Approximation by Power Series of Functions

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Abstract

Derivative-matching approximations are constructed as power series built from functions. The method assumes the knowledge of special values of the Bell polynomials of the second kind, we refer to the literature where such formulas can be found.

Introduction

Given a function f and a point of expansion x_0, it is customary to say that the Taylor polynomial of degree one is the best linear approximation of f at x_0, that the Taylor polynomial of degree two is the best quadratic approximation, etc... In this spirit[1], we present here several new approximations $\mathcal{A}_i$ of f such that

$$\frac{d^n}{dx^n} f\left(x\right)|_{x=0} = \frac{d^n}{dx^n} \mathcal{A}_i\left(x\right)|_{x=0}, \quad n \in \mathbb{N}_0, \tag{1}$$

where, without loss of generality, we assume that the expansion is done at $x = 0$ (the expansion can be shifted to arbitrary point x_0 by shifting its argument). We denote the equality (1) by $f \approx \mathcal{A}_i$.

1 Power series built from functions

We build $\mathcal{A}_i$ as a power series of some properly chosen function g following the construction from Sec. 4.1.2 of [1]. We propose

$$\mathcal{A}_i\left(x\right) = \sum_{n=0}^{\infty} a_n \left[g\left(x\right)\right]^n \approx f(x) \text{ with } g\left(0\right) = 0 \text{ and } g'\left(0\right) \neq 0. \tag{2}$$

The existence of a non-zero derivative at zero implies g can be inverted on some neighborhood of zero $x \equiv g^{-1}\left(y\right)$. We have

$$f\left[g^{-1}\left(y\right)\right] \approx \sum_{n=0}^{\infty} a_n y^n,$$

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[1] We do not address questions of the convergence for $x \neq 0$.

i.e. the expansion coefficients a_n are given by the power expansion coefficients of $f\left(g^{-1}\right)$

$$a_n = \frac{1}{n!}\frac{d^n}{dx^n} f\left(g^{-1}\left(x\right)\right)|_{x=0}. \tag{3}$$

This can be written in terms of the Faà di Bruno's formula, where the Bell polynomials of the second kind $B_{n,k}$ appear

$$a_n = \frac{1}{n!}\sum_{k=0}^{n} d_k^f B_{n,k}(d_1^{g^{-1}}, d_2^{g^{-1}}, \ldots, d_{n-k+1}^{g^{-1}}); \quad d_n^h \equiv \frac{d^n}{dx^n} h\left(x\right)|_{x=0}.$$

In [1] we presented only few expansions, here we systematically review the existing formulas for special values of the Bell polynomials [2] and propose a larger number of them[2].

With the aim to keep the text brief, we provide our results as a list where only the necessary information is summarized. We present (when possible) explicit forms of g and g^{-1} and also the formula for the Belle polynomial values[3]. We use

$$\begin{aligned}
B_{n,k}(\overrightarrow{d}^{g^{-1}}) &\equiv B_{n,k}(d_1^{g^{-1}}, d_2^{g^{-1}}, \ldots, d_{n-k+1}^{g^{-1}}),\\
\left\langle\alpha\right\rangle_n &= \prod_{k=0}^{n-1}\left(x-k\right) \qquad \text{(falling factorial)},\\
0^0 &= 1,\\
W\left(x\right) &\to \text{principal branch of the Lambert W function},\\
\left[\begin{array}{c} n\\ m \end{array}\right] &\to \text{Stirling number of the first kind},\\
\left[\!\!\left[\begin{array}{c} n\\ m \end{array}\right]\!\!\right] &\to \text{Stirling number of the seconf kind},
\end{aligned}$$

where a closed formula is available only for the latter Stirling numbers

$$\left[\!\!\left[\begin{array}{c} n\\ m \end{array}\right]\!\!\right] = \frac{1}{m!}\sum_{k=0}^{m}\left(-1\right)^k \binom{m}{k}\left(m-k\right)^n.$$

When it is necessary to extend the definition of f to x_0 we use the notation

$$f(x_0) \doteq h\left(x_0\right) \Leftrightarrow f(x_0) = \lim_{y\to x_0^{(\pm)}} h\left(y\right),$$

where the exact version of the limit (left, right, both sides) depends on the context.

[2] We include also those from [1], so that we provide a complete list of approximations of this type.

[3] We want to provide the full information needed for an eventual implementation so that the reader does not need to look into the literature we cite.

2 List of expansions

We separate cases where an explicit formula for g is found and those where it is not. The expansion is for all cases constructed as

$$\mathcal{A}_i\left(x\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{k=0}^{n}d_k^f B_{n,k}(\overrightarrow{d}^{\,g^{-1}})\right]\left[g\left(x\right)\right]^n.$$

2.1 Formulas with explicit expression for g

1. Logarithm-based expansion

$$\begin{aligned}g\left(x\right)=\ln\left(x+1\right); \quad g^{-1}\left(x\right)=\exp\left(x\right)-1,\\ B_{n,k}(\overrightarrow{d}^{\,g^{-1}})=B_{n,k}(1,1,1,\ldots)=\left[\!\!\left[\begin{matrix}n\\k\end{matrix}\right]\!\!\right].\end{aligned}$$

2. Exponential-based expansion

$$\begin{aligned}g\left(x\right)=1-e^{-x}; \quad g^{-1}\left(x\right)=-\ln\left(1-x\right),\\ B_{n,k}(\overrightarrow{d}^{\,g^{-1}})=B_{n,k}(0!,1!,2!,\ldots)=(-1)^{n-k}\left[\begin{matrix}n\\k\end{matrix}\right].\end{aligned}$$

3. Expansion with inverse hyperbolic sine

$$\begin{aligned}g\left(x\right)=\operatorname{asinh}(x); \quad g^{-1}\left(x\right)=\sinh(x),\\ B_{n,k}(\overrightarrow{d}^{\,g^{-1}})=B_{n,k}(1,0,1,0,1\ldots)=\frac{1}{2^k k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}(k-2l)^n.\end{aligned}$$

4. Arcus-sine-based expansion

$$\begin{aligned}g\left(x\right)&=\arcsin(x); \quad g^{-1}\left(x\right)=\sin(x),\\ B_{n,k}(\overrightarrow{d}^{\,g^{-1}})&=B_{n,k}(1,0,-1,0,1\ldots),\\ &=\frac{(-1)^k}{2^k k!}\cos\left[\frac{(n-k)\,\pi}{2}\right]\sum_{q=0}^{k}(-1)^q\binom{k}{q}(2q-k)^n.\end{aligned}$$

5. Expansion in powers of $\sqrt[\alpha]{x+1}-1$

$$\begin{aligned}g\left(x\right)&=\sqrt[\alpha]{x+1}-1; \quad g^{-1}\left(x\right)=(1+x)^\alpha-1; \quad \alpha\in\mathbb{R}\setminus\{0\}\\ B_{n,k}(\overrightarrow{d}^{\,g^{-1}})&=B_{n,k}\left(\langle\alpha\rangle_1,\langle\alpha\rangle_2,\langle\alpha\rangle_3,\ldots\right),\\ &=\frac{(-1)^k}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\langle\alpha l\rangle_n.\end{aligned}$$

Notable spacial cases (polynomial and rational) happen for $\alpha=\pm 1/n$, $n\in\mathbb{N}$.

6. Square-root-based expansion

$$
\begin{aligned}
g\left(x\right) &= \sqrt{2x+w^2}-w; \quad g^{-1}\left(x\right)=\frac{1}{2}x^2+wx; \quad w\in\mathbb{R}\backslash\left\{0\right\},\\
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}\left(w,1,0,0,0,\ldots\right),\\
&= \frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}w^{2k-n}.
\end{aligned}
$$

7. Polynomial expansion

$$
\begin{aligned}
g\left(x\right) &= \frac{x^2-2\sqrt{\alpha}x}{\beta}; \quad g^{-1}\left(x\right)=\sqrt{\alpha+\beta x}-\sqrt{\alpha}; \quad \alpha,\beta\in\mathbb{R}\backslash\left\{0\right\},\\
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}\left(d_1^{g^{-1}},d_2^{g^{-1}},\ldots,d_{n-k+1}^{g^{-1}}\right),\\
&= \left(-1\right)^{n+k}\left[2\left(n-k\right)-1\right]!!\left(\frac{\beta}{2}\right)^n\binom{2n-k-1}{2\left(n-k\right)}\frac{1}{\alpha^{n-k/2}},
\end{aligned}
$$

where

$$
d_n^{g^{-1}}=\alpha^{\frac{1}{2}-n}\beta^n\prod_{k=1}^{n}\left(k+\frac{1}{2}-n\right).
$$

8. Expansion with the square root in the denominator

$$
\begin{aligned}
g\left(x\right) &= 1-\frac{1}{\sqrt{x+1}}; \quad g^{-1}\left(x\right)=\frac{1}{\left(x-1\right)^2}-1,\\
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}(2!,3!,4!,\ldots)=\frac{n!}{k!}\sum_{l=0}^{k}\left(-1\right)^{k-l}\binom{k}{l}\binom{n+2l-1}{n}.
\end{aligned}
$$

9. Expansion with fraction including square root

$$
\begin{aligned}
g\left(x\right) &= \frac{-1+\sqrt{4x^2+1}}{2x}; \quad g^{-1}\left(x\right)=\frac{x}{1-x^2},\\
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}(1!,0,3!,0,5!,0\ldots)=\frac{1+\left(-1\right)^{n+k}}{2}\frac{n!}{k!}\binom{\frac{n+k}{2}-1}{k-1}.
\end{aligned}
$$

10. Expansion with the Lambert function

$$
\begin{aligned}
g\left(x\right) &= W\left[e^{w-1}\left(w+x-1\right)\right]+1-w, \quad w\in\mathbb{R}\backslash\left\{0\right\},\\
g^{-1}\left(x\right) &= \left(w+x-1\right)e^x+1-w,\\
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}(w,w+1,w+2,\ldots),\\
&= k^{n-k}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}\left[\sum_{q=0}^{n-k}\frac{\left(-1\right)^q}{k^q}\binom{n-k}{q}\frac{\left[\!\left[\begin{matrix}l+q\\ l\end{matrix}\right]\!\right]}{\binom{l+q}{l}}\right]\left(w-1\right)^l.
\end{aligned}
$$

11. Second expansion with the Lambert function

$$g(x) \doteq \frac{W\left[-e^{-(x+1)}(x+1)\right]}{x+1}+1; \quad g^{-1}(x) \doteq -\frac{\ln(1-x)}{x}-1,$$

$$B_{n,k}(\overrightarrow{d}^{g^{-1}}) = B_{n,k}\left(\frac{1!}{2},\frac{2!}{3},\frac{3!}{4},\ldots\right),$$

$$= \frac{(-1)^{n-k}}{k!}\sum_{m=0}^{k}(-1)^m\binom{k}{m}\frac{\left[\begin{matrix} n+m \\ m \end{matrix}\right]}{\binom{n+m}{m}}.$$

As readily seen form the argument of the function W(which is defined from $-1/e$ to ∞), this approximation is valid in the right neighborhood of zero.

12. Third expansion with the Lambert function

$$g(x) \doteq -\frac{W\left(-\frac{\exp\left(-\frac{1}{1+x}\right)}{1+x}\right)+xW\left(-\frac{\exp\left(-\frac{1}{1+x}\right)}{1+x}\right)+1}{1+x},$$

$$g^{-1}(x) \doteq \frac{e^x-1}{x}-1,$$

$$B_{n,k}(\overrightarrow{d}^{g^{-1}}) = B_{n,k}\left(\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right),$$

$$= \frac{n!}{(n+k)!}\sum_{l=0}^{k}(-1)^{k-l}\binom{n+k}{k-l}\left[\!\left[\begin{matrix} n+l \\ l \end{matrix}\right]\!\right].$$

As readily seen form the argument of the function W(which is defined from $-1/e$ to ∞), this approximation is valid in the left neighborhood of zero.

2.2 Formulas without explicit expression for g

With the function g^{-1} known, one can use numerical or approximation methods to get g in the proximity of zero.

1. $g^{-1}(x) = (w-1+e^x)x; \quad w \neq 0$

$$B_{n,k}(\overrightarrow{d}^{g^{-1}}) = B_{n,k}(w,2,3,4,\ldots) = \binom{n}{k}\sum_{r=0}^{k}\binom{k}{r}(k-r)^{n-k}(w-1)^r.$$

2. $g^{-1}(x) = e^x(x-2)-x+2$

$$B_{n,k}(\overrightarrow{d}^{g^{-1}}) = B_{n,k}(-2,0,1,2,3,\ldots),$$

$$= \sum_{r=0}^{k} r!\binom{n}{r}\binom{k}{r}(-2)^{k-r}\left[\!\left[\begin{matrix} n-r \\ k \end{matrix}\right]\!\right].$$

3. $g^{-1}(x) = \left(2e^x - x^2 - 2x - 2\right) / \left(2x^2\right)$

$$
\begin{aligned}
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}(\frac{1}{2.3}, \frac{1}{3.4}, \ldots), \\
&= \frac{(-1)^k}{[2\,(n+k)]!!} \binom{n}{k} \sum_{m=0}^{n-k} m!\, \langle -2k \rangle_{n-k-m} \binom{n-k}{m} \binom{n+k}{m} \\
&\times \sum_{l=0}^{k} (-1)^l \binom{k}{l} \sum_{q=0}^{n+k} 2^q \, \langle k-l \rangle_{n+k-q} \binom{n+k}{q} \frac{\left[\!\left[\begin{matrix} l+q \\ l \end{matrix} \right]\!\right]}{\binom{l+q}{l}}.
\end{aligned}
$$

4. $\left(6xe^x - 12e^x - x^3 + 6x + 12\right) / \left(6x^3\right)$

$$
\begin{aligned}
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}(\frac{1}{3.4}, \frac{1}{4.5}, \ldots), \\
&= \frac{(-1)^k \binom{n}{k}}{(n+2k)!6^k} \sum_{q=0}^{n-k} \langle -3k \rangle_q \,(n-k-q)! \binom{n-k}{q} \binom{n+2k}{n-k-q} \\
&\times \sum_{l=0}^{k} 12^l \binom{k}{l} \sum_{p=0}^{n+2k} (n+2k-p)! \binom{n+2k}{p} \frac{\left[\!\left[\begin{matrix} p+l \\ l \end{matrix} \right]\!\right]}{\binom{p+l}{l}} \\
&\times \sum_{s=0}^{k-l} \binom{k-l}{s} \frac{(-6)^s}{(n-p+2l+2s)!} \sum_{\beta=1}^{s} \binom{s}{\beta} \beta^{n-p+2l+2s}.
\end{aligned}
$$

5. $g^{-1}(x) = \alpha + (\alpha + a_1 - 1)x + \frac{1}{2}(\alpha + a_2 - 2)x^2 + (x - \alpha)e^x; \quad a_1 \neq 0$

$$
\begin{aligned}
B_{n,k}(\overrightarrow{d}^{\,g^{-1}}) &= B_{n,k}(a_1, a_2, 3-\alpha, 4-\alpha, 5-\alpha, \ldots), \\
&= \frac{n!}{k!} \sum_{m=0}^{k} \binom{k}{m} \sum_{p+q+r=n-k}^{k} (-\alpha)^{m-q} \,(a_1+\alpha)^{k-m-p} \\
&\times \left(\frac{a_2+\alpha}{2} - 1\right)^p \frac{\langle k-m \rangle_p \,\langle m \rangle_q}{p!q!r!} \frac{\left[\!\left[\begin{matrix} r+m \\ m \end{matrix} \right]\!\right]}{\binom{r+m}{m}}.
\end{aligned}
$$

Function g can be expressed in terms of the Lambert W for $a_1 = 1 - \alpha$, $a_2 = 2 - \alpha$, which however corresponds to the case 11 from the previous section.

References

[1] Liptaj Andrej. General approach to function approximation. — 2022. — Access mode: https://arxiv.org/abs/2201.07983.

[2] Special values of the Bell polynomials of the second kind for some sequences and functions / Feng Qi, Da-Wei Niu, Dongkyu Lim, Yong-Hong Yao // Journal of Mathematical Analysis and Applications. — 2020. — Vol. 491, no. 2. — P. 124382. — Access mode: https://www.sciencedirect.com/science/article/pii/S0022247X20305448.