sigma}(\tau^\varepsilon - \chi(q))\| = \|d\chi|_{q_\sigma}\partial_s q_\sigma\| \le \mu_\infty\sqrt{c^*}$. Integrate $\frac{d}{d\tilde\sigma}\|X(\tilde\sigma)\|$, just as we did to estimate $\|X(\sigma)\|$, to obtain

$$\|\ell(\sigma)\| \le |\sigma| \max_{\tilde\sigma\in[0,\sigma]} \|\tfrac{d}{d\tilde\sigma}\ell(\tilde\sigma)\| + \|\ell^\varepsilon\| \stackrel{(6.60)}{\le} |\sigma|\mu_\infty\sqrt{c^*} + \tfrac{c_0}{\tilde m_H} + c_5\Delta\sqrt{\varepsilon}.$$

Since $|\sigma|, \Delta, \varepsilon \le 1$ this proves estimate five. The proof of Step 3 is complete.

Step 4. *Shrinking σ_0 and ε_0, if necessary, we have*

$$\tfrac{d}{d\sigma}\theta^\varepsilon(\sigma) \ge \tfrac{c^*}{2}$$

for $\varepsilon \in (0, \varepsilon_0]$ and $|\sigma| \le \sigma_0$, where c^* is defined in (6.48).

We will investigate the two terms in the sum

$$\tfrac{d}{d\sigma}\theta^\varepsilon(\sigma) = -\tfrac{d}{d\sigma}\langle \Xi^\varepsilon_\sigma, X(\sigma)\rangle - \varepsilon^2 \tfrac{d}{d\sigma}\langle \lambda^\varepsilon_\sigma, \ell(\sigma)\rangle \tag{6.61}$$

separately. The key term is $-\langle \Xi^\varepsilon_\sigma, \overline\nabla_\sigma X\rangle$: In Step 1 we defined $\mathcal{Z}^\varepsilon_\sigma := I_{q_\sigma}\partial_s q_\sigma - \zeta^*_\sigma$ and we have seen that $\|\zeta^*\|_{0,2\varepsilon} \le c_1\varepsilon^2$ tends to zero as $\varepsilon \to 0$. This tells that the first component Ξ^ε_σ of $\mathcal{Z}^\varepsilon_\sigma$ and the first component $\partial_s q_\sigma$ of $I_{q_\sigma}\partial_s q_\sigma$ are L^2-close to one another. In Step 3 we have seen that $\overline\nabla_\sigma X$ is L^2-close to $-\partial_s q_\sigma$. We shall prove that all the other terms are arbitrarily small and hence $\partial_\sigma\theta^\varepsilon$ is arbitrarily close to $\|\partial_s q\|^2 = c^*$. More precisely, for the first term in (6.61) we obtain (using repeatedly $\partial_\sigma u_\sigma = \partial_s u_\sigma$ and similar)

$$\begin{aligned}
-\tfrac{d}{d\sigma}\langle \Xi^\varepsilon_\sigma, X\rangle &= -\langle \Xi^\varepsilon_\sigma, \overline\nabla_\sigma X\rangle - \langle \overline\nabla_s \Xi^\varepsilon_\sigma, X\rangle \\
&= \|\partial_s q_\sigma\|^2 - \langle \Xi^\varepsilon_\sigma, \partial_s q_\sigma + \overline\nabla_\sigma X\rangle - \langle X^*_\sigma, \partial_s q_\sigma\rangle \\
&\quad - \langle \overline\nabla_s \partial_s q_\sigma, X\rangle - \langle X^*_\sigma, \overline\nabla_s X\rangle \\
&\ge \|\partial_s q\|^2 - c_{10}\big(\|\partial_s q_\sigma + \overline\nabla_\sigma X\| + \|X\| + \|X^*\|\big) \\
&\ge c^* - c_{11}\big(|\sigma| + \Delta\varepsilon^{1/2} + \varepsilon^2\big).
\end{aligned}$$

The second step adds zero (Ξ^ε_σ in Step 1) to summand one, applies integration by parts to summand two. Step three uses Cauchy-Schwarz, shift invariance of norms, the inequalities $\|\Xi^\varepsilon\| \le c_3$ (6.58), $\|\partial_s q\|^2 = c^*$, $\|\nabla_s\partial_s q\| \le c_0$ (6.54), and $\|\overline\nabla_s X\| \le c_6$ (Step 3). The last step uses Step 3 and (6.57).

For the second term in (6.61) the following preparation is in order

$$\varepsilon^3\|\ell'\|^2 = \varepsilon^3\|\tau^{\varepsilon\prime} - d\chi|_{q_\sigma}\partial_s q_\sigma\|^2 \le 2\varepsilon\big(\varepsilon^2\|\tau^{\varepsilon\prime}\|^2 + \mu_\infty^2 c^*\big) \stackrel{(6.48)}{\le} 2\varepsilon c^*\big(1 + \mu_\infty^2\big)$$

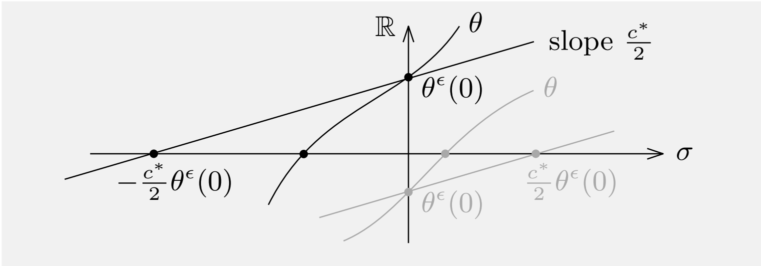

Figure 2: Step 5 detects zero of θ^ε – two cases depending on sign of $\theta^\varepsilon(0)$

where here and in the following we use that $\|d\chi|_q\partial_s q\|^2 \le \mu_\infty^2 c^*$. We have

$$\begin{aligned}
&\varepsilon^2 \tfrac{d}{d\sigma}\left\langle \lambda_\sigma^\varepsilon, \ell\right\rangle \\
&= \varepsilon^2\left\langle \tfrac{d}{ds}(d\chi|_{q_\sigma}\partial_s q_\sigma - \ell_\sigma^*), \ell\right\rangle + \varepsilon^2\left\langle d\chi|_{q_\sigma}\partial_s q_\sigma - \ell_\sigma^*, \tfrac{d}{d\sigma}(\tau^\varepsilon - \chi(q_\sigma))\right\rangle \\
&\le \varepsilon^{1/2}\left(\|d\chi|_q\partial_s q\| + \|\ell^*\|\right)\cdot \varepsilon^{3/2}\|\tfrac{d}{ds}\ell\| - \varepsilon^2\|d\chi|_q\partial_s q\|^2 + \varepsilon\|\ell^*\|\cdot\varepsilon\|d\chi|_q\partial_s q\| \\
&\le \varepsilon\left(\mu_\infty^2 c^* + \|\ell^*\|^2\right) + \tfrac{\varepsilon^3}{2}\|\ell'\|^2 + \varepsilon^2\mu_\infty^2 c^* + \tfrac{\varepsilon^2}{2}\|\ell^*\|^2 + \tfrac{\varepsilon^2}{2}\mu_\infty^2 c^* \\
&\le c^*\varepsilon c_{12} \quad \text{with } c_{12} = 1 + \mu_\infty^2(2 + 3\varepsilon/2).
\end{aligned}$$

Step one uses that on shifted maps $\frac{d}{d\sigma}$ acts like $\frac{d}{ds} = {}'$. Step two uses integration by parts, Cauchy-Schwarz and triangle inequality, shift invariance of the norms and $\frac{d}{d\sigma}\tau^\varepsilon = 0$. We already distributed $\varepsilon^2 = \varepsilon^{1/2}\varepsilon^{3/2}$ suitably in order to apply Young's inequality $ab \le a^2/2 + b^2/2$ in step three. Step four uses the prepared estimate for $\varepsilon^3\|\ell'\|^2$ and that $\|\ell^*\| \le c_1\varepsilon$, by (6.57).

Putting things together, since we fixed $\Delta \le \delta_0 \le 1$, we obtain

$$\tfrac{d}{d\sigma}\theta^\varepsilon(\sigma) \ge c^*(1 - \varepsilon c_{12}) - c_{11}(|\sigma| + 2\varepsilon^{1/2}).$$

So Step 4 is true after shrinking $\sigma_0, \varepsilon_0 > 0$, if necessary, such that $\varepsilon_0 c_{12} \le 1/4$ and $c_{11}(|\sigma_0| + 2\varepsilon_0^{1/2}) \le c^*/4$.

Step 5. *We prove Theorem 6.1.*

Let the $(u^\varepsilon, \tau^\varepsilon)$ satisfy the requirements of the theorem. Estimate $\|\ell^\varepsilon\|_\infty + \|\ell^\varepsilon\| \le c$ holds by Step 3 with $\sigma = 0$ and by (6.60), enlarging c, if necessary. By Steps 2 and 4, see Figure 2, there is a time shift $\sigma \in [-\sigma_0, \sigma_0]$ such that

$$\theta^\varepsilon(\sigma) = 0, \qquad |\sigma| \le c_{14}\left(\|X^\varepsilon\| + \varepsilon^2\right), \quad c_{14} := \tfrac{2c_2}{c^*}.$$

Let $X := X(\sigma)$ and $\ell := \ell(\sigma)$. Then, by Step 3, we have $X, \ell \in L^2$, and by Step 1, we have $\zeta := (X, \ell) \in \mathrm{im}\,(D^\varepsilon_{q_\sigma})^*$ and, by Step 3 again, it holds

$$\|X\|_\infty \le \left(\Delta + c_0 c_{14}\left(\Delta + \varepsilon^{3/2}\right)\right)\varepsilon^{1/2}, \qquad \|\ell\|_\infty \le c_0 < \infty. \tag{6.62}$$

If $\Delta + c_0 c_{14}\left(\Delta + \varepsilon^{3/2}\right) \le \delta_0$ is less than the constant δ_0 in the hypothesis of the uniqueness part of the implicit function theorem I [FW, Sec. 6], then $(u^\varepsilon, \tau^\varepsilon) = \mathcal{T}^\varepsilon(q_\sigma)$, see Figure 1. □

7. Surjectivity of $\mathcal{T}^\varepsilon$

The proof of Theorem C uses local convergence (Lemma 3.5), convergence to broken orbits (Lemma 3.6), the base implicit function Theorem A.1 (IFT II), some of the uniform L^∞ and L^2 gradient bounds from Section 4, uniform exponential decay (5.40) of $|\partial_s u(s)|$ will be crucial to satisfy the IFT II assumption (A.1), and the local surjectivity Theorem 6.1.

Proof. The map $\mathcal{T}^\varepsilon$ is injective for $\varepsilon > 0$ sufficiently small by part I [FW]. Fix a constant $\kappa > 0$ as in the local properness Hypothesis 2.1. To prove surjectivity assume by contradiction the result was false. Then there is a pair of critical points $x^\mp \in \mathrm{Crit}\, f$ of Morse index difference one and sequences $\varepsilon_i \in (0, \varepsilon_\kappa)$ and $(u_i, \tau_i) \in \mathcal{M}^{\varepsilon_i}_{x^-,x^+}$ such that

$$\lim_{i\to\infty} \varepsilon_i = 0, \qquad (u_i, \tau_i) \notin \mathcal{T}^{\varepsilon_i}\left(\mathcal{M}^0_{x^-,x^+}\right). \tag{7.63}$$

Applying time shifts we assume without loss of generality that

$$F_H\left(u_i(0), \tau_i(0)\right) = \frac{c_*}{2} := \frac{f(x^-) - f(x^+)}{2}. \tag{7.64}$$

We prove in four steps that, after passing to a subsequence, there is a sequence $q_i \in \mathcal{M}^0_{x^-,x^+}$ such that

$$u_i = \mathrm{Exp}_{q_i} X_i$$

and the vector field sequence $X_i \in C^\infty(\mathbb{R}, q_i^* TM)$ satisfies

$$\lim_{i\to\infty} \varepsilon_i^{-1/2}\left(\|X_i\|_\infty + \|X_i\|\right) = 0. \tag{7.65}$$

Thus, by Theorem 6.1, for i sufficiently large, there is a time shift $\sigma_i \in \mathbb{R}$ with $(u_i, \tau_i) = \mathcal{T}^{\varepsilon_i}(q_i(\sigma_i + \cdot))$. This contradicts (7.63) and thus proves Theorem C.

Step 1 (Uniform exponential decay (5.39))**.** $\forall \delta > 0$ $\exists T_0 > 0$ *such that*

$$E^{\varepsilon_i}_{\mathbb{R}\setminus[-T_0,T_0]}(u_i, v_i) < \delta \tag{7.66}$$

for every $i \in \mathbb{N}$ *where* $E^{\varepsilon}_I(u,\tau)$ *denotes the energy along an interval* $I \subset \mathbb{R}$*.*

Proof by contradiction. Assume the result is false. Then there is a constant $\delta > 0$, a sequence of positive reals $T_i \to \infty$, and a subsequence, still denoted by $(\varepsilon_i, u_i, \tau_i)$, such that, for every $i \in \mathbb{N}$, the increasing-interval energies

$$E^{\varepsilon_i}_{[-T_i,T_i]}(u_i, \tau_i) \le f(x^-) - f(x^+) - \delta \tag{7.67}$$

are uniformly bounded away from the total energy $E^{\varepsilon_i}(u_i, \tau_i) = f(x^-) - f(x^+)$. According to Lemma 3.6 there is a further subsequence, still denoted by (u_i, τ_i), that converges to a broken base connecting trajectory, more precisely, a finite collection of base connecting trajectories $q^k \in \mathcal{M}^0_{x^{k-1},x^k}$, $k = 1, \dots, \ell$, with $x^- = x^0, x^1, \dots, x^{\ell-1}, x^\ell = x^+ \in \mathrm{Crit}\, f$.

We prove that $\ell \ge 2$: Otherwise $v_i := u_i(s_i + \cdot)$ converges to $q := q^1 \in \mathcal{M}^0_{x^-,x^+}$ and $\vartheta_i := \tau_i(s_i + \cdot) \to \chi(q)$ as in the local convergence Lemma 3.5, for some sequence $s_i \in \mathbb{R}$ provided by Lemma 3.6 (i).

We claim that the sequence s_i is bounded. We first exclude the case that (s_i) is unbounded from above. To see this, we assume, by contradiction, that s_i diverges to $+\infty$. We abbreviate $z_i := (u_i, \tau_i)$. By the gauge condition (7.64) and the fact that downward gradient trajectories flow downhill, it holds $F_H(z_i(s)) \le \frac{c_*}{2}$ for every $s \in [0, \infty)$. Therefore $F_H((s_i)_* z_i(s)) \le \frac{c_*}{2}$ for every $s \in [-s_i, \infty)$. Since $s_i \to \infty$, for every $T > 0$ there exists i_0 such that $F_H((s_i)_* z_i(s)) \le \frac{c_*}{2}$ for all $s \in [-T, T]$ and $i \ge i_0$. Since $(s_i)_* z_i = (v_i, \vartheta_i)$ converges to $(q, \chi(q))$ in C^1_{loc} we have for each $s \in [-T, T]$ that

$$f(q(s)) \overset{(3.25)}{=} \lim_{i \to \infty} F_H(v_i(s), \vartheta_i(s)) \le \tfrac{c_*}{2}$$

for every $s \in [-T, T]$. Since T was arbitrary, we have $f(q(s)) \le \frac{c_*}{2}$ for every $s \in \mathbb{R}$. In particular, we have $f(x^-) \le \frac{c_*}{2}$. Contradiction.

The case where $s \to -\infty$ leads in a similar way to a contradiction, just use the interval $(-\infty, 0]$ instead of $[0, \infty)$ and $(-\infty, -s_i]$ instead of $[-s_i, \infty)$ and look at the positive asymptotic x^+ instead of x^-. This proves the claim.

Since the sequence s_i is bounded, given a constant $T > 0$, there is the inclusion $[s_i - T, s_i + T] \subset [-T_i, T_i]$ for all large i, so (7.67) applies. We obtain

$$\begin{aligned} E^0_{[-T,T]}(q) &\overset{(3.25)}{=} \lim_{i\to\infty} E^{\varepsilon_i}_{[-T,T]}(v_i, \vartheta_i) = \lim_{i\to\infty} E^{\varepsilon_i}_{[s_i-T,s_i+T]}(u_i, \tau_i) \\ &\overset{(7.67)}{\le} f(x^-) - f(x^+) - \delta \end{aligned}$$

where identity two is by variable substitution $\sigma(s) := s_i + s$. On the other hand, since q connects x^- to x^+ we get $E^0(q) = f(x^-) - f(x^+)$ by (6.48). Contradiction. Thus we have proved that $\ell \ge 2$.

As f is Morse-Smale for g, the Morse index difference of x^- and x^+ must be at least two. This contradicts our assumption and proves Step 1.

To formulate Step 2 we first need the following discussion. The a priori Theorem 3.1 provides $c_1 > 0$ such that

$$\|\tau_i\|_\infty + \|\tau_i'\|_\infty \le c_1 \tag{7.68}$$

for every i. Let $\kappa > 0$ be a constant as in the local properness Hypothesis 2.1. The vector field $V := \overline{\nabla} H/|\overline{\nabla} H|^2$ along the open subset $M_{\mathrm{reg}} := \{dH \neq 0\}$ of M generates a local flow $\{\varphi_r\}$ on M_{reg}. Because Σ is compact, for $\delta \in (0, \kappa)$ small enough, the following map is a diffeomorphism onto its image

$$\varphi \colon \Sigma \times (-\delta, \delta) \to U_\Sigma = U_\Sigma(\delta) := \operatorname{im} \varphi \subset M, \quad (q, r) \mapsto \varphi_r q.$$

(The map φ provides a retraction $\rho = \rho^2 \colon U_\Sigma \to U_\Sigma$. The identities

$$H(\varphi_0 q) = 0, \qquad \tfrac{d}{dr} H(\varphi_r q) = dH|_{\varphi_r q} \tfrac{d}{dr} \varphi_r q = \left\langle \overline{\nabla} H|_{\varphi_r q}, V|_{\varphi_r q} \right\rangle = 1,$$

show that $H(\varphi_r(q)) = r$ for every $(q, r) \in \Sigma \times (-\delta, \delta)$.

By the ε_i-equations (1.3) and (7.68) we have $|H(u_i)| = \varepsilon_i^2 |\tau_i'| \le c_1 \varepsilon_i^2$. As ε_i goes to zero, we can assume, maybe after forgetting the first elements of the sequence, that u_i takes values in the image of φ for every i. Hence there exist maps $\mathfrak{q}_i \colon \mathbb{R} \to \Sigma$ and $r_i \colon \mathbb{R} \to (-\delta, \delta)$ with

$$u_i = \varphi_{r_i}(\mathfrak{q}_i), \qquad r_i = H(u_i) = -\varepsilon_i^2 \tau_i', \qquad |r_i| \le c_1 \varepsilon_i^2 \le c_1, \tag{7.69}$$

pointwise at $s \in \mathbb{R}$ and for every i.

Step 2. *$\forall i$ the derivative satisfies $|\partial_s \mathfrak{q}_i| \le 2C_K |\partial_s u_i|$, so it decays exponentially. Here $C_K := \|(d\varphi_{r_i}|_{\mathfrak{q}_i})^{-1}\|_{L^\infty(K)}$ and K is compact by Theorem 3.1.*

By the identity (7.69) and definition of the local flow φ we have

$$\begin{aligned}\partial_s u_i &= \tfrac{\overline{\nabla} H(u_i)}{|\overline{\nabla} H(u_i)|^2}\, r_i' + d\varphi_{r_i}(\mathfrak{q}_i)\partial_s \mathfrak{q}_i \\ &= \tfrac{\overline{\nabla} H(u_i)}{|\overline{\nabla} H(u_i)|^2} \left\langle \overline{\nabla} H(u_i), \partial_s u_i \right\rangle + d\varphi_{r_i}|_{\mathfrak{q}_i} \partial_s \mathfrak{q}_i.\end{aligned}$$

Hence

$$|\partial_s \mathfrak{q}_i| = \left| (d\varphi_{r_i}|_{\mathfrak{q}_i})^{-1} \left(\partial_s u_i - \tfrac{\overline{\nabla} H(u_i)}{|\overline{\nabla} H(u_i)|^2} \left\langle \overline{\nabla} H(u_i), \partial_s u_i \right\rangle \right) \right| \le 2C_K |\partial_s u_i|.$$

Thus exponential decay of $\partial_s u_i$, see (5.40), implies exponential decay of $\partial_s \mathfrak{q}_i$.

Step 3. *Let $\mathfrak{q}_i \colon \mathbb{R} \to \Sigma$ be as in (7.69). For each i sufficiently large there is a base trajectory $q_i \in \mathcal{M}^0_{x^-,x^+}$ and a vector field $\xi_i \in C^\infty(\mathbb{R}, q_i^* T\Sigma)$ such that*

$$\mathfrak{q}_i = \exp_{q_i} \xi_i, \qquad \|\xi_i\|_\infty \le \|\xi_i\|_{1,2} \le c_3 \varepsilon_i$$

for some positive constant c_3 which does not depend on i.

Let δ, c, and ρ be the positive constants in the uniform exponential decay Theorem 5.1 and choose $T_0 > 0$ in Step 1 such that (7.66) holds for this δ. By Step 2 and uniform exponential decay (5.40) we have

$$|\partial_s \mathfrak{q}_i(s)| \le 2C_K |\partial_s u_i(s)| \le 2C_K c e^{-\rho(|s| - T_0 - 1)}$$

for every $|s| \ge T_0 + 1$. Note that the right hand side does not depend on i. As $e^s \ge 1 + s^2$ for all sufficiently large s, there is a constant $c_2 > 0$ such that

$$|\partial_s \mathfrak{q}_i(s)| \le \frac{c_2}{1+s^2} \tag{7.70}$$

for every $s \in \mathbb{R}$ and every $i \in \mathbb{N}$. In the following add zero in step one, then in step two for summand one use (2.10) and to summand two apply [FW, Le. 6.8 i)][7] for some constant $d = d(c_1)$ to get the L^2 estimate

$$\begin{aligned}\|\tau_i - \chi(\mathfrak{q}_i)\| &\le \|\tau_i - \chi(u_i)\| + \|\chi(\varphi_{r_i} \mathfrak{q}_i) - \chi(\mathfrak{q}_i)\| \\ &\le \left\| \frac{\left\langle \overline{\nabla} H(u_i), \partial_s u_i \right\rangle}{|\overline{\nabla} H(u_i)|^2} \right\| + d\|r_i\| \\ &\le \frac{\varepsilon_i^2 \|\tau_i''\|}{c_\kappa^2} + d\varepsilon_i^2 \|\tau_i'\| \\ &\le \varepsilon_i \left(\tfrac{C}{c_\kappa^2} + d\sqrt{c^*} \right).\end{aligned} \tag{7.71}$$

[7] apply lemma ($\delta = 1$, $X = 0$, $\hat{X} = r_i$) pointwise at s, so $|\hat{X}| = |r_i| \le c_1$ by (7.69)

In step three, to deal with summand one, we used the identity $\hat{\tau}2$ in (2.8) and (2.9) with constant $c_\kappa > 0$. For summand two we used the identity in (7.69) for r_i. Step four holds by the L^2 estimate (4.30) for $\varepsilon_i \tau_i''$ with constant $C > 0$ and by the ε-energy identity (1.7). So

$$\begin{aligned} \|\partial_s \mathfrak{q}_i + \nabla f(\mathfrak{q}_i)\| &\overset{(1.4)}{=} \|\partial_s \mathfrak{q}_i + \overline{\nabla} F(\mathfrak{q}_i) + \chi(\mathfrak{q}_i) \overline{\nabla} H(\mathfrak{q}_i)\| \\ &\overset{(1.3)}{=} \|(\chi(\mathfrak{q}_i) - \tau_i) \overline{\nabla} H(\mathfrak{q}_i)\| \\ &\le \ C_1 \varepsilon_i, \quad C_1 := \left(\tfrac{C}{c_\kappa^2} + d\sqrt{c^*}\right) \|\overline{\nabla} H\|_{L^\infty(K)} \end{aligned} \tag{7.72}$$

where step one is the base equation in (Σ, g) expressed in the ambience (M, G) and step two is by the ε_i-equation.

Now let $\delta_0 = \delta_0(c_2)$ and $C_2 = C_2(c_2)$ be the constants in the base implicit function Theorem A.1. Then the map $\mathfrak{q}_i$ satisfies the hypotheses of Theorem A.1 whenever $C_1 \varepsilon_i < \delta_0$. Thus, for each i sufficiently large, there is a base trajectory $q_i \in \mathcal{M}^0_{x^-,x^+}$ and a vector field $\xi_i \in C^\infty(\mathbb{R}, q_i^* TM)$ with

$$\mathfrak{q}_i = \exp_{q_i} \xi_i, \ \|\xi_i\| \le \|\xi_i\|_{1,2} \le C_2 \|\partial_s \mathfrak{q}_i + \overline{\nabla} F(\mathfrak{q}_i) + \chi(\mathfrak{q}_i) \overline{\nabla} H(\mathfrak{q}_i)\| \le C_1 C_2 \varepsilon_i.$$

But $\|\xi_i\|_\infty \le \|\xi_i\|_{1,2}$, see e.g. [FW, Sec. 4.2.5]. This proves Step 3.

Step 4. *For each i sufficiently large there is a unique vector field $X_i \in C^\infty(\mathbb{R}, q_i^* TM)$ that satisfies the identity $u_i = \mathrm{Exp}_{q_i} X_i$ and the limit (7.65).*

We prove Step 4 based on Step 3. Pointwise at every $s \in \mathbb{R}$ it holds that

$$\begin{aligned} |X_i| &= d_M(q_i, u_i) \\ &\le d_M(q_i, \mathfrak{q}_i) + d_G(\mathfrak{q}_i, u_i) \\ &\le d_\Sigma(q_i, \mathfrak{q}_i) + \mathrm{length}(\varphi_{[0,r_i]} q_i) \\ &\le |\xi_i| + \tfrac{r_i}{c_\kappa} \\ &\le c_3 \varepsilon_i + \tfrac{c_1}{c_\kappa} \varepsilon_i^2 \\ &< \iota(T_\Sigma M, G) \end{aligned}$$

where the final inequality holds for all sufficiently large i. To see the other inequalities start with inequality one, this is the triangle inequality for the Riemannian distance in (M, G). Inequality two holds since there are more curves in M than in the subset Σ and since $r \mapsto \varphi_r q_i$ is a particular curve among many. Inequality three holds since $[0, 1] \ni t \mapsto \exp_{q_i} t\xi_i$ is a particular curve among many and since the length of a curve is bounded by the length of the parametrization interval, here r_i, times the largest speed, here

$|\frac{d}{dt}\varphi_t q_i| = |\overline{\nabla} H(\varphi_t q_i)|^{-1} \le \frac{1}{c_\kappa}$ by (2.9). Inequality four holds for summand 1 by Step 3, for summand two by (7.69). As $d_M(q_i, u_i) < \iota(T_\Sigma M, G)$, the injectivity radius of (M, G) along Σ, and $u_i = \mathrm{Exp}_{q_i} X_i$, it holds $|X_i| = d_M(q_i, u_i)$.

Of the above estimate for $|X_i(s)|$ take the supremum over $s \in \mathbb{R}$ to get the L^∞ limit in (7.65). For the L^2 limit take the L^2 norm and use Step 3 and (7.69) to get $\|X_i\| \le \|\xi_i\| + \frac{1}{c_\kappa}\|r_i\| \le c_3\varepsilon_i + \frac{\varepsilon_i^2}{c_\kappa}\|\tau_i'\| \le \varepsilon_i\left(c_3 + \sqrt{c^*}/c_\kappa\right)$. This proves Step 4 and Theorem C. □

Appendix A. Implicit function theorem II – Base

The following theorem IFT II is used in the proof of surjectivity of the map $\mathcal{T}^\varepsilon \colon \mathcal{M}^0 \to \mathcal{M}^\varepsilon$ in Theorem C, see (7.72).

We remind the reader that $|\cdot|$ denotes – depending on context – absolute value or the length of a tangent vector with respect to a Riemannian metric.

Theorem A.1 (IFT II). *Let (Σ, g) be a compact Riemannian manifold (no boundary). Suppose f is Morse and D^0_q is onto whenever $q \in \mathcal{M}^0_{y^-,y^+}$ and $y^\mp \in \mathrm{Crit}\, f$.*[8] *Pick two critical points $x^\mp \in \mathrm{Crit}\, f$ of index difference one. Then, for any $c_0 > 0$, there are constants $\delta_0, C > 0$ such that the following holds. If $\mathfrak{q} \colon \mathbb{R} \to M$ is a smooth map with $\lim_{s\to\mp\infty} \mathfrak{q}(s) = x^\mp$ and*

$$|\partial_s \mathfrak{q}(s)| \le \frac{c_0}{1+s^2} \tag{A.1}$$

for all $s \in \mathbb{R}$ and

$$\|\partial_s \mathfrak{q} + \overline{\nabla} F(\mathfrak{q}) + \chi(\mathfrak{q})\overline{\nabla} H(\mathfrak{q})\| \le \delta_0, \tag{A.2}$$

then there is a trajectory $q \in \mathcal{M}^0_{x^-,x^+}$ and a $\xi \in \mathrm{im}\,(D^0_q)^* \cap W^{1,2}$ with

$$\mathfrak{q} = \exp_q \xi, \qquad \|\xi\|_{1,2} \le C\|\partial_s \mathfrak{q} + \overline{\nabla} F(\mathfrak{q}) + \chi(\mathfrak{q})\overline{\nabla} H(\mathfrak{q})\|.$$

The Sobolev $W^{1,2}$ norm is defined by $\|\xi\|_{1,2}^2 := \|\xi\|^2 + \|\nabla_s \xi\|^2$.

To prove the base implicit function theorem IFT II requires analogous inputs as the ε-ambient implicit function theorem, IFT I in [FW], namely a-priori estimates, exponential decay, Fredholm property, bounded right inverse, and quadratic estimates. In addition, IFT II requires similar steps as in the proof of local and global surjectivity in Sections 6 and 7, just without

[8] in other words (f, g) is Morse Smale.

the additional difficulty of the ε parameter dependency. The steps of the present section have been carried out in great detail in [Web13] in a slightly more general context. Thus we shall detail here only those parts of proof that use the specific nature of the differential equation at hand such as, for example, quadratic estimates.

A.1. A priori gradient bounds

Proposition A.2 (A priori estimates). *Fix a smooth $f\colon \Sigma \to \mathbb{R}$. Then there is a constant $C > 0$ such that the following is true for each smooth path $q\colon \mathbb{R} \to \Sigma$ and every smooth, compactly supported, vector field ξ along q.*

(i) *Suppose ξ satisfies the linear equation $\nabla_s \xi + \nabla_\xi \nabla f(q) = 0$. Then*

$$|\xi(s)| + |\nabla_s \xi(s)| \le C \|\xi\|_{L^2([s-1,s])}$$
$$|\nabla_s \nabla_s \xi(s)| \le C \|\xi\|_{L^2([s-1,s])} \cdot \|\partial_s q\|_\infty$$

for every $s \in \mathbb{R}$ and where $L^2([s-1,s]) = L^2([s-1,s], q^*T\Sigma)$.

(ii) *Suppose q solves $\partial_s q + \nabla f(q) = 0$. Then*

$$|\partial_s q(s)|^2 + |\nabla_s \partial_s q(s)|^2 \le C E^0_{[s-1,s]}(q) \tag{A.3}$$

for every $s \in \mathbb{R}$ where E^0_I is the energy over the domain $I \subset \mathbb{R}$.

Proof of Proposition A.2. (i) Pick $\sigma \in \mathbb{R}$ and define $\xi^\sigma(s) := \xi(s+\sigma)$ and $\mathfrak{f}(s) := \frac{1}{2}|\xi(s+\sigma)|^2 = \frac{1}{2}|\xi^\sigma(s)|^2$ for every $s \in \mathbb{R}$. As $\nabla_s \xi = -\nabla_\xi \nabla f(q)$ we get

$$|\nabla_s \xi| \le \|\nabla\nabla f\|_\infty |\xi|, \quad \mathfrak{f}' = \langle \nabla_s \xi^\sigma, \xi^\sigma \rangle \le |\nabla_s \xi^\sigma| \cdot |\xi^\sigma| \le \mu \mathfrak{f}, \quad \mu = 2\|\nabla\nabla f\|_\infty,$$

pointwise at $s \in \mathbb{R}$. Thus, by Lemma 4.3 with $r = 1$, we obtain the estimate

$$|\xi(\sigma)|^2 = 2\mathfrak{f}(0) \le 4de^\mu \int_{-1}^0 \mathfrak{f}(s)\, ds = 2de^{2\|\nabla\nabla f\|_\infty} \|\xi\|^2_{L^2([\sigma-1,\sigma])}$$

for every $\sigma \in \mathbb{R}$. Let R be the Riemannian curvature tensor of (Σ, g), then

$$\nabla_s \nabla_s \xi = -\nabla_s \nabla_\xi \nabla f(q) = R(\xi, \partial_s q)\nabla f(q) - \nabla_\xi \nabla_s \nabla f(q)$$

pointwise at $s \in \mathbb{R}$. (ii) Apply part (i) to $\xi = \partial_s q$. □

A.2. Exponential decay

Two common methods to establish exponential decay are

- the energy method;
- the action-energy inequality.

The energy method was applied with great success in infinite dimensions, see e.g. [Sal99], even in infinite dimensional adiabatic limits, see [DS94, GS05, SW06]. We utilize the energy method in the present finite dimensional adiabatic limit, Section 5, having in mind the infinite dimensional case, namely Rabinowitz-Floer homology; see [FW, Introduction].

While the energy method in general involves huge calculations, the hypotheses for the action-energy inequality seem easier to check, at least in the present finite dimensional context. However, the technique of the energy method, namely to establish differential inequalities for densities, at the same time provides a priori estimates simply utilizing some of the exponential decay density calculations. Moreover, the a priori estimates arrive together with interval L^2 estimates of higher derivatives in terms of the first derivative – (a part of) the energy, as is illustrated by Theorem 4.1.

The action-energy inequality is discussed next.

A.2.1. Action-energy inequality. Advantages of the action-energy inequality method, see [GS05, Zil09, AF13], are the following

- usable for functions that are only C^1 (part (i));
- stable under change of equivalent metric;
- at a non-degenerate critical point the action-energy hypothesis (A.4) follows from Taylor's theorem; see Lemma A.4.

Let f be a smooth function on a Riemannian manifold Σ. The **covariant Hessian operator** of f at a point $q \in \Sigma$ is the linear map given by

$$A_q^0 \colon T_q\Sigma \to T_q\Sigma, \quad \xi \mapsto \nabla_\xi \nabla f(q).$$

The linear map A_q^0 is symmetric; see e.g. [FW, Sec. 4.1.2]. For convenience of the reader we state and prove the action-energy inequality next.

Proposition A.3. *Let $f \colon \Sigma \to \mathbb{R}$ be a smooth function on a Riemannian manifold, let x be a zero $f(x) = 0$. Assume there are constants $c, c_1, c_2 > 0$*

and $\delta < \iota_x$ (injectivity radius at x) such that the ***action-energy inequality***

$$|f(q)| \le c|\nabla f(q)|^2, \qquad \forall q \in \bar{B}_\delta = \{q = \exp_x \xi \colon |\xi| \le \delta\} \tag{A.4}$$

holds. Then the following is true. All solutions $q\colon [0,\infty) \to B_\delta$ of

$$q' = -\nabla f(q), \qquad q(s) \overset{s\to\infty}{\longrightarrow} x,$$

approach the critical point x exponentially with the same rate, namely

$$|\xi_s| = d(q_s, x) \le 2\sqrt{cf(q_s)} \le 2e^{-s/2c}\sqrt{cf(q_0)}$$

$\forall s \ge 0$. Moreover, the derivatives of the trajectory q decay exponentially

$$|\partial_s q_s| \le c_1 f(q_0)^{1/3} e^{-s/3c}, \qquad |\nabla_s \partial_s q_s| \le c_2 f(q_0)^{1/3} e^{-s/3c}, \tag{A.5}$$

for every $s \ge 0$.

Proof. Let $q \not\equiv 0$. Pick two times $s_+ \ge s \ge 0$. So $f(q_s) > f(x) = 0$. We have

$$\begin{aligned}
\operatorname{dist}(q_s, q_{s_+}) &\le \ell(q|_{[s,s_+]}) \\
&= \int_s^{s_+} |\partial_s q_s|\, ds \overset{\text{(sol)}}{=} \int_s^{s_+} |\nabla f(q_s)|\, ds = \int_s^{s_+} \frac{|\nabla f(q_s)|^2}{|\nabla f(q_s)|}\, ds \\
&\overset{\text{(A.4)}}{\le} \int_s^{s_+} \frac{|\nabla f(q_s)|^2}{\sqrt{f(q_s)/c}}\, ds \overset{\text{(sol)}}{=} \sqrt{c}\int_s^{s_+} \frac{-\frac{d}{ds} f(q_s)}{\sqrt{f(q_s)}}\, ds \\
&= -2\sqrt{c}\int_s^{s_+} \tfrac{d}{ds}\sqrt{f(q_s)}\, ds \\
&= 2\sqrt{c}\left(\sqrt{f(q_s)} - \sqrt{f(q_{s_+})}\right)
\end{aligned}$$

for every $s \ge 0$. By hypothesis $f(q_{s_+}) \to f(x) = 0$, as $s_+ \to \infty$. Thus

$$|\xi_s| = \operatorname{dist}(q_s, x) \le 2\sqrt{cf(q_s)}, \quad q_s = \exp_x \xi_s.$$

By the action-energy inequality (A.4) and as q is a solution (sol), we get

$$\tfrac{d}{ds} f(q_s) = df|_{q_s} \tfrac{d}{ds} q_s \overset{\text{(sol)}}{=} -|\nabla f(q_s)|^2 \overset{(A.4)}{\le} \tfrac{-1}{c} f(q_s) < 0$$

$\forall s \ge 0$. Thus there is the inequality $\frac{d}{ds} \ln f(q_s) = \frac{(f\circ q)'(s)}{f\circ q(s)} \le \frac{-1}{c}$ whose integral over $[0,s]$ produces the first assertion $f(q_s) \le e^{-s/c} f(q_0)$ for $s \ge 0$.

Next we show that $\partial_s q$ decays exponentially. From the formula $\nabla_s \partial_s q_s = -A_{q_s}\partial_s q_s$ and the hypothesis that q takes values in the compact ball $\bar{B}_\delta$ we deduce that there exist $C > 0$ such that $|\nabla_s \partial_s q_s| < C$ for every $s \geq 0$. Given $T > 0$, we abbreviate $\mu := |\partial_s q_T|$. Apply the fundamental theorem of calculus to the derivative

$$\tfrac{d}{ds}|\partial_s q_s| = \tfrac{d}{ds}\sqrt{\langle \partial_s q_s, \partial_s q_s\rangle} = \frac{\langle \nabla_s \partial_s q_s, \partial_s q_s\rangle}{|\partial_s q_s|} \geq -|\nabla_s \partial_s q_s| \geq -C$$

in order to obtain for $s \in [T, T + \frac{\mu}{2C}]$ the estimate

$$|\partial_s q_s| - \mu = |\partial_s q_s| - |\partial_s q_T| = \int_T^s \tfrac{d}{ds}|\partial_s q_s| \geq -C(s-T) \geq -\tfrac{\mu}{2},$$

thus $|\partial_s q_s| \geq \frac{\mu}{2}$. By assertion one we obtain

$$E_{[T,\infty)}(q) = \int_T^\infty |\partial_s q_s|^2 \, ds = f(q_T) - f(x) \leq e^{-T/c} f(q_0).$$

On the other hand, we have that

$$E_{[T,\infty)}(q) = \int_T^\infty |\partial_s q_s|^2 \, ds \geq \int_T^{T+\frac{\mu}{2C}} \underbrace{|\partial_s q_s|}_{\geq \mu/2}{}^2 \, ds \geq \frac{\mu^3}{8C} = \frac{|\partial_s q_T|^3}{8C}.$$

Combine both inequalities to get $|\partial_s q_T| \leq (8Cf(q_0))^{1/3} e^{-T/3c}$. Because $\nabla_s \partial_s q_s = -A_{q_s}\partial_s q_s$, inequality two of (A.5) follows from the first one and since the Hessian is bounded along the compact set $\bar{B}_\delta$. This concludes the proof of Proposition A.3. □

Lemma A.4. *At a non-degenerate critical point (A.4) is true.*

Proof. By the Morse Lemma we can assume that in local coordinates around the critical point the function has the form $f(q) = \sum_{i=1}^n a_i q_i^2$ for reals $a_i \neq 0$.

Step 1. *(Flat case) Assume the metric is standard in Morse coordinates.*

Under this assumption the gradient is given by $\nabla f(q) = (2a_1 q_1, \ldots, 2a_n q_n)$, therefore $|\nabla f(q)|^2 = 4\sum_i a_i^2 q_i^2$. We abbreviate $a := \min\{a_1, \ldots, a_n\}$, then

$$|f(q)| \leq \sum_i |a_i| q_i^2 = \frac{1}{a}\sum_i \underbrace{a|a_i|}_{\leq a_i^2} q_i^2 \leq \frac{1}{a}\sum_i a_i^2 q_i^2 = \frac{1}{4a}|\nabla f(q)|^2.$$

Let $g_1 = g$ be the given metric on Σ and g_2 the euclidean metric on $\mathbb{R}^n$. In any compact neighborhood the two metrics are equivalent to each other.

Step 2. *Suppose we have two metrics and a constant c that satisfy the norm estimate $|\cdot|_1 \le c|\cdot|_2$. Then we have the estimate for the norm of the gradients*

$$|{}^2\nabla f(q)|_2 \le c|{}^1\nabla f(q)|_1$$

Step 2 follows from the following estimate

$$\begin{aligned}
0 &\le |c^2 \cdot {}^1\nabla f(q) - {}^2\nabla f(q)|^2 \\
&= c^4|{}^1\nabla f(q)|_1^2 - 2c^2 \left\langle {}^1\nabla f(q), {}^2\nabla f(q)\right\rangle_1 + |{}^2\nabla f(q)|_1^2 \\
&= c^4|{}^1\nabla f(q)|_1^2 - 2c^2 df_q({}^2\nabla f_q) + |{}^2\nabla f(q)|_1^2 \\
&\le c^4|{}^1\nabla f(q)|_1^2 - 2c^2|{}^2\nabla f(q)|_2^2 + c^2|{}^2\nabla f(q)|_2^2 \\
&= c^4|{}^1\nabla f(q)|_1^2 - c^2|{}^2\nabla f(q)|_2^2.
\end{aligned}$$

As a consequence of Step 2, maybe after changing the constant c, the action-energy inequality (A.4) continues to hold if one replaces the metric by an equivalent one. By Step 1 this proves (A.4) for the given metric g. □

A.2.2. Finite energy trajectories. In this article all solutions of the 0-equation, (1.4), have finite energy, indeed $E^0(q) \le \operatorname{osc} f$; cf. [FW, Sec. 3.1.1]. For general problems, this is not so, but it is usually still the set of finite energy solutions which carries desired information. Usually in more general cases, see e.g. [Sal99] in a PDE setting, finite energy solutions are characterized by equivalence of properties (i), (ii), (iii) in the next proposition.

Proposition A.5 (Finite energy and Morse). *Let $f\colon \Sigma \to \mathbb{R}$ be Morse and $q\colon \mathbb{R} \to \Sigma$ a solution of the downward gradient equation (1.4). Then*

(i) $E^0(q) < \infty$;
(ii) *there exist critical points $x\mp \in \operatorname{Crit} f$ such that $\lim_{s\to\mp\infty} q(s) = x^\mp$;*
(iii) *there are constants $\delta, c > 0$ such that $|\partial_s q(s)| \le ce^{-\delta|s|}$ for every $s \in \mathbb{R}$.*

Existence of asymptotic limits, (ii), enters the proof of the surjectivity Theorem C, right after (3.28). The proof of Proposition A.5 uses

Lemma A.6 (Critical point detection). *Let $f\colon \Sigma \to \mathbb{R}$ be Morse. Then, for every $\delta_0 \in (0, \iota_\Sigma)$, there is a constant $\delta_1 > 0$, such that the following is true. If $q \in \Sigma$ is an almost critical point of f in the sense that $|\nabla f(q)| < \delta_1$, then there is a critical point x of f and a δ_0-short difference vector $\xi \in T_x\Sigma$ such that $q = \exp_x \xi$ and $|\xi| \le \delta_0$.*

Proof. Like Lemma 5.3, but shorter by lack of a second component. □

Proof of Proposition A.5. We show equivalence of (i), (ii), and (iii). The implication (iii) ⇒ (i) of finite energy is clear. (ii) ⇒ (i) is the base energy identity, prior to (1.7). We prove (i) ⇒ (ii). For any $s \in \mathbb{R}$ it holds that

$$|\nabla f(q(s))|^2 \stackrel{(1.4)}{=} |\partial_s q(s)|^2 \stackrel{(A.3)}{\leq} C E^0_{[s-1,s]}(q) \stackrel{s\to\mp\infty}{\longrightarrow} 0$$

where the limit is zero since the energy (integral) over the whole real line $\mathbb{R}$ is finite by hypothesis (i). Hence it remains to show that the asymptotic limits $\lim_{s\to\mp\infty} q(s)$ exist, in which case they are critical points automatically.
To this end fix a constant $4\delta_0 > 0$ smaller than the injectivity radius $\iota_\Sigma > 0$ and smaller than the minimal distance $\min_{x,y\in \mathrm{Crit}\, f} \mathrm{dist}(x, y) > 0$ of any two of the finitely many critical points. Let δ_1 be the constant in Lemma A.6 with this choice of δ_0. Pick a large time $T > 0$ such that $|\nabla f(q(s))| < \delta_1$ whenever $s > T$. Given $s > T$, Lemma A.6 associates to the point $q(s)$ a critical point x_s at distance $\leq \delta_0$. But the radius δ_0 ball about x_s contains no other critical points. Thus $x_0 := x_s$ does not depend on $s > T$. Hence the whole forward trajectory $q|_{(T,\infty)}$ stays in the δ_0 ball about x_0. But this remains true for any smaller δ_0, hence $q(s) \to x_0$ as $s \to \infty$. Same for the backward limit $s \to -\infty$.

That (ii) ⇒ (iii) holds by Proposition A.3 based on Lemma A.4. Part (i) is true since $E^0(q) \leq \max f - \min f$ by [FW, Le. 3.3]. □

A.3. Quadratic estimate and right inverse

Proposition A.7 (Quadratic estimate). *There is a constant $\delta \in (0, 1]$ with the following significance. For every $C_0 > 0$, there is a constant $c > 0$ such that the following is true. Let $q\colon \mathbb{R} \to \Sigma$ be a map and ξ a smooth compactly supported vector field along q such that*

$$\|\partial_s q\|_\infty \leq C_0, \qquad \|\xi\|_\infty \leq \delta,$$

then

$$\|\mathcal{F}^0_q(\xi) - \mathcal{F}^0_q(0) - d\mathcal{F}^0_q(0)\xi\| \leq c\|\xi\|_\infty \left(\|\xi\| + \|\nabla_s\xi\| \cdot \|\xi\|_\infty\right). \tag{A.6}$$

Here the map $\mathcal{F}^0_q$ is defined by

$$\mathcal{F}^0_q(\xi) := \phi(q, \xi)^{-1} \left(\partial_s \left(\exp_q \xi\right) + \nabla f \left(\exp_q \xi\right)\right) \tag{A.7}$$

where $\phi = \phi(q,\xi)\colon T_q\Sigma \to T_{\exp_q(\xi)}\Sigma$ is parallel transport in (Σ, g), pointwise at $s \in \mathbb{R}$, along the geodesic $r \mapsto \exp_q(r\xi)$; see [FW, §4.1.3].

As Σ is compact the injectivity radius of the Riemannian manifold (Σ, G) is positive. Choose $\delta = \iota(\Sigma, g)/2 > 0$, so X is in the domain of exp.

Proof. Let lower case e_i be the covariant derivatives of the exponential map exp of the Riemannian manifold (Σ, g), see [FW, Thm. 6.7], then

$$\partial_s(exp_q\xi) = e_1(q,\xi)\partial_s q + e_2(q,\xi)\nabla_s\xi. \tag{A.8}$$

By definition of $\mathcal{F}_q^0$ and since $d\mathcal{F}_q^0(0)\xi = D_q^0\xi = \nabla_s\xi + \nabla_\xi\nabla f(q)$, see e.g. [FW, Sec. 4.1.2], we obtain

$$\begin{aligned}
\mathcal{F}_q^0(\xi) &= \phi(q,\xi)^{-1}\left(e_1(q,\xi)\partial_s q + e_2(q,\xi)\nabla_s\xi + \nabla f(\exp_q \xi)\right)\\
-\mathcal{F}_q^0(0) &= -\partial_s q - \nabla f(q)\\
-D_q^0\xi &= -\nabla_s\xi - \nabla_\xi\nabla f(q).
\end{aligned}$$

Now we write $\mathcal{F}_q^0(\xi) - \mathcal{F}_q^0(0) - D_q^0\xi = F_1(\xi) + F_2(\xi) + F_3(\xi)$ where

$$\begin{aligned}
F_1(\xi) &:= \left(\phi(q,\xi)^{-1}e_1(q,\xi) - \mathbb{1}\right)\partial_s q\\
F_2(\xi) &:= \left(\phi(q,\xi)^{-1}e_2(q,\xi) - \mathbb{1}\right)\nabla_s \xi\\
F_3(\xi) &:= \phi(q,\xi)^{-1}\nabla f(\exp_q \xi) - \nabla f(q) - \nabla_\xi\nabla f(q).
\end{aligned}$$

In [FW, Sec. 6.1] in the proof of Quadratic Estimate I we showed, in the present notation and pointwise at $s \in \mathbb{R}$, that $h_2(\xi) := \left(\phi(q,\xi)^{-1}e_2(q,\xi) - \mathbb{1}\right)\xi$ satisfies $h_2(0) = 0$ and $Dh_2(0)\xi = 0$. The proof carries over literally if e_1 replaces e_2. Use Lemma [FW, Le. 6.8] ii) pointwise, then integrate, to get

$$\|F_1\| \le c_1 C_0 \|\xi\|_\infty \|\xi\|, \quad \|F_2\| \le c_1 \|\xi\|_\infty^2 \|\nabla_s\xi\|, \quad \|F_3\| \le c_1\|\xi\|_\infty\|\xi\|,$$

for some constant $c_1 > 0$ that depends on the $C^2(\Sigma)$ norm of f. ☐

Right inverse.

Proposition A.8 (Right inverse). *Let $x^{\mp} \in \mathrm{Crit} f$ be non-degenerate critical points and $q \in \mathcal{M}^0_{x^-,x^+}$ such that $D^0_q \colon W^{1,2} \to L^2$ is surjective. Then there is a positive constant $c = c(q)$, invariant under s-shifts of q, such that*

$$\|\xi^*\|_{1,2} \le c \|D^0_q \xi^*\| \tag{A.9}$$

for every $\xi^* \in \mathrm{im}\,(D^0_q)^* \cap W^{1,2}$.

Proof. By non-degeneracy of the asymptotic boundary conditions $x^{\mp}$, both operators D^0_q and $(D^0_q)^*$ are Fredholm. Since $\ker(D^0_q)^* = \mathrm{coker}\, D^0_q = \{0\}$, see e.g. [FW, Sec. 4.1.6], the bounded linear map D^0_q is injective. Hence, by the open mapping theorem, there is a positive constant $c_1 = c_1(q)$ such that

$$\|\eta\| + \|\nabla_s \eta\| \le c_1 \|(D^0_q)^* \eta\|$$

for every $\eta \in W^{1,2}(\mathbb{R}, q^*TM)$. Both sides of the inequality are invariant under time shift, thus so is the constant c_1. $\square$

A.4. Proof of IFT II

Assume by contradiction the result was false. Then there exists a pair of critical points $x^{\mp} \in \mathrm{Crit} f$ of Morse index difference one and a sequence of smooth maps $\mathfrak{q}_i \colon \mathbb{R} \to \Sigma$ such that $\lim_{s\to\mp\infty} \mathfrak{q}_i(s) = x^{\mp}$ and

$$|\partial_s \mathfrak{q}_i(s)| \le \frac{c_0}{1+s^2} \tag{A.10}$$

for every $s \in \mathbb{R}$ and

$$\|\partial_s \mathfrak{q}_i + \nabla f|_{\mathfrak{q}_i}\| \le \frac{1}{i}, \tag{A.11}$$

but $\mathfrak{q}_i$ does not satisfy the conclusion of Theorem A.1 for $c = i$. This means that if $q \in \mathcal{M}^0_{x^-,x^+}$ and $\xi^i \in \mathrm{im}\,(D^0_q)^* \cap W^{1,2}$ satisfy $\mathfrak{q}_i = \exp_q \xi^i$, then

$$\|\partial_s \mathfrak{q}_i + \nabla f|_{\mathfrak{q}_i}\| < \frac{1}{i} \|\xi^i\|_{1,2}. \tag{A.12}$$

Fix a regular value d_* of f between $f(x^+)$ and $f(x^-)$. Applying time shifts, if necessary, see (6.49), we may assume without loss of generality that

$$f(\mathfrak{q}_i(0)) = d_*. \tag{A.13}$$

Recall that, by (A.3), finite energy 0-solutions $\tilde{q}$ satisfy the a priori bound

$$\|\partial_s \tilde{q}\|_\infty \le C_0 \tag{A.14}$$

for some constant $C_0 > 0$ of the form $C_0 = \tilde{C}(f) \cdot \mathrm{osc} f$.

Claim. *There exists a subsequence, still denoted by $\mathfrak{q}_i$, a constant $C > 0$, a connecting trajectory $q \in \mathcal{M}^0_{x^-,x^+}$, and a sequence of time shifts $\sigma_i \in \mathbb{R}$, see (6.49), such that the sequence of difference vector fields η_i determined by*

$$\mathfrak{q}_i = \exp_{q_{\sigma_i}} \eta_i$$

is in the $W^{1,2}$ part of the adjoint image, L^∞ null, and $W^{1,2}$ bounded, i.e.

$$\eta_i \in \mathrm{im}\,(D^0_{q_{\sigma_i}})^* \cap W^{1,2}, \qquad \lim_{i\to\infty} \|\eta_i\|_\infty = 0, \qquad \|\eta_i\|_{1,2} \le C \;\; \forall i. \tag{A.15}$$

First we show how the claim leads to a contradiction. Consider the shifted connecting trajectories q_{σ_i} and vector fields η_i along them, as provided by the claim. These satisfy the assumptions of the quadratic estimate, Proposition A.7, by (A.14) with constant C_0 and by choosing a further subsequence, if necessary, so that $\|\eta_i\|_\infty \le \delta$. Let $C_2 := c(C_0) > 0$ be the constant provided by Proposition A.7. Since $\mathcal{M}^0_{x^-,x^+}$ is a finite set, and so is $Crit f$, the estimate for the right inverse, Proposition A.8, applies with constant $C_1 = C_1(f) > 0$. By (A.7) and since parallel transport is an isometry we obtain step one

$$\begin{aligned}
\|\partial_s \mathfrak{q}_i + \nabla f|_{\mathfrak{q}_i}\| &= \|\mathcal{F}^0_{q_{\sigma_i}}(\eta_i)\| \\
&\ge \|D^0_{q_{\sigma_i}} \eta_i\| - \|\mathcal{F}^0_{q_{\sigma_i}}(\eta_i) - \mathcal{F}^0_{q_{\sigma_i}}(0) - d\mathcal{F}^0_{q_{\sigma_i}}(0)\,\eta_i\| \\
&\ge \|\eta_i\|_{1,2} \left(\tfrac{1}{C_1} - C_2 \|\eta_i\|_\infty (1 + \|\eta_i\|_{1,2}) \right) \\
&\ge \tfrac{1}{2C_1} \|\eta_i\|_{1,2}.
\end{aligned}$$

In step two we added twice zero, namely $D^0_{q_{\sigma_i}} - d\mathcal{F}^0_{q_{\sigma_i}}(0) = 0$ and $\mathcal{F}^0_{q_{\sigma_i}}(0) = 0$. Step three is by (A.6) and (A.9). By the null sequence in (A.15) the final step holds for sufficiently large i. For $i > 2C_1$ the estimate contradicts (A.12), this proves Theorem A.1. It remains to prove the claim. This takes four steps.

Step 1. (Compactness and limit $q \in \mathcal{M}^0_{x^-,x^+}$)**.** *There is a subsequence of $\mathfrak{q}_i$, still denoted by $\mathfrak{q}_i$, whose L^∞ limit is a connecting trajectory q, in symbols*

$$\exists q \in \mathcal{M}^0_{x^-,x^+}: \qquad \mathfrak{q}_i = \exp_q \xi_i, \qquad \lim_{i\to\infty} (\|\xi_i\|_\infty + \|\xi_i\|) = 0. \tag{A.16}$$

Proof. By the Nash embedding theorem we may suppose without loss of generality that $(\Sigma, g) \hookrightarrow \mathbb{R}^N$ is isometrically embedded in some Euclidean $\mathbb{R}^N$.

The sequence $\mathfrak{q}_i\colon \mathbb{R} \to \Sigma \hookrightarrow \mathbb{R}^N$ has a uniform $W^{1,2}(I_T)$ bound: By compactness of Σ there is a uniform C^0 bound. For $r \geq 1$ there is the estimate

$$\int_{-\infty}^{\infty} \left(\frac{1}{1+s^2}\right)^r ds \leq 2 + 2\int_1^{\infty} \frac{1}{s^{2r}}\, ds = 2\left(1 + \tfrac{1}{2r-1}\right) \leq 4.$$

Thus, by assumption (A.10), we get the bound

$$\|\partial_s \mathfrak{q}_i\|_{L^r(I_T)} \leq \|\partial_s \mathfrak{q}_i\|_r \leq 4c_0, \qquad r \in [1,\infty], \tag{A.17}$$

uniformly for all i and $T > 0$.
Thus, by the Arzelà-Ascoli and the Banach-Alaoglu theorems, a suitable subsequence, still denoted by $\mathfrak{q}_i$, converges strongly in C^0 and weakly in $W^{1,2}$ on every compact interval I_T to some continuous map $q\colon \mathbb{R} \to \Sigma$ which is locally of class $W^{1,2}$. Hence the sequence $\partial_s \mathfrak{q}_i + \nabla f(\mathfrak{q}_i)$ converges weakly to $\partial_s q + \nabla f(q)$. On the other hand, by (A.11), it converges to zero in L^2. By uniqueness of limits q satisfies the base equation $\partial_s q = -\nabla f(q) \in C^0(\mathbb{R}, \Sigma)$ almost everywhere. Thus q is smooth by bootstrapping.
The limit q is non-constant: Since $\mathfrak{q}_i(0)$ converges to $q(0)$ and $f \in C^0$ we get

$$f(q(0)) = \lim_i f(\mathfrak{q}_i(0)) = d_*$$

by (A.13). But d_* is a regular value of f, thus $\partial_s q(0) = -\nabla f(q(0)) \neq 0$.
The limit of the sequence $\mathfrak{q}_i$, the smooth solution q of (1.4), has finite energy and distinct asymptotic limits $y^\mp \in \operatorname{Crit} f$: Any solution is of finite energy, see [FW, Le. 3.3]. Proposition A.5 (ii) provides $y^\mp \in \operatorname{Crit} f$ and these are distinct since $\partial_s q \neq 0$ and f decays strictly along non-constant trajectories.
Broken trajectory: By standard arguments, see e.g. Lemma 3.6, a subsequence of $\mathfrak{q}_i$ converges to a k-fold broken trajectory which, in fact, must be unbroken by Morse index difference 1 of $x^\mp$.[9] Thus $q \in \mathcal{M}^0_{x^-,x^+}$ and this proves the first assertion of Step 1.

It remains to prove nullity of the limits in (A.16). We abbreviate $d :=$ dist. Since we have only C^0_{loc} convergence $\mathfrak{q}_i \to q$, we decompose, for $T > 0$, the real line $\mathbb{R} = (-\infty, -T) \cup [-T, T] \cup (T, \infty) = I_T^- \cup I_T \cup I_T^+$ into

[9] for any two next neighbor break points $\operatorname{ind}_f(y^-) - \operatorname{ind}_f(y^+) = \operatorname{index} D^0_q = \dim\ker D^0_q \geq 1$ since D^0_q is surjective by Morse-Smale and $\partial_s q \in \ker D^0_q$, but the difference between $x^\mp$ is 1

three pieces. Along the non-compact ends $I_T^\mp$ we define tangent vectors $\xi_i^\mp(s), \xi^\mp(s) \in T_{x^\mp}\Sigma$ by

$$\mathfrak{q}_i(s) = \exp_{x^\mp} \xi_i^\mp(s), \qquad q(s) = \exp_{x^\mp} \xi^\mp(s),$$

for $\mp s > 1$ large. Given $\varepsilon \in (0, \iota_\Sigma)$, we show that there is $T = T(\varepsilon) > 0$ with

$$|\xi_i(s)| = d(\mathfrak{q}_i(s), q(s)) \le d(\mathfrak{q}_i(s), x^+) + d(x^+, q(s)) < \tfrac{\varepsilon}{3} \tag{A.18}$$

for all i and $s > T$; similarly for x^-. On the other hand, by $C^0_{\rm loc}$ convergence, there is i_ε with $\|\xi_i\|_{L^\infty(I_T)} < \varepsilon/3$ whenever $i \ge i_\varepsilon$. Putting things together

$$\begin{aligned}
\|\xi_i\|_\infty &= \|\xi_i\|_{L^\infty(I_T^-)} + \|\xi_i\|_{L^\infty(I_T)} + \|\xi_i\|_{L^\infty(I_T^+)} \\
&\le \sup_{I_T^-} \big(d(\mathfrak{q}_i, x^-) + d(x^-, q)\big) + \|\xi_i\|_{L^\infty(I_T)} + \sup_{I_T^+} \big(d(\mathfrak{q}_i, x^+) + d(x^+, q)\big) \\
&\le \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

$\forall i \ge i_\varepsilon$. To conclude the proof that the L^∞ limit in (A.16) is zero it remains to show (A.18). To this end use the decay hypothesis (A.10) to obtain

$$d(\mathfrak{q}_i(\sigma), x^+) \le \operatorname{length}(\mathfrak{q}_i|_{[\sigma,\infty)}) = \int_\sigma^\infty \frac{c_0}{1+s^2}\, ds \le \int_\sigma^\infty \frac{c_0}{s^2}\, ds = \frac{c_0}{\sigma} < \frac{\varepsilon}{6}$$

for all i and $\sigma > 6c_0/\varepsilon$. Concerning q use exponential decay

$$|\partial_s q(s)| \le c e^{-\delta |s|}, \quad s \in \mathbb{R}, \tag{A.19}$$

with constants[10] $c, \delta > 0$, as provided by Proposition A.5, in order to obtain

$$d(x^+, q(\sigma)) \le \operatorname{length}(q|_{[\sigma,\infty)}) \le \int_\sigma^\infty c e^{-\delta s}\, ds = \frac{c}{\delta}\, e^{-\delta\sigma} < \frac{\varepsilon}{6}$$

for every $\sigma > -\delta^{-1} \ln(\varepsilon\delta/6c)$. The two distance estimates prove (A.18).

To prove nullity of the L^2 limit in (A.16) we proceed similarly as above, but now we integrate over (T, ∞) the square of (A.18). Using the distance

[10] A priori the constants c, δ depend on $q \in \mathcal{M}^0_{x^-,x^+}$, but due to the gauge condition (A.13) and the index difference one assumption there are only finitely many such q. Moreover, there are only finitely many critical points of f, so we can choose c and δ to only depend on f, g.

estimates we get

$$\int_T^\infty d(\mathfrak{q}_i(\sigma), x^+)^2\, d\sigma \le \int_T^\infty \frac{c_0^2}{\sigma^2}\, d\sigma = \frac{c_0^2}{T}$$

whose right hand side is independent of i and goes to zero, as $T \to \infty$, and

$$\int_T^\infty d(x^+, q(\sigma))^2\, d\sigma \le \frac{c^2}{\delta^2} \int_T^\infty e^{-\delta\sigma}\, d\sigma = \frac{c^2}{\delta^3}\, e^{-\delta T}$$

which also goes to zero, as $T \to \infty$. This proves (A.16) and Step 1. □

Step 2 (Distance to shifted limit q_σ)**.** *Let $\Delta_i := \|\xi_i\|_\infty + \|\xi_i\|$ be the null sequence from (A.16) and C_0 the constant in (A.14). Then there are a constant $\sigma_0 > 0$ and an integer $i_0 \ge 1$ such that for all $\sigma \in [-\sigma_0, \sigma_0]$ and $i \ge i_0$ the following is true. Firstly, there is a vector field $\eta = \eta(\sigma, i)$ along the shifted connecting trajectory $q_\sigma := q(\cdot + \sigma)$, of bounded length and determined by*

$$\mathfrak{q}_i = \exp_{q_\sigma} \eta, \qquad \|\eta\|_\infty < \iota_\Sigma. \tag{A.20}$$

Furthermore, there is a constant $c_2 = c_2(\sigma_0) > 0$ such that

$$\|\eta\|_\infty \le \Delta_i + C_0|\sigma|, \qquad \|\eta\| \le 2\Delta_i + c_2|\sigma|, \qquad \|\nabla_s \eta\| \le c_2,$$

whenever $\sigma \in [-\sigma_0, \sigma_0]$ and $i \ge i_0$.

Proof. Abbreviate $d :=$ dist. Given $\sigma \in \mathbb{R}$, then for every $s \in \mathbb{R}$ it holds that

$$d(q(s), q(s+\sigma)) \le |\sigma| \int_0^1 |\partial_s q(s + r\sigma)|\, dr \le \begin{cases} |\sigma| \cdot \|\partial_s q\|_\infty \le |\sigma| C_0 \\ |\sigma| \cdot c e^{\delta|\sigma|} e^{-\delta|s|} \end{cases} \tag{A.21}$$

where inequality one involves the length of the path $[0,1] \ni r \mapsto q(s + r\sigma)$ and the upper estimate uses (A.14), whereas the lower estimate uses (A.19). By the triangle inequality and the upper estimate we obtain that

$$d(\mathfrak{q}_i(s), q_\sigma(s)) \le d(\mathfrak{q}_i(s), q(s)) + d(q(s), q_\sigma(s)) \le \Delta_i + C_0|\sigma| \tag{A.22}$$

for all i and σ. Existence of σ_0 and i_0: For $|\sigma|$ small and i large the right hand side is $< \iota_\Sigma$. In this case the left hand side equals $|\eta(s)|$ and this proves the L^∞ estimate for $\eta = \eta(\sigma, i)$. Concerning the L^2 estimate, by (A.22), we get

$$\|\eta\|^2 \le 2\int_{\mathbb{R}} d(\mathfrak{q}_i(s), q(s))^2 + d(q(s), q_\sigma(s))^2\, ds \le 2\Delta_i^2 + 2|\sigma|^2 \tfrac{2c^2 e^{2\delta\sigma_0}}{\delta}$$

where inequality two uses the lower estimate in (A.21). As in (A.8) we get

$$\partial_s \mathfrak{q}_i = e_1(q_\sigma, \eta)\partial_s q_\sigma + e_2(q_\sigma, \eta)\nabla_s \eta. \tag{A.23}$$

Because $e_2(q_\sigma, 0) = \mathbb{1}$ is the identity and $\|\eta\|_\infty$ is as small as we wish by choosing $\sigma_0 > 0$ smaller and i_0 larger, we can resolve for $\nabla_s \eta$ and obtain an L^2 bound, uniform in i and σ, since $\|\partial_s \mathfrak{q}_i\| \le 4c_0$, by (A.17), and $\|\partial_s q_\sigma\|^2 = \|\partial_s q\|^2 = E^0(q) \le \operatorname{osc} f$. This proves Step 2. □

Step 3 (Time shift to adjoint image)**.** *For $\sigma \in [-\sigma_0, \sigma_0]$ consider the function $\theta_i(\sigma) := -\langle \partial_s q_\sigma, \eta\rangle$ where $\eta = \eta(\sigma, i)$ is the vector field along q_σ determined by (A.20) and $\langle\cdot,\cdot\rangle$ is the $L^2(\mathbb{R}, q_\sigma^* T\Sigma)$ inner product. The zeroes of the function are characterized by*

$$\theta_i(\sigma) = 0 \quad \Longleftrightarrow \quad \eta \in \operatorname{im}(D^0_{q_\sigma})^*.$$

Moreover, there exist new constants $\sigma_0, c_3 > 0$ and $i_0 \in \mathbb{N}$ such that

$$|\theta_i(0)| \le c_3 \Delta_i, \qquad \tfrac{d}{d\sigma}\theta_i(\sigma) \ge \tfrac{c^*}{2},$$

whenever $\sigma \in [-\sigma_0, \sigma_0]$ and $i \ge i_0$ and where $c^* := f(x^-) - f(x^+) > 0$.

Proof. The derivative $\partial_s q_\sigma$ generates $\ker D^0_{q_\sigma}$: Firstly, it is element, as follows by linearizing (1.4) at q_σ. Secondly, the kernel has dimension 1 as follows from the Fredholm index being the asymptotic's Morse index difference, that is one, and $D^0_{q_\sigma}$ being surjective by Morse-Smale. '$\Leftarrow$' true since $\partial_s q_\sigma \in \ker D^0_{q_\sigma}$. '$\Rightarrow$' true since $\eta \perp \ker D^0_{q_\sigma} = \operatorname{coker}(D^0_{q_\sigma})^*$.

Since $\eta = \eta(\sigma, i)$ reduces to ξ_i at $\sigma = 0$, by Cauchy-Schwarz we get that

$$|\theta_i(0)| \le \|\partial_s q_\sigma\| \cdot \|\eta\| = \sqrt{E^0(q_\sigma)} \cdot \|\xi_i\| \le \operatorname{osc} f \cdot \Delta_i.$$

We abbreviate $e_i = e_i(q_\sigma, \eta)$. Differentiate $\mathfrak{q}_i = \exp_{q_\sigma} \eta$ with respect to σ and resolve for $\nabla_\sigma \eta = -e_2^{-1} e_1 \partial_s q_\sigma$, then add zero to obtain

$$\begin{aligned} \tfrac{d}{d\sigma}\theta_i(\sigma) &= -\langle \nabla_s \partial_s q_\sigma, \eta\rangle - \left\langle \partial_s q_\sigma, -\partial_s q_\sigma + (\mathbb{1} - e_2^{-1} e_1)\partial_s q_\sigma \right\rangle \\ &\ge \|\partial_s q\|^2 - \|\nabla_s \partial_s q\| \cdot \|\eta\| - \tfrac{1}{4}\|\partial_s q\|^2 \\ &\ge \tfrac{1}{2}\|\partial_s q\|^2 = \tfrac{1}{2} E^0(q) = \tfrac{1}{2} c^*. \end{aligned}$$

Inequality one uses Cauchy-Schwarz, that $e_1(q_\sigma, 0) = \mathbb{1} = e_2(q_\sigma, 0)$, and that $\|\eta\|_\infty$ is arbitrarily small by enlarging i_0 and shrinking σ_0 in Step 2 so that the linear map $\mathbb{1} - e_2(q_\sigma(s), \eta(s))^{-1} e_1(q_\sigma(s), \eta(s)))$ is of norm $\le 1/4$ at every time $s \in \mathbb{R}$.
For inequality two we used the following. By exponential decay (A.19) in combination with (A.5) we get that $\|\nabla_s \partial_s q\| \le c_4$ for some constant c_4. Now enlarge i_0 and shrink σ_0 so that by Step 2 it holds $\|\eta\| \le c^*/4c_4$. We also use the energy identity. This proves Step 3. □

Step 4. *We prove the claim.*

Proof. By Step 3 there exists, for every $i \ge i_0$, an element $\sigma_i \in [-\sigma_0, \sigma_0]$ such that $\theta_i(\sigma_i) = 0$ and $|\sigma_i| \le c_3 \Delta_i \frac{2}{c_*}$. Set $\eta_i := \eta(\sigma_i, i)$. Then $\eta_i \in \mathrm{im}\,(D^0_{q_{\sigma_i}})^*$, still by Step 3, and by Step 2 we have, enlarge i_0 if necessary, the estimates

$$\|\eta_i\|_\infty \le \Delta_i \left(1 + C_0 2 c_3 / c^*\right), \qquad \|\eta_i\|_{1,2}^2 \le \Delta_i \left(2 + c_2 2 c_3 / c^*\right) + c_2^2 \le 2 c_2^2.$$

This proves (A.15), hence the claim, so Step 4, and Theorem A.1. □

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