REDUCTION FORMULAS OF THE COSINE OF INTEGER FRACTIONS OF π

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Abstract. The power of some cosines of integer fractions π/n allows a reduction to lower powers of the same angle. These are tabulated in the format

$$\sum_{i=0}^{\lfloor n/2\rfloor} a_i^{(n)} \cos^i \frac{\pi}{n} = 0; \quad (n = 2, 3, 4, \ldots).$$

Related expansions of Chebyshev Polynomials $T_n(x)$ and factorizations of $T_n(x)+1$ are also given. [viXra:2210.0026]

1. Description

1.1. **Chebyshev Coefficients.** Each first formula in the list given in Section 2 is an expansion [12, 1.331.3][9] of

(1)
$$\cos(n\varphi) = 2^{n-1}\cos^n\varphi + n\sum_{k=1}^{\lfloor n/2\rfloor}(-1)^k\frac{1}{k}\binom{n-k-1}{k-1}2^{n-2k-1}\cos^{n-2k}\varphi; \quad (n>0),$$

providing an extension of the Tables [12, 1.335] which cover the cases up to $n = 7$. With

$$\Re e^{in\varphi} = \cos(n\varphi) = T_n(x), \tag{2}$$

this is also an extension of the Table [1, 22.3] of the coefficients of the Chebyshev Polynomials [1, 22.3.6][2, (7)], and of sequence A053120 of the OEIS [10]. The shortcut $x \equiv \cos\varphi$ is used throughout.

1.2. **Cosine Reductions.** Each second formula for a n is a Gröbner-base reduction [13, 17] of the equation $\cos(n\varphi) = -1$ for $\varphi = \pi/n$, i.e., obtained through factorization of the polynomial $T_n(x) + 1$ trivially derived from each first formula.

Example 1. *For $n = 9$, Eq. (22), the factorization is $T_9(x)+1 = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x + 1 = (x + 1)(2x - 1)^2(8x^3 - 6x - 1)^2$.*

The existence of a non-trivial factorization is demonstrated via [1, 22.7.24] and [1, 22.12.5], see [8, 3, 21, 14]:

(3)
$$T_n(x)+1 = \begin{cases} T_{2m}(x)+1 = 2T_m^2(x) & ; \quad (n\equiv 2m, \text{ even}); \\ T_{2m+1}(x)+1 & \\ = (x+1)[\sum_{i=0}^{2m-1} 2(-1)^i(i+1)T_{2m-i}(x) + (2m+1)] & \\ = (x+1)[1-2(1-x)\sum_{k=0}^{m-1} U_{2k+1}(x)] & , \quad (n\equiv 2m+1, \text{ odd}). \end{cases}$$

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If n is not a prime number, the second line tabulated comprises the formula which appears for a multiple earlier in the table implicitly.

Example 2. *Eq. (29) for* $n = 14$ *is an implicit equation for* $\cos(\pi/14)$ *which is reduced to Eq. (19) by use of the duplication formula* [12, 1.323] $\cos^2 \frac{\pi}{14} = \frac{1}{2}[1 + \cos\frac{\pi}{7}]$ *and its powers* $\cos^4 \frac{\pi}{14} = \frac{1}{4}[1+\cos\frac{\pi}{7}]^2$ *and* $\cos^6\frac{\pi}{14} = \frac{1}{8}[1+\cos\frac{\pi}{7}]^3$.

Construction of the results would also work for composite n in the other direction.

Higher cosine powers than the leading power of the second equations follow by taking the remainder of a polynomial long division modulo the one actually shown.

Example 3. *Eq. (27) has a leading exponent 4 in* $\cos\frac{\pi}{12}$*, so polynomials of degree 4 and higher can be reduced to polynomials of degree 3 or less:*

$$\cos^5\frac{\pi}{12} = \frac{1}{16}\cos\frac{\pi}{12}\left[16\cos^4\frac{\pi}{12} - 16\cos^2\frac{\pi}{12}+1\right]+\cos^3\frac{\pi}{12}-\frac{1}{16}\cos\frac{\pi}{12} = \cos^3\frac{\pi}{12}-\frac{1}{16}\cos\frac{\pi}{12}; \tag{4}$$

$$\cos^6\frac{\pi}{12} = \left[\frac{1}{16}+\frac{1}{16}\cos^2\frac{\pi}{12}\right]\left[16\cos^4\frac{\pi}{12} - 16\cos^2\frac{\pi}{12}+1\right]+\frac{15}{16}\cos^2\frac{\pi}{12}-\frac{1}{16} = \frac{15}{16}\cos^2\frac{\pi}{12}-\frac{1}{16}. \tag{5}$$

The degrees of the minimal polynomials of $\cos(\pi/n)$ are tabulated in [10, A055034].

1.3. Representations by Irrational Numbers.

The table is also a quick guide to construct cosines in terms of square and cubic roots of rational numbers: if the second equation is quadratic, bi-quadratic etc. in $\cos(\pi/n)$, it can be solved for $\cos(\pi/n)$ with the standard formulas for this type of polynomial roots [16, 4].

Example 4. *Eq. (23) is bi-quadratic in* $\cos(\pi/10)$*, with a first root*

$$\cos^4\frac{\pi}{10} - \frac{5}{4}\cos^2\frac{\pi}{10} + \frac{5}{16} = 0 \Rightarrow \cos^2\frac{\pi}{10} = \frac{5}{8} + \sqrt{(\frac{5}{8})^2 - \frac{5}{16}} = \frac{5}{8} + \frac{\sqrt{5}}{8}, \tag{6}$$

and therefore a second root

$$\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}. \tag{7}$$

Bootstrapping from the expressions for $\cos\frac{\pi}{2}$, $\cos\frac{\pi}{3}$, $\cos\frac{\pi}{5}$ and $\cos\frac{\pi}{15}$ given below, and with [1, 4.3.21]

$$\cos\frac{z}{2} = \pm\sqrt{\frac{1+\cos z}{2}}, \tag{8}$$

one finds algebraic representations for all cosines of the form $\cos\frac{\pi}{2^k}$, $\cos\frac{\pi}{3\times 2^k}$, $\cos\frac{\pi}{5\times 2^k}$, and $\cos\frac{\pi}{3\times 5\times 2^k}$, $k = 0, 1, 2, \ldots$ [5, 11, 15]. Explicit values for $n = 5, 8, 12, 15\ldots$ have been given by Vidūnas [19, 20]. We adopt his notation:

$$\phi \equiv 1+\sqrt{5}; \quad \phi^* \equiv 1-\sqrt{5}; \quad \psi \equiv \sqrt{5+2\sqrt{5}}; \quad \psi^* \equiv \sqrt{5-2\sqrt{5}}. \tag{9}$$

The values of ψ and ψ^* are entries A019970 and A019934 of the OEIS [10]. All but the first of the decimal digits of ϕ and ϕ^* are those of entry A002163. Other ways to combine these square roots are known [6, 7].

2. Cosines

$n = 2$:

$$\Re e^{2i\varphi} = 2x^2 - 1; \quad \cos\frac{\pi}{2} = 0. \tag{10}$$

$n = 3$:

$$\Re e^{3i\varphi} = 4x^3 - 3x; \quad 2\cos\frac{\pi}{3} - 1 = 0. \tag{11}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} = \sin\frac{\pi}{6}. \tag{12}$$

$n = 4$:

$$\Re e^{4i\varphi} = 8x^4 - 8x^2 + 1; \quad 2\cos^2\frac{\pi}{4} - 1 = 0. \tag{13}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \tag{14}$$

is entry [10, A010503].

$n = 5$:

$$\Re e^{5i\varphi} = 16x^5 - 20x^3 + 5x; \quad 4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0. \tag{15}$$

$$\cos\frac{\pi}{5} = \frac{\phi}{4} = \sin\frac{3\pi}{10} \tag{16}$$

is entry [10, A019863].

$n = 6$:

$$\Re e^{6i\varphi} = 32x^6 - 48x^4 + 18x^2 - 1; \quad 4\cos^2\frac{\pi}{6} - 3 = 0. \tag{17}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \tag{18}$$

is entry [10, A010527].

$n = 7$:

$$\Re e^{7i\varphi} = 64x^7 - 112x^5 + 56x^3 - 7x; \quad 8\cos^3\frac{\pi}{7} - 4\cos^2\frac{\pi}{7} - 4\cos\frac{\pi}{7} + 1 = 0. \tag{19}$$

$\cos\frac{\pi}{7} = \sin\frac{5\pi}{14}$ is entry [10, A073052].

$n = 8$:

$$\Re e^{8i\varphi} = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1; \quad 8\cos^4\frac{\pi}{8} - 8\cos^2\frac{\pi}{8} + 1 = 0. \tag{20}$$

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2} = \sin\frac{3\pi}{8} \tag{21}$$

is entry [10, A144981].

$n = 9$:

$$\Re e^{9i\varphi} = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x; \quad 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1 = 0. \tag{22}$$

$\cos\frac{\pi}{9} = \sin\frac{7\pi}{18}$ is entry [10, A019879].

$n = 10$:

$$\Re e^{10i\varphi} = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1;$$
$$16\cos^4\frac{\pi}{10} - 20\cos^2\frac{\pi}{10} + 5 = 0. \tag{23}$$

$$\cos\frac{\pi}{10} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} = \sin\frac{2\pi}{5} \tag{24}$$

is entry [10, A019881].

$n = 11$:

$$\Re e^{11i\varphi} = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x; \tag{25}$$
$$32\cos^5\frac{\pi}{11} - 16\cos^4\frac{\pi}{11} - 32\cos^3\frac{\pi}{11} + 12\cos^2\frac{\pi}{11} + 6\cos\frac{\pi}{11} - 1 = 0. \tag{26}$$

$\cos\frac{\pi}{11} = \sin\frac{9\pi}{22}$ is entry [10, A387440].

$n = 12$:

$$\Re e^{12i\varphi} = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1;$$
$$16\cos^4\frac{\pi}{12} - 16\cos^2\frac{\pi}{12} + 1 = 0. \tag{27}$$

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin\frac{5\pi}{12} \tag{28}$$

is entry [10, A019884].

$n = 13$:

$$\Re e^{13i\varphi} = 4096x^{13} - 13312x^{11} + 16640x^9 - 9984x^7 + 2912x^5 - 364x^3 + 13x;$$
$$64\cos^6\frac{\pi}{13} - 32\cos^5\frac{\pi}{13} - 80\cos^4\frac{\pi}{13} + 32\cos^3\frac{\pi}{13} + 24\cos^2\frac{\pi}{13} - 6\cos\frac{\pi}{13} - 1 = 0.$$

$\cos\frac{\pi}{13} = \sin\frac{11\pi}{26}$ is entry [10, A387441].

$n = 14$:

$$\Re e^{14i\varphi} = 8192x^{14} - 28672x^{12} + 39424x^{10} - 26880x^8 + 9408x^6 - 1568x^4 + 98x^2 - 1;$$
$$64\cos^6\frac{\pi}{14} - 112\cos^4\frac{\pi}{14} + 56\cos^2\frac{\pi}{14} - 7 = 0. \tag{29}$$

$\cos\frac{\pi}{14} = \sin\frac{3\pi}{7}$ is entry A232735 in [10].

$n = 15$:

$$\Re e^{15i\varphi} = 16384x^{15} - 61440x^{13} + 92160x^{11} - 70400x^9 + 28800x^7 - 6048x^5 + 560x^3 - 15x;$$
$$16\cos^4\frac{\pi}{15} + 8\cos^3\frac{\pi}{15} - 16\cos^2\frac{\pi}{15} - 8\cos\frac{\pi}{15} + 1 = 0.$$

$$\cos\frac{\pi}{15} = \phi^*\frac{\sqrt{3}\psi+1}{8\sqrt{5}} = \sin\frac{13\pi}{30} \tag{30}$$

is entry A019887 in [10].

$n = 16$:

$$\Re e^{16i\varphi} = 32768x^{16} - 131072x^{14} + 212992x^{12} - 180224x^{10} + 84480x^8 - 21504x^6 + 2688x^4 - 128x^2 + 1;$$
$$128\cos^8\frac{\pi}{16} - 256\cos^6\frac{\pi}{16} + 160\cos^4\frac{\pi}{16} - 32\cos^2\frac{\pi}{16} + 1 = 0.$$

$$\cos\frac{\pi}{16} = \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{2+\sqrt{2}}} = \sin\frac{7\pi}{16} \tag{31}$$

is entry A232737 in [10].

$n = 17$:

$$\Re e^{17i\varphi} = 65536x^{17} - 278528x^{15} + 487424x^{13} - 452608x^{11} + 239360x^9 - 71808x^7 + 11424x^5 - 816x^3 + 17x;$$

$$256\cos^8\frac{\pi}{17} - 128\cos^7\frac{\pi}{17} - 448\cos^6\frac{\pi}{17} + 192\cos^5\frac{\pi}{17} + 240\cos^4\frac{\pi}{17} - 80\cos^3\frac{\pi}{17} - 40\cos^2\frac{\pi}{17} + 8\cos\frac{\pi}{17} + 1 = 0.$$

$\cos\frac{\pi}{17} = \sin\frac{15\pi}{34}$ is entry A210649 in [10].

$n = 18$:

$$\Re e^{18i\varphi} = 131072x^{18} - 589824x^{16} + 1105920x^{14} - 1118208x^{12} + 658944x^{10} - 228096x^8 + 44352x^6 - 4320x^4 + 162x^2 - 1;$$

$$64\cos^6\frac{\pi}{18} - 96\cos^4\frac{\pi}{18} + 36\cos^2\frac{\pi}{18} - 3 = 0.$$

$\cos\frac{\pi}{18} = \sin\frac{4\pi}{9}$ is entry A019889 in [10].

$n = 19$:

$$\Re e^{19i\varphi} = 262144x^{19} - 1245184x^{17} + 2490368x^{15} - 2723840x^{13} + 1770496x^{11} - 695552x^9 + 160512x^7 - 20064x^5 + 1140x^3 - 19x;$$

$$512\cos^9\frac{\pi}{19} - 256\cos^8\frac{\pi}{19} - 1024\cos^7\frac{\pi}{19} + 448\cos^6\frac{\pi}{19} + 672\cos^5\frac{\pi}{19} - 240\cos^4\frac{\pi}{19} - 160\cos^3\frac{\pi}{19} + 40\cos^2\frac{\pi}{19} + 10\cos\frac{\pi}{19} - 1 = 0.$$

$\cos\frac{\pi}{19} = \sin\frac{17\pi}{38}$ is entry A387442 in [10].

$n = 20$:

$$\Re e^{20i\varphi} = 524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14} + 4659200x^{12} - 2050048x^{10} + 549120x^8 - 84480x^6 + 6600x^4 - 200x^2 + 1;$$

$$256\cos^8\frac{\pi}{20} - 512\cos^6\frac{\pi}{20} + 304\cos^4\frac{\pi}{20} - 48\cos^2\frac{\pi}{20} + 1 = 0.$$

$$\cos\frac{\pi}{20} = \sqrt{\phi^*}\frac{\psi+\sqrt{5}}{4\sqrt{5}} = \sin\frac{9\pi}{20} \tag{32}$$

is entry A019890 in [10].

$n = 21$:

$$\Re e^{21i\varphi} = 1048576x^{21} - 5505024x^{19} + 12386304x^{17} - 15597568x^{15} + 12042240x^{13} - 5870592x^{11} + 1793792x^9 - 329472x^7 + 33264x^5 - 1540x^3 + 21x;$$

$$64\cos^6\frac{\pi}{21} + 32\cos^5\frac{\pi}{21} - 96\cos^4\frac{\pi}{21} - 48\cos^3\frac{\pi}{21} + 32\cos^2\frac{\pi}{21} + 16\cos\frac{\pi}{21} + 1 = 0.$$

$\cos\frac{\pi}{21} = \sin\frac{19\pi}{42}$ is entry A387454 in [10].

$n = 22$:

$$\Re e^{22i\varphi} = 2097152x^{22} - 11534336x^{20} + 27394048x^{18} - 36765696x^{16} + 30638080x^{14} - 16400384x^{12} + 5637632x^{10} - 1208064x^8 + 151008x^6 - 9680x^4 + 242x^2 - 1;$$

$$1024\cos^{10}\frac{\pi}{22} - 2816\cos^8\frac{\pi}{22} + 2816\cos^6\frac{\pi}{22} - 1232\cos^4\frac{\pi}{22} + 220\cos^2\frac{\pi}{22} - 11 = 0.$$

$\cos\frac{\pi}{22} = \sin\frac{5\pi}{11}$ is entry A387443 in [10].

$n = 23$:

$$\Re e^{23i\varphi} = 4194304x^{23} - 24117248x^{21} + 60293120x^{19} - 85917696x^{17} + 76873728x^{15} - 44843008x^{13} +17145856x^{11} - 4209920x^9 + 631488x^7 - 52624x^5 + 2024x^3 - 23x;$$

$$2048\cos^{11}\frac{\pi}{23} - 1024\cos^{10}\frac{\pi}{23} - 5120\cos^9\frac{\pi}{23} + 2304\cos^8\frac{\pi}{23} + 4608\cos^7\frac{\pi}{23} - 1792\cos^6\frac{\pi}{23} -1792\cos^5\frac{\pi}{23} + 560\cos^4\frac{\pi}{23} + 280\cos^3\frac{\pi}{23} - 60\cos^2\frac{\pi}{23} - 12\cos\frac{\pi}{23} + 1 = 0.$$

$\cos\frac{\pi}{23} = \sin\frac{21\pi}{46}$ is entry A387444 in [10].

$n = 24$:

$$\Re e^{24i\varphi} = 8388608x^{24} - 50331648x^{22} + 132120576x^{20} - 199229440x^{18} + 190513152x^{16} - 120324096x^{14} +50692096x^{12} - 14057472x^{10} + 2471040x^8 - 256256x^6 + 13728x^4 - 288x^2 + 1;$$

$$256\cos^8\frac{\pi}{24} - 512\cos^6\frac{\pi}{24} + 320\cos^4\frac{\pi}{24} - 64\cos^2\frac{\pi}{24} + 1 = 0.$$

$$\cos\frac{\pi}{24} = \frac{\sqrt{2\sqrt{2}+\sqrt{3}+1}}{2^{5/4}} = \sin\frac{11\pi}{24} \tag{33}$$

is entry A144982 in [10].

$n = 25$:

$$\Re e^{25i\varphi} = 16777216x^{25} - 104857600x^{23} + 288358400x^{21} - 458752000x^{19} + 466944000x^{17} -317521920x^{15} + 146227200x^{13} - 45260800x^{11} + 9152000x^9 - 1144000x^7 + 80080x^5 -2600x^3 + 25x;$$

$$1024\cos^{10}\frac{\pi}{25} - 2560\cos^8\frac{\pi}{25} + 2240\cos^6\frac{\pi}{25} - 32\cos^5\frac{\pi}{25} - 800\cos^4\frac{\pi}{25} + 40\cos^3\frac{\pi}{25} +100\cos^2\frac{\pi}{25} - 10\cos\frac{\pi}{25} - 1 = 0.$$

$\cos\frac{\pi}{25} = \sin\frac{23\pi}{50}$ is entry A387445 in [10].

$n = 26$:

$$\Re e^{26i\varphi} = 33554432x^{26} - 218103808x^{24} + 627048448x^{22} - 1049624576x^{20} + 1133117440x^{18} -825556992x^{16} + 412778496x^{14} - 141213696x^{12} + 32361472x^{10} - 4759040x^8 + 416416x^6 -18928x^4 + 338x^2 - 1;$$

$$4096\cos^{12}\frac{\pi}{26} - 13312\cos^{10}\frac{\pi}{26} + 16640\cos^8\frac{\pi}{26} - 9984\cos^6\frac{\pi}{26} + 2912\cos^4\frac{\pi}{26} - 364\cos^2\frac{\pi}{26} + 13 = 0.$$

$\cos\frac{\pi}{26} = \sin\frac{6\pi}{13}$ is entry A387446 in [10].

$n = 27$:

$$\Re e^{27i\varphi} = 67108864x^{27} - 452984832x^{25} + 1358954496x^{23} - 2387607552x^{21} + 2724986880x^{19} -2118057984x^{17} + 1143078912x^{15} - 428654592x^{13} + 109983744x^{11} - 18670080x^9 +1976832x^7 - 117936x^5 + 3276x^3 - 27x;$$

$$512\cos^9\frac{\pi}{27} - 1152\cos^7\frac{\pi}{27} + 864\cos^5\frac{\pi}{27} - 240\cos^3\frac{\pi}{27} + 18\cos\frac{\pi}{27} - 1 = 0.$$

$\cos\frac{\pi}{27} = \sin\frac{25\pi}{54}$ is entry A387447 in [10].

$n = 28$:

$$\Re e^{28i\varphi} = 134217728x^{28} - 939524096x^{26} + 2936012800x^{24} - 5402263552x^{22} + 6499598336x^{20} - 5369233408x^{18} + 3111714816x^{16} - 1270087680x^{14} + 361181184x^{12} - 69701632x^{10} + 8712704x^{8} - 652288x^{6} + 25480x^{4} - 392x^{2} + 1;$$

$$4096\cos^{12}\frac{\pi}{28} - 12288\cos^{10}\frac{\pi}{28} + 13568\cos^{8}\frac{\pi}{28} - 6656\cos^{6}\frac{\pi}{28} + 1376\cos^{4}\frac{\pi}{28} - 96\cos^{2}\frac{\pi}{28} + 1 = 0.$$

$\cos\frac{\pi}{28} = \sin\frac{13\pi}{28}$ is entry A387448 in [10].

$n = 29$:

$$\Re e^{29i\varphi} = 268435456x^{29} - 1946157056x^{27} + 6325010432x^{25} - 12163481600x^{23} + 15386804224x^{21} - 13463453696x^{19} + 8341487616x^{17} - 3683254272x^{15} + 1151016960x^{13} - 249387008x^{11} + 36095488x^{9} - 3281408x^{7} + 168896x^{5} - 4060x^{3} + 29x;$$

$$16384\cos^{14}\frac{\pi}{29} - 8192\cos^{13}\frac{\pi}{29} - 53248\cos^{12}\frac{\pi}{29} + 24576\cos^{11}\frac{\pi}{29} + 67584\cos^{10}\frac{\pi}{29} - 28160\cos^{9}\frac{\pi}{29} - 42240\cos^{8}\frac{\pi}{29} + 15360\cos^{7}\frac{\pi}{29} + 13440\cos^{6}\frac{\pi}{29} - 4032\cos^{5}\frac{\pi}{29} - 2016\cos^{4}\frac{\pi}{29} + 448\cos^{3}\frac{\pi}{29} + 112\cos^{2}\frac{\pi}{29} - 14\cos\frac{\pi}{29} - 1 = 0.$$

$\cos\frac{\pi}{29} = \sin\frac{27\pi}{58}$ is entry A387453 in [10].

$n = 30$:

$$\Re e^{30i\varphi} = 536870912x^{30} - 4026531840x^{28} + 13589544960x^{26} - 27262976000x^{24} + 36175872000x^{22} - 33426505728x^{20} + 22052208640x^{18} - 10478223360x^{16} + 3572121600x^{14} - 859955200x^{12} + 141892608x^{10} - 15275520x^{8} + 990080x^{6} - 33600x^{4} + 450x^{2} - 1;$$

$$256\cos^{8}\frac{\pi}{30} - 448\cos^{6}\frac{\pi}{30} + 224\cos^{4}\frac{\pi}{30} - 32\cos^{2}\frac{\pi}{30} + 1 = 0.$$

$\cos\frac{\pi}{30} = \sin\frac{7\pi}{15}$ is entry A019893 in [10].

$n = 31$:

$$\Re e^{31i\varphi} = 1073741824x^{31} - 8321499136x^{29} + 29125246976x^{27} - 60850962432x^{25} + 84515225600x^{23} - 82239815680x^{21} + 57567870976x^{19} - 29297934336x^{17} + 10827497472x^{15} - 2870927360x^{13} + 533172224x^{11} - 66646528x^{9} + 5261568x^{7} - 236096x^{5} + 4960x^{3} - 31x;$$

$$32768\cos^{15}\frac{\pi}{31} - 16384\cos^{14}\frac{\pi}{31} - 114688\cos^{13}\frac{\pi}{31} + 53248\cos^{12}\frac{\pi}{31} + 159744\cos^{11}\frac{\pi}{31} - 67584\cos^{10}\frac{\pi}{31} - 112640\cos^{9}\frac{\pi}{31} + 42240\cos^{8}\frac{\pi}{31} + 42240\cos^{7}\frac{\pi}{31} - 13440\cos^{6}\frac{\pi}{31} - 8064\cos^{5}\frac{\pi}{31} + 2016\cos^{4}\frac{\pi}{31} + 672\cos^{3}\frac{\pi}{31} - 112\cos^{2}\frac{\pi}{31} - 16\cos\frac{\pi}{31} + 1 = 0.$$

$\cos\frac{\pi}{31} = \sin\frac{29\pi}{62}$ is entry A387449 in [10].

$n = 32$:

$$\Re e^{32i\varphi} = 2147483648x^{32} - 17179869184x^{30} + 62277025792x^{28} - 135291469824x^{26} + 196293427200x^{24} - 200655503360x^{22} + 148562247680x^{20} - 80648077312x^{18} + 32133218304x^{16} - 9313976320x^{14} + 1926299648x^{12} - 275185664x^{10} + 25798656x^{8} - 1462272x^{6} + 43520x^{4} - 512x^{2} + 1;$$

$$32768\cos^{16}\frac{\pi}{32} - 131072\cos^{14}\frac{\pi}{32} + 212992\cos^{12}\frac{\pi}{32} - 180224\cos^{10}\frac{\pi}{32} + 84480\cos^{8}\frac{\pi}{32} - 21504\cos^{6}\frac{\pi}{32} + 2688\cos^{4}\frac{\pi}{32} - 128\cos^{2}\frac{\pi}{32} + 1 = 0.$$

$\cos\frac{\pi}{32} = \sin\frac{15\pi}{32}$ is entry A343056 in [10].

$n = 33$:

$$\Re e^{33i\varphi} = 4294967296x^{33} - 35433480192x^{31} + 132875550720x^{29} - 299708186624x^{27} +453437816832x^{25} - 485826232320x^{23} + 379364311040x^{21} - 218864025600x^{19} +93564370944x^{17} - 29455450112x^{15} + 6723526656x^{13} - 1083543552x^{11} +118243840x^9 - 8186112x^7 + 323136x^5 - 5984x^3 + 33x;$$

$$1024\cos^{10}\frac{\pi}{33} + 512\cos^9\frac{\pi}{33} - 2560\cos^8\frac{\pi}{33} - 1280\cos^7\frac{\pi}{33} + 2176\cos^6\frac{\pi}{33} +1088\cos^5\frac{\pi}{33} - 688\cos^4\frac{\pi}{33} - 344\cos^3\frac{\pi}{33} + 48\cos^2\frac{\pi}{33} + 24\cos\frac{\pi}{33} + 1 = 0$$

$\cos\frac{\pi}{33} = \sin\frac{31\pi}{66}$ is entry A387450 in [10].

$n = 34$:

$$\Re e^{34i\varphi} = 8589934592x^{34} - 73014444032x^{32} + 282930970624x^{30} - 661693399040x^{28} +1042167103488x^{26} - 1167945891840x^{24} + 959384125440x^{22} - 586290298880x^{20} +267776819200x^{18} - 91044118528x^{16} + 22761029632x^{14} - 4093386752x^{12} + 511673344x^{10} -42170880x^8 + 2108544x^6 - 55488x^4 + 578x^2 - 1;$$

$$65536\cos^{16}\frac{\pi}{34} - 278528\cos^{14}\frac{\pi}{34} + 487424\cos^{12}\frac{\pi}{34} - 452608\cos^{10}\frac{\pi}{34} + 239360\cos^8\frac{\pi}{34} -71808\cos^6\frac{\pi}{34} + 11424\cos^4\frac{\pi}{34} - 816\cos^2\frac{\pi}{34} + 17 = 0.$$

$\cos\frac{\pi}{34} = \sin\frac{8\pi}{17}$ is entry A387451 in [10].

3. Double Angle Cosines

Replacing cosines of the previous section with the double angle formula

$$\cos\alpha = \sqrt{\frac{\cos(2\alpha)+1}{2}} \tag{34}$$

extends the reduction formulas to angles which are generalized fractions $2\pi/n$.

- If the minimal polynomial is not biquadratic/even but has terms with odd exponents, further steps of squaring may be needed to remove a square root, and this may increase the order of the minimal polynomial and/or produce spurious extra solutions.
- If the minimal polynomial is even, the polynomial for the generalized fraction may split in more elementary factors and the order may become smaller.

The algebraic degrees of the polynomials are sequence [10, A023022]. If n is prime, a closed-from representation of the minimal polynomial is known [18].

2/3:

$$2\cos\frac{2\pi}{3} + 1 = 0. \tag{35}$$

$$\cos\frac{2\pi}{3} = -1/2. \tag{36}$$

2/5:

$$4\cos^2\frac{2\pi}{5} + 2\cos(\frac{2\pi}{5}) - 1 = 0. \tag{37}$$

$$\cos\frac{2\pi}{5} = 1/\phi = \sin\frac{\pi}{10} \tag{38}$$

is [10, A019827].

2/7:

$$8\cos^3\frac{2\pi}{7} + 4\cos^2(\frac{2\pi}{7}) - 4\cos(\frac{2\pi}{7}) - 1 = 0. \tag{39}$$

$$\cos\frac{2\pi}{7} = \sin\frac{3\pi}{14} \approx 0.62348980 \tag{40}$$

is [10, A362922].

2/9:

$$8\cos^3\frac{2\pi}{9} - 6\cos(\frac{2\pi}{9}) + 1 = 0. \tag{41}$$

$$\cos\frac{2\pi}{9} = \sin\frac{5\pi}{18} \approx 0.766044443 \tag{42}$$

is [10, A019859].

2/11:

$$32\cos^5\frac{2\pi}{11} + 16\cos^4\frac{2\pi}{11} - 32\cos^3\frac{2\pi}{11} - 12\cos^2(\frac{2\pi}{11}) + 6\cos(\frac{2\pi}{11}) + 1 = 0. \tag{43}$$

$$\cos\frac{2\pi}{11} = \sin\frac{7\pi}{22} \approx 0.841253533. \tag{44}$$

2/13:

(45)

$$64\cos^6(\frac{2\pi}{13}) + 32\cos^5(\frac{2\pi}{13}) - 80\cos^4(\frac{2\pi}{13}) - 32\cos^3(\frac{2\pi}{13}) + 24\cos^2(\frac{2\pi}{13}) + 6\cos(\frac{2\pi}{13}) - 1 = 0.$$

$$\cos\frac{2\pi}{13} = \sin\frac{9\pi}{26} \approx 0.8854560. \tag{46}$$

2/15:

$$16\cos^4(\frac{2\pi}{15}) - 8\cos^3(\frac{2\pi}{15}) - 16\cos^2(\frac{2\pi}{15}) + 8\cos(\frac{2\pi}{15}) + 1 = 0. \tag{47}$$

$$\cos\frac{2\pi}{15} = \sin\frac{11\pi}{30} \approx 0.913545458 \tag{48}$$

is [10, A019875].

2/17:

$$\begin{aligned} 256\cos^8(\frac{2\pi}{17}) + 128\cos^7(\frac{2\pi}{17}) - 448\cos^6(\frac{2\pi}{17}) - 192\cos^5(\frac{2\pi}{17}) + 240\cos^4(\frac{2\pi}{17}) \\ + 80\cos^3(\frac{2\pi}{17}) - 40\cos^2(\frac{2\pi}{17}) - 8\cos(\frac{2\pi}{17}) + 1 = 0. \end{aligned} \tag{49}$$

$$\cos\frac{2\pi}{17} = \sin\frac{13\pi}{34} \approx 0.932472229 \tag{50}$$

is [10, A210644].

2/19:

$$512\cos^9(\frac{2\pi}{19})+256\cos^8(\frac{2\pi}{19})-1024\cos^7(\frac{2\pi}{19})-448\cos^6(\frac{2\pi}{19})+672\cos^5(\frac{2\pi}{19}) +240\cos^4(\frac{2\pi}{19})-160\cos^3(\frac{2\pi}{19})-40\cos^2(\frac{2\pi}{19})+10\cos(\frac{2\pi}{19})+1=0. \quad (51)$$

$$\cos\frac{2\pi}{19}=\sin\frac{15\pi}{38}\approx 0.945817242. \quad (52)$$

2/21:

$$64\cos^6(\frac{2\pi}{21})-32\cos^5(\frac{2\pi}{21})-96\cos^4(\frac{2\pi}{21})+48\cos^3(\frac{2\pi}{21})+32\cos^2(\frac{2\pi}{21})-16\cos(\frac{2\pi}{21})+1=0. \quad (53)$$

$$\cos\frac{2\pi}{21}=\sin\frac{17\pi}{42}\approx 0.955572806. \quad (54)$$

2/23:

$$2048\cos^{11}(\frac{2\pi}{23})+1024\cos^{10}(\frac{2\pi}{23})-5120\cos^9(\frac{2\pi}{23})-2304\cos^8(\frac{2\pi}{23})+4608\cos^7(\frac{2\pi}{23}) +1792\cos^6(\frac{2\pi}{23})-1792\cos^5(\frac{2\pi}{23})-560\cos^4(\frac{2\pi}{23})+280\cos^3(\frac{2\pi}{23})+60\cos^2(\frac{2\pi}{23})-12\cos(\frac{2\pi}{23})-1=0. \quad (55)$$

$$\cos\frac{2\pi}{23}=\sin\frac{19\pi}{46}\approx 0.962917287 \quad (56)$$

2/25:

$$1024\cos^{10}(\frac{2\pi}{25})-2560\cos^8(\frac{2\pi}{25})+2240\cos^6(\frac{2\pi}{25})+32\cos^5(\frac{2\pi}{25})-800\cos^4(\frac{2\pi}{25}) -40\cos^3(\frac{2\pi}{25})+100\cos^2(\frac{2\pi}{25})+10\cos(\frac{2\pi}{25})-1=0. \quad (57)$$

$$\cos\frac{2\pi}{25}=\sin\frac{21\pi}{50}\approx 0.968583161. \quad (58)$$

4. Triple Angle Cosines

For angles that are three times an angle of Section 2, a substitution akin to (34) is complicated because it requires cube roots. The alternative recipe here is to start with an algebraic ansatz with a vector of unknown a_i

$$a_i\cos^i(3\alpha)+a_{i-1}\cos^{i-1}(3\alpha)+\cdots+a_0=0, \quad (59)$$

to expand all powers of $\cos(3\alpha)=4\cos^3\alpha-3\cos\alpha$, reduce them with long division to the remainder of the polynomials obtained in Section 2, and gather the coefficients in a square matrix. The computation increases the maximum exponent i until the null space of this matrix is not empty. The vector a_i of this null space reduces the value of $\cos(3\alpha)$ to zero as required above.

Example 5. *$\alpha = \pi/11$ in (26) has a minimum polynomial $z(x) = 32x^5 - 16x^4 - 32x^3 + 12x^2 + 6x - 1 = 0$. The calculation of $\cos(3\alpha)$ starts with*

$$
\begin{aligned}
(4x^3-3x)^0 &= 1;\\
(4x^3-3x)^1 &= 4x^3-3x;\\
(4x^3-3x)^2 &= 16x^6-24x^4+9x^2 \equiv \frac{1}{4} - x + 3x^2 + 2x^3 - 4x^4 \pmod{z(x)}\\
(4x^3-3x)^3 &= 64x^9-144x^7+180x^5-27x^3 \equiv \frac{1}{4} - \frac{9}{4}x - \frac{1}{2}x^2 + 3x^3 \pmod{z(x)}
\end{aligned}
$$

The coefficients $[x^r]$ of the i-th power remainders are inserted into the ith column and r-th row of the matrix

$$
\left(\begin{array}{ccccccc}
1 & 0 & 1/4 & 1/4 & 1/8 & 1/4 & \cdots\\
0 & -3 & -1 & -9/4 & -9/8 & -15/8 & \cdots\\
0 & 0 & 3 & -1/2 & 3 & -1/8 & \cdots\\
0 & 4 & 2 & 3 & 2 & 5/2 & \cdots\\
0 & 0 & -4 & 0 & -4 & -1/2 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}\right), \tag{60}
$$

which becomes a matrix with a non-trivial null space if it is grown to that 6×6 size as shown; the smaller upper-left submatrices all have nonzero determinant. The vector a_i that spans this space is $(-1/32, 3/16, 3/8, -1, -1/2, 1)$, conveniently multiplied by 32 to obtain a polynomial with integer coefficients. It turns out that this is $(-1, 6, 12, -32, -16, 32)$, the same as the minimum polynomial $z(x)$ for $\cos\alpha$ we started with.

Remark 1. *The same algorithm can be employed based on (1) to create reduction formulas for $\cos(4\alpha)$, $\cos(5\alpha)$ once the minimum polynomials for $\cos\alpha$ are known.*

3/5:

$$-1 - 2\cos(\frac{3\pi}{5}) + 4\cos^2(\frac{3\pi}{5}) = 0. \tag{61}$$

$$\cos\frac{3\pi}{5} = -\sin\frac{\pi}{10} = -\frac{\sqrt{5}-1}{4} \approx -0.3090169943749474 \tag{62}$$

is [10, A019827].

3/7:

$$1 - 4\cos(\frac{3\pi}{7}) - 4\cos^2(\frac{3\pi}{7}) + 8\cos^3(\frac{3\pi}{7}) = 0. \tag{63}$$

$$\cos\frac{3\pi}{7} = \sin\frac{\pi}{14} \approx 0.222520933956314 \tag{64}$$

is [10, A232736].

3/8:

$$1 - 8\cos^2(\frac{3\pi}{8}) + 8\cos^4(\frac{3\pi}{8}) = 0. \tag{65}$$

$$\cos\frac{3\pi}{8} = \sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}} \approx 0.38268343236508977 \tag{66}$$

is [10, A182168].

3/10:

$$5-20\cos^2(\frac{3\pi}{10})+16\cos^4(\frac{3\pi}{10})=0. \tag{67}$$

$$\cos\frac{3\pi}{10}=\sin\frac{\pi}{5}=\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\approx 0.58778525229247 \tag{68}$$

is [10, A019845].

3/11:

$$-1+6\cos(\frac{3\pi}{11})+12\cos^2(\frac{3\pi}{11})-32\cos^3(\frac{3\pi}{11})-16\cos^4(\frac{3\pi}{11})+32\cos^5(\frac{3\pi}{11})=0. \tag{69}$$

$$\cos\frac{3\pi}{11}=\sin\frac{5\pi}{11}\approx 0.654860733945285064. \tag{70}$$

3/13:

$$1-6\cos(\frac{3\pi}{13})+24\cos^2(\frac{3\pi}{13})+32\cos^3(\frac{3\pi}{13})-80\cos^4(\frac{3\pi}{13})-32\cos^5(\frac{3\pi}{13})+64\cos^6(\frac{3\pi}{13})=0. \tag{71}$$

$$\cos\frac{3\pi}{13}=\sin\frac{7\pi}{26}\approx 0.7485107481711010. \tag{72}$$

5. Sines

Replacing $x\to\sqrt{1-x^2}$ transforms the minimal polynomials of the cosines of Section 2 to sines. The manipulations mentioned at the start of Section 3 are used to remove square roots of the polynomial. Some sines of the format $\sin\frac{\pi}{n}$ have already been obtained in (12), (14), (18), (62), (64), (66) etc. and will not be resurrected here.

The degrees of the minimal polynomials of $\sin(\pi/n)$ are tabulated in [10, A055035].

$n=7$:

$$64\sin^6(\frac{\pi}{7})-112\sin^4(\frac{\pi}{7})+56\sin^2(\frac{\pi}{7})-7=0. \tag{73}$$

$$\sin\frac{\pi}{7}=\cos\frac{5\pi}{14}\approx 0.433883739 \tag{74}$$

is entry [10, A323601].

$n=9$:

$$64\sin^6(\frac{\pi}{9})-96\sin^4(\frac{\pi}{9})+36\sin^2(\frac{\pi}{9})-3=0. \tag{75}$$

$$\sin\frac{\pi}{9}=\cos\frac{7\pi}{18}\approx 0.342020143 \tag{76}$$

is entry [10, A019829].

$n=11$:

$$1024\sin^{10}(\frac{\pi}{11})-2816\sin^8(\frac{\pi}{11})+2816\sin^6(\frac{\pi}{11})-1232\sin^4(\frac{\pi}{11})+220\sin^2(\frac{\pi}{11})-11=0. \tag{77}$$

$$\sin\frac{\pi}{11}=\cos\frac{9\pi}{22}\approx 0.281732556. \tag{78}$$

$n = 12$:

$$1 - 16\sin^2(\frac{\pi}{12}) + 16\sin^4(\frac{\pi}{12}) = 0. \quad (79)$$

$$\sin\frac{\pi}{12} = \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \approx 0.258819045 \quad (80)$$

is entry [10, A019824].

$n = 13$:

$$4096\sin^{12}(\frac{\pi}{13}) - 13312\sin^{10}(\frac{\pi}{13}) + 16640\sin^8(\frac{\pi}{13}) - 9984\sin^6(\frac{\pi}{13}) + 2912\sin^4(\frac{\pi}{13}) - 364\sin^2(\frac{\pi}{13}) + 13 = 0. \quad (81)$$

$$\sin\frac{\pi}{13} = \cos\frac{11\pi}{26} \approx 0.239315664. \quad (82)$$

$n = 15$:

$$256\sin^8(\frac{\pi}{15}) - 448\sin^6(\frac{\pi}{15}) + 224\sin^4(\frac{\pi}{15}) - 32\sin^2(\frac{\pi}{15}) + 1 = 0. \quad (83)$$

$$\sin\frac{\pi}{15} = \cos\frac{13\pi}{30} = -\frac{\phi^*(\psi-\sqrt{3})}{8} \approx 0.207911690 \quad (84)$$

is entry [10, A019821].

$n = 16$:

$$1 - 32\sin^2(\frac{\pi}{16}) + 160\sin^4(\frac{\pi}{16}) - 256\sin^6(\frac{\pi}{16}) + 128\sin^8(\frac{\pi}{16}) = 0. \quad (85)$$

$$\sin\frac{\pi}{16} = \cos\frac{7\pi}{16} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2} \approx 0.195090322 \quad (86)$$

is entry [10, A232738].

$n = 17$:

(87)

$$65536\sin^{16}(\frac{\pi}{17}) - 278528\sin^{14}(\frac{\pi}{17}) + 487424\sin^{12}(\frac{\pi}{17}) - 452608\sin^{10}(\frac{\pi}{17}) + 239360\sin^8(\frac{\pi}{17}) - 71808\sin^6(\frac{\pi}{17}) + 11424\sin^4(\frac{\pi}{17}) - 816\sin^2(\frac{\pi}{17}) + 17 = 0.$$

$$\sin\frac{\pi}{17} = \cos\frac{15\pi}{34} \approx 0.183749517 \quad (88)$$

is entry [10, A241243].

$n = 18$:

$$8\sin^3(\frac{\pi}{18}) - 6\sin(\frac{\pi}{18}) + 1 = 0. \quad (89)$$

$$\sin\frac{\pi}{18} = \cos\frac{4\pi}{9} \approx 0.173648177 \quad (90)$$

is entry [10, A019819].

$n = 19$:

$$262144\sin^{18}(\frac{\pi}{19})-1245184\sin^{16}(\frac{\pi}{19})+2490368\sin^{14}(\frac{\pi}{19})-2723840\sin^{12}(\frac{\pi}{19})+1770496\sin^{10}(\frac{\pi}{19}) - 695552\sin^{8}(\frac{\pi}{19}) + 160512\sin^{6}(\frac{\pi}{19}) - 20064\sin^{4}(\frac{\pi}{19}) + 1140\sin^{2}(\frac{\pi}{19}) - 19 = 0. \quad (91)$$

$$\sin\frac{\pi}{19} = \cos\frac{17\pi}{38} \approx 0.164594590. \quad (92)$$

$n = 20$:

$$1 - 48\sin^2(\frac{\pi}{20}) + 304\sin^4(\frac{\pi}{20}) - 512\sin^6(\frac{\pi}{20}) + 256\sin^8(\frac{\pi}{20}) = 0. \quad (93)$$

$$\sin\frac{\pi}{20} = \cos\frac{9\pi}{20} = \frac{\sqrt{8-2\sqrt{10+2\sqrt{5}}}}{4} \approx 0.156434465 \quad (94)$$

is entry [10, A019818].

$n = 21$:

$$4096\sin^{12}(\frac{\pi}{21})-11264\sin^{10}(\frac{\pi}{21})+11264\sin^{8}(\frac{\pi}{21})-4992\sin^{6}(\frac{\pi}{21})+960\sin^{4}(\frac{\pi}{21})-64\sin^{2}(\frac{\pi}{21})+1 = 0. \quad (95)$$

$$\sin\frac{\pi}{21} = \cos\frac{19\pi}{42} \approx 0.149042266. \quad (96)$$

$n = 22$:

$$32\sin^5(\frac{\pi}{22}) - 16\sin^4(\frac{\pi}{22}) - 32\sin^3(\frac{\pi}{22}) + 12\sin^2(\frac{\pi}{22}) + 6\sin(\frac{\pi}{22}) - 1 = 0. \quad (97)$$

$$\sin\frac{\pi}{22} = \cos\frac{5\pi}{11} \approx 0.142314838. \quad (98)$$

$n = 23$:

$$4194304\sin^{22}(\frac{\pi}{23})-24117248\sin^{20}(\frac{\pi}{23})+60293120\sin^{18}(\frac{\pi}{23})-85917696\sin^{16}(\frac{\pi}{23}) +76873728\sin^{14}(\frac{\pi}{23})-44843008\sin^{12}(\frac{\pi}{23})+17145856\sin^{10}(\frac{\pi}{23})-4209920\sin^{8}(\frac{\pi}{23}) + 631488\sin^{6}(\frac{\pi}{23}) - 52624\sin^{4}(\frac{\pi}{23}) + 2024\sin^{2}(\frac{\pi}{23}) - 23 = 0. \quad (99)$$

$$\sin\frac{\pi}{23} = \cos\frac{21\pi}{46} \approx 0.136166649. \quad (100)$$

$n = 24$:

$$1 - 64\sin^2(\frac{\pi}{24}) + 320\sin^4(\frac{\pi}{24}) - 512\sin^6(\frac{\pi}{24}) + 256\sin^8(\frac{\pi}{24}) = 0. \quad (101)$$

$$\sin\frac{\pi}{24} = \cos\frac{11\pi}{24} \approx 0.130526192 \quad (102)$$

is entry [10, A343054].

$n = 25$:

(103)
$$1048576 \sin^{20}(\frac{\pi}{25}) - 5242880 \sin^{18}(\frac{\pi}{25}) + 11141120 \sin^{16}(\frac{\pi}{25}) - 13107200 \sin^{14}(\frac{\pi}{25}) + 9318400 \sin^{12}(\frac{\pi}{25}) - 4101120 \sin^{10}(\frac{\pi}{25}) + 1100800 \sin^{8}(\frac{\pi}{25}) - 171200 \sin^{6}(\frac{\pi}{25}) + 14000 \sin^{4}(\frac{\pi}{25}) - 500 \sin^{2}(\frac{\pi}{25}) + 5 = 0.$$

(104)
$$\sin\frac{\pi}{25} = \cos\frac{23\pi}{50} \approx 0.12533323.$$

$n = 26$:

(105)
$$64 \sin^{6}(\frac{\pi}{26}) + 32 \sin^{5}(\frac{\pi}{26}) - 80 \sin^{4}(\frac{\pi}{26}) - 32 \sin^{3}(\frac{\pi}{26}) + 24 \sin^{2}(\frac{\pi}{26}) + 6 \sin(\frac{\pi}{26}) - 1 = 0.$$

(106)
$$\sin\frac{\pi}{26} = \cos\frac{6\pi}{13} \approx 0.120536680.$$

$n = 27$:

(107)
$$262144 \sin^{18}(\frac{\pi}{27}) - 1179648 \sin^{16}(\frac{\pi}{27}) + 2211840 \sin^{14}(\frac{\pi}{27}) - 2236416 \sin^{12}(\frac{\pi}{27}) + 1317888 \sin^{10}(\frac{\pi}{27}) - 456192 \sin^{8}(\frac{\pi}{27}) + 88704 \sin^{6}(\frac{\pi}{27}) - 8640 \sin^{4}(\frac{\pi}{27}) + 324 \sin^{2}(\frac{\pi}{27}) - 3 = 0.$$

(108)
$$\sin\frac{\pi}{27} = \cos\frac{25\pi}{54} \approx 0.116092914.$$

$n = 28$:

(109)
$$1 - 96 \sin^{2}(\frac{\pi}{28}) + 1376 \sin^{4}(\frac{\pi}{28}) - 6656 \sin^{6}(\frac{\pi}{28}) + 13568 \sin^{8}(\frac{\pi}{28}) - 12288 \sin^{10}(\frac{\pi}{28}) + 4096 \sin^{12}(\frac{\pi}{28}) = 0.$$

(110)
$$\sin\frac{\pi}{28} = \cos\frac{13\pi}{28} \approx 0.111964476.$$

$n = 29$:

(111)
$$268435456 \sin^{28}(\frac{\pi}{29}) - 1946157056 \sin^{26}(\frac{\pi}{29}) + 6325010432 \sin^{24}(\frac{\pi}{29}) - 12163481600 \sin^{22}(\frac{\pi}{29}) + 15386804224 \sin^{20}(\frac{\pi}{29}) - 13463453696 \sin^{18}(\frac{\pi}{29}) + 8341487616 \sin^{16}(\frac{\pi}{29}) - 3683254272 \sin^{14}(\frac{\pi}{29}) + 1151016960 \sin^{12}(\frac{\pi}{29}) - 249387008 \sin^{10}(\frac{\pi}{29}) + 36095488 \sin^{8}(\frac{\pi}{29}) - 3281408 \sin^{6}(\frac{\pi}{29}) + 168896 \sin^{4}(\frac{\pi}{29}) - 4060 \sin^{2}(\frac{\pi}{29}) + 29 = 0.$$

(112)
$$\sin\frac{\pi}{29} = \cos\frac{27\pi}{58} \approx 0.108119018.$$

$n = 30$:

$$16\sin^4(\frac{\pi}{30}) + 8\sin^3(\frac{\pi}{30}) - 16\sin^2(\frac{\pi}{30}) - 8\sin(\frac{\pi}{30}) + 1 = 0. \tag{113}$$

$$\sin\frac{\pi}{30} = \cos\frac{7\pi}{15} \approx 0.104528463 \tag{114}$$

is entry [10, A019815].

6. Double Angle Sines

Starting from the minimal polynomials for $\cos(2\pi/n)$ of Section 3, the substitution $x \to \sqrt{1-x^2}$ produces the minimal polynomials of the sines of $2\pi/n$. The algebraic degrees of the polynomials are sequence [10, A093819].

2/3: $\sin\frac{2\pi}{3} = \cos\frac{\pi}{6}$ is already covered by (18).

2/5: $\sin\frac{2\pi}{5} = \cos\frac{\pi}{10}$ is already covered by (23).

2/7:

$$64\sin^6(\frac{2\pi}{7}) - 112\sin^4(\frac{2\pi}{7}) + 56\sin^2(\frac{2\pi}{7}) - 7 = 0. \tag{115}$$

$$\sin\frac{2\pi}{7} = \cos\frac{3\pi}{14} \approx 0.781831482. \tag{116}$$

2/9:

$$64\sin^6(\frac{2\pi}{9}) - 96\sin^4(\frac{2\pi}{9}) + 36\sin^2(\frac{2\pi}{9}) - 3 = 0. \tag{117}$$

$$\sin\frac{2\pi}{9} = \cos\frac{5\pi}{18} \approx 0.642787609. \tag{118}$$

is entry [10, A019849].

2/11:

$$1024\sin^{10}(\frac{2\pi}{11}) - 2816\sin^8(\frac{2\pi}{11}) + 2816\sin^6(\frac{2\pi}{11}) - 1232\sin^4(\frac{2\pi}{11}) + 220\sin^2(\frac{2\pi}{11}) - 11 = 0. \tag{119}$$

$$\sin\frac{2\pi}{11} = \cos\frac{7\pi}{22} \approx 0.540640817. \tag{120}$$

2/13:

$$4096\sin^{12}(\frac{2\pi}{13}) - 13312\sin^{10}(\frac{2\pi}{13}) + 16640\sin^8(\frac{2\pi}{13}) - 9984\sin^6(\frac{2\pi}{13}) + 2912\sin^4(\frac{2\pi}{13}) - 364\sin^2(\frac{2\pi}{13}) + 13 = 0. \tag{121}$$

$$\sin\frac{2\pi}{13} = \cos\frac{9\pi}{26} \approx 0.464723171. \tag{122}$$

2/15:

$$256\sin^8(\frac{2\pi}{15}) - 448\sin^6(\frac{2\pi}{15}) + 224\sin^4(\frac{2\pi}{15}) - 32\sin^2(\frac{2\pi}{15}) + 1 = 0. \tag{123}$$

$$\sin\frac{2\pi}{15} = \cos\frac{11\pi}{30} \approx 0.406736643 \tag{124}$$

is entry [10, A019833].

2/17:

$$65536\sin^{16}(\frac{2\pi}{17}) - 278528\sin^{14}(\frac{2\pi}{17}) + 487424\sin^{12}(\frac{2\pi}{17}) - 452608\sin^{10}(\frac{2\pi}{17}) + 239360\sin^{8}(\frac{2\pi}{17}) - 71808\sin^{6}(\frac{2\pi}{17}) + 11424\sin^{4}(\frac{2\pi}{17}) - 816\sin^{2}(\frac{2\pi}{17}) + 17 = 0. \tag{125}$$

$$\sin\frac{2\pi}{17} = \cos\frac{13\pi}{34} \approx 0.361241666. \tag{126}$$

2/19:

$$262144\sin^{18}(\frac{2\pi}{19}) - 1245184\sin^{16}(\frac{2\pi}{19}) + 2490368\sin^{14}(\frac{2\pi}{19}) - 2723840\sin^{12}(\frac{2\pi}{19}) + 1770496\sin^{10}(\frac{2\pi}{19}) - 695552\sin^{8}(\frac{2\pi}{19}) + 160512\sin^{6}(\frac{2\pi}{19}) - 20064\sin^{4}(\frac{2\pi}{19}) + 1140\sin^{2}(\frac{2\pi}{19}) - 19 = 0. \tag{127}$$

$$\sin\frac{2\pi}{19} = \cos\frac{15\pi}{38} \approx 0.324699469. \tag{128}$$

2/21:

$$4096\sin^{12}(\frac{2\pi}{21}) - 11264\sin^{10}(\frac{2\pi}{21}) + 11264\sin^{8}(\frac{2\pi}{21}) - 4992\sin^{6}(\frac{2\pi}{21}) + 960\sin^{4}(\frac{2\pi}{21}) - 64\sin^{2}(\frac{2\pi}{21}) + 1 = 0. \tag{129}$$

$$\sin\frac{2\pi}{21} = \cos\frac{17\pi}{42} \approx 0.294755174. \tag{130}$$

2/23:

$$194304\sin^{22}(\frac{2\pi}{23}) - 24117248\sin^{20}(\frac{2\pi}{23}) + 60293120\sin^{18}(\frac{2\pi}{23}) - 85917696\sin^{16}(\frac{2\pi}{23}) + 76873728\sin^{14}(\frac{2\pi}{23}) - 44843008\sin^{12}(\frac{2\pi}{23}) + 17145856\sin^{10}(\frac{2\pi}{23}) - 4209920\sin^{8}(\frac{2\pi}{23}) + 631488\sin^{6}(\frac{2\pi}{23}) - 52624\sin^{4}(\frac{2\pi}{23}) + 2024\sin^{2}(\frac{2\pi}{23}) - 23 = 0. \tag{131}$$

$$\sin\frac{2\pi}{23} = \cos\frac{19\pi}{46} \approx 0.269796771. \tag{132}$$

7. Tangents

The substitution $x \to 1/\sqrt{1+x^2}$ in the minimal polynomials in Section 2 and elimination of the square roots generates minimal polynomials for $\tan(\pi/n)$-values, because $\cos\alpha = 1/\sqrt{1+\tan^2\alpha}$. The sequence of the algebraic degrees of the polynomials is [10, A089929].

$n = 3$:

$$3 - \tan^2 \frac{\pi}{3} = 0. \tag{133}$$

$$\tan \frac{\pi}{3} = \sqrt{3} \tag{134}$$

is entry [10, A002194].

$n = 4$:

$$\tan \frac{\pi}{4} = 1. \tag{135}$$

$n = 5$:

$$\tan^4(\frac{\pi}{5}) - 10\tan^2(\frac{\pi}{5}) + 5 = 0. \tag{136}$$

$$\tan \frac{\pi}{5} = \psi^* \approx 0.72654 \tag{137}$$

is entry [10, A019934].

$n = 6$:

$$1 - 3\tan^2 \frac{\pi}{6} = 0. \tag{138}$$

$$\tan \frac{\pi}{6} = 1/\sqrt{3} \tag{139}$$

is entry [10, A020760].

$n = 7$:

$$\tan^6(\frac{\pi}{7}) - 21\tan^4(\frac{\pi}{7}) + 35\tan^2(\frac{\pi}{7}) - 7 = 0. \tag{140}$$

$$\tan \frac{\pi}{7} \approx 0.481574619 \tag{141}$$

is entry [10, A343058].

$n = 8$:

$$\tan^2(\frac{\pi}{8}) + 2\tan(\frac{\pi}{8}) - 1 = 0. \tag{142}$$

$$\tan \frac{\pi}{8} = \sqrt{2} - 1 \approx 0.414213562 \tag{143}$$

is entry [10, A188582].

$n = 9$:

$$\tan^6(\frac{\pi}{9}) - 33\tan^4(\frac{\pi}{9}) + 27\tan^2(\frac{\pi}{9}) - 3 = 0. \tag{144}$$

$$\tan \frac{\pi}{9} \approx 0.363970234 \tag{145}$$

is entry [10, A019918].

$n = 10$:

$$1 - 10\tan^2(\frac{\pi}{10}) + 5\tan^4(\frac{\pi}{10}) = 0. \tag{146}$$

$$\tan \frac{\pi}{10} \approx 0.324919696 \tag{147}$$

is entry [10, A019916].

$n = 11$:

(148)
$$\tan^{10}(\frac{\pi}{11}) - 55\tan^8(\frac{\pi}{11}) + 330\tan^6(\frac{\pi}{11}) - 462\tan^4(\frac{\pi}{11}) + 165\tan^2(\frac{\pi}{11}) - 11 = 0.$$

(149)
$$\tan\frac{\pi}{11} \approx 0.293626493.$$

$n = 12$:

(150)
$$\tan^2(\frac{\pi}{12}) - 4\tan(\frac{\pi}{12}) + 1 = 0.$$

(151)
$$\tan\frac{\pi}{12} = 2 - \sqrt{3} \approx 0.267949192$$

is entry [10, A019913].

$n = 13$:

(152)
$$\tan^{12}(\frac{\pi}{13}) - 78\tan^{10}(\frac{\pi}{13}) + 715\tan^8(\frac{\pi}{13}) - 1716\tan^6(\frac{\pi}{13}) + 1287\tan^4(\frac{\pi}{13}) - 286\tan^2(\frac{\pi}{13}) + 13 = 0.$$

(153)
$$\tan\frac{\pi}{13} \approx 0.246477863.$$

$n = 14$:

(154)
$$1 - 21\tan^2(\frac{\pi}{14}) + 35\tan^4(\frac{\pi}{14}) - 7\tan^6(\frac{\pi}{14}) = 0.$$

(155)
$$\tan\frac{\pi}{14} \approx 0.228243474.$$

is entry [10, A343059].

$n = 15$:

(156)
$$\tan^8(\frac{\pi}{15}) - 92\tan^6(\frac{\pi}{15}) + 134\tan^4(\frac{\pi}{15}) - 28\tan^2(\frac{\pi}{15}) + 1 = 0.$$

(157)
$$\tan\frac{\pi}{15} \approx 0.212556561$$

is entry [10, A019910].

$n = 16$:

(158)
$$\tan^4(\frac{\pi}{16}) + 4\tan^3(\frac{\pi}{16}) - 6\tan^2(\frac{\pi}{16}) - 4\tan(\frac{\pi}{16}) + 1 = 0.$$

(159)
$$\tan\frac{\pi}{16} \approx 0.19891236.$$

is entry [10, A343060].

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