Octonion Automorphisms as Algebraic Basis Gauge Transformations

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Abstract

An algebraic basis gauge transformation is defined here as a transformation on the set of intrinsic Octonion Algebra basis elements. They are linear combinations of these structured such that a reversible bijection is produced between each indexed intrinsic basis element and the same index gauge basis element. The gauge transformation is required to map any orientation for original Octonion Algebra to a gauge basis with identical index matched orientation. Orientation is required to be a global and local gauge invariant.

These gauge transformation matrices are found to be lower 7x7 block diagonal members of the group SO(7). Any global gauge transformed Octonion covariant derivative is form invariant with the intrinsic basis representation. Allowing local parametrization variation, fields in the physics sense are added to the still present form invariant content through addition of the covariant differentiation connection, whose general form derivation is provided.

Subgroups of PSL(2,7) give two methods for creating Octonion algebraic basis gauge transformations. Both are shown to be expressible as circle group fibrations over the basic quad basis subspace defined for a choice of Quaternion subalgebra. The chosen subalgebra gauge basis components are then produced from the basic quad fibration by a process called basic quad algebraic completion.

One method uses permutation subgroups of PSL(2,7) that leave one non-scalar basis element unchanged. This is shown to produce a gauge comparable to the direct product U(1)xU(1)xU(1). This method provides a smooth map between any of the four sets of Quaternion subalgebra basis triplets that exclude the unchanged basis element, and each of the other three. This gauges out a four-fold Octonion symmetry on basis element choices representing 3D axial (closed products) and polar (open products) vector types.

The other method uses permutation subgroups of PSL(2,7) that leave the set of basis elements in one Quaternion subalgebra triplet intact. Here, half-angle 2-torus fibrations on the basic quad subspace embed a standard orthonormal whole-angle spherical-polar basis in the preserved subalgebra after algebraic completion. Half-angle 3-torus basic quad fibrations embed a whole angle Euler Angle basis in the preserved subalgebra after algebraic completion.

A composition between any two algebraic basis gauge transformations is shown to produce a third, forming a group operation with closure on algebraic basis gauge types. A parallelism between this composition and fiber product structure is demonstrated.

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There are several types of gauge transformations. They typically leave some feature of the mathematical system and/or expressions invariant after application of either a structural or functional transformation. This screed is about transformations applied to the structural basis system of Octonion Algebra, which in turn can induce functional transformations if certain differential equations are desired to be invariant to the structural modifications.

Octonion Algebra is commonly referred to as $\mathbb{O}$. I refer to the transformations that follow as *algebraic basis gauge transformations*. The modifier *algebraic basis* is added to expressly point out that the basis transformations we seek here are not the simple linear algebra vector space basis coefficient modifications that are only required to continue spanning the space. We will be transforming $\mathbb{O}$ algebra's fundamental basis element system, whose member products define the operation of 8-dimensional algebraic element multiplication, specified as * here. The feature fundamentally required to remain invariant will be transformations to a new basis system which exhibits the same orientation structure of the original intrinsic algebra basis system e_n. This orientation structure is the full set of rules defining *, and different orientations for Octonion Algebras define different rule sets.

Define this gauge transformed basis as $g_i = M_{ij}\, e_j$ where M_{ij} is an 8x8 matrix of scalar values specifying the linear combination of the intrinsic Octonion basis element set e_j for each resultant gauge basis element g_i. To simplify identification and use of the isomorphism between the e and g bases, we will look to structure matrix M such that there is a reversable algebraic structure bijection relating same indexed e and g bases: $e_n \rightarrow g_n$. This means after choosing any of 16 Octonion Algebra orientations defining $e_a * e_b = s_{abc}\, e_c$ where the s_{abc} are its structure constants, we must also have $g_a * g_b = s_{abc}\, g_c$. The e_x in each g of course multiply as usual within the g basis products.

One requirement for the g basis to represent an $\mathbb{O}$ algebra is the product $g_a * g_b$ for $a \neq b \neq 0$ must anti-commute. This forces all g_n for $n \neq 0$ to have no intrinsic basis scalar content. We must also have no intrinsic basis scalar content for every product $g_a * g_b$ for a, $b \neq 0$ unless a = b. We can write the scalar portion of the product $g_a * g_b$ for a, $b \neq 0$ as $M_{aj}\, e_j * M_{bj}\, e_j = -M_{aj}\, M_{bj}$. If a=b, this must equal –1 and if $a \neq b$ this must equal 0. Therefore, we require $M_{aj}\, M_{bj} = \delta_{ab}$. This restricts M to be an orthonormal matrix.

We must also have $g_0 * g_n = g_n * g_0$, and this forces g_0 to have no non-scalar content, so we must have $M_{00} = 1$, as well as $M_{0a} = 0$ and $M_{a0} = 0$ for $a \neq 0$. M then is restricted to a lower block diagonal 7x7 orthonormal matrix which we will restrict to a +1 determinant or Jacobian as the case may be. This block diagonal portion of T will then be a member of the group SO(7). It might be desirable to make $M_{00} = c$, the speed of light in order to cast the scalar basis with dimension length like the others. This will give the common and appropriate 1/c and $1/c^2$ scalings for first and second order time partial derivatives respectively. With no loss of generality take it here as c = 1.

Not every member of the group SO(7) for this M portion restriction will produce a desirable isomorphism. For instance, the SO(7) subgroup of all 7! 7x7 permutation matrices will include members that will violate my desire to use only one choice of the 30 possible ways to partition the Quaternion subalgebra triplet enumerations defined below. Pick one, then move on since the differences are basis element naming conventions, aliases which are structurally irrelevant. The meat on the structural bones begins with the Quaternion subalgebras and is fully disclosed using any single choice. The complexity of 480 different Octonion multiplication tables is unnecessary, only 480/30 = 16 are required. Each of the full complement of 7! = 5040 permutation matrices will however provide one of 480 legitimate Octonion representations, but if we want to stay within one way to partition the triplets, we must stay within the order 5040/30 = 168 subgroup of permutations that do this. This group is of course PSL(2,7), the automorphism group of the Fano Plane. A subset of SO(7), not any member of the full group, will therefore provide us all *desirable* isomorphic algebraic basis gauge transformations.

Beyond consideration of the simple required orthogonality conditions outlined above and living within a single Quaternion subalgebra triplet enumeration, we must complete the full set of gauge basis element product comparisons. We must step outside the domain of linear algebra, matrix manipulation and group theory of SO(7) to find such transformations. Methods to achieve this are presented below.

When the algebraic basis defining the operation of algebraic element multiplication * is changed up through some transformation, we need to understand how to do calculus within the new basis. A proper general definition of differentiation should explicitly tell us how to do this in *any* basis. This form is called the covariant derivative. The proper covariant derivative definition for Octonion Algebra is the Ensemble Derivative **E** defined in references [1], [6] as:

$\mathbf{E}(\mathbf{A}(\mathbf{v})) = 1/J\ \partial/\partial v_i\ [\ C_{ij}\ \ T_{kl}\ \ A_k\ ]\ \ e_j * e_l$

This definition is more general than our area of interest here. T_{kl} is the transformation matrix between the intrinsic Octonion Algebra e basis and the transformed basis we take here to be the gauge transformation g basis. The algebraic basis gauge transformation is then defined to be $g_k = T_{kl}\ e_l$. Variable **v** is the position algebraic element within the transformed basis, defined as $v_i\ g_i$. Matrix C_{ij} holds the cofactor of each T_{ij}, and J is the Jacobian of T. Since we restrict g to an algebraic isomorphism, T is required to be as with M above, lower block diagonal limited member of SO(7).

Limiting T to J = +1 orthonormal, matrix C will equal matrix T. The covariant derivative form may then be written as

$\mathbf{E}(\mathbf{A}(\mathbf{v})) = \partial/\partial v_i\ [\ A_k\ g_i * g_k\ ]$

If the g basis description is independent of the gauge transformation position algebraic element **v**, that is the gauge transformation is a global gauge, we can take g_i and g_k outside the differentiation. In this case, the differentiation over **v** can be written as a simple g system * product of the algebraic element del operator given by $\boldsymbol{\nabla}_{(\mathbf{v})} = g_i\ \partial/\partial v_i$ acting on the g basis functional algebraic element $A_k\ g_k$, and this may be written as $\boldsymbol{\nabla}_{(\mathbf{v})} * \mathbf{A}(\mathbf{v})$.

The covariant Ensemble Derivative in the intrinsic e basis with position algebraic element $u_i\ e_i$ defines $\mathbf{u} = \mathbf{v}$, $T_{kl} = \delta_{kl}$, $C_{ij} = \delta_{ij}$, J = +1 so we can write the intrinsic basis covariant derivative as

$\mathbf{E}(\mathbf{A}(\mathbf{u})) = \partial/\partial u_i\ [\ A_k\ ]\ e_i * e_k = \boldsymbol{\nabla}_{(\mathbf{u})} * \mathbf{A}(\mathbf{u})$.

This is seen to be form invariant with the g basis representation, and do remember that * in both are isomorphic definitions of basis element multiplication for each side of the reversable bijection e ↔ g. Since all Octonion covariant differential equations are required to be constructed from full applications of the Ensemble Derivative, any such equation will exhibit form invariance for any proper global algebraic basis gauge transformation.

If we allow the parametrization of the SO(7) portion of T to vary with **v** position, we now have a *local algebraic basis gauge transformation*. The * isomorphism is still required to hold at each **v** position, but for the covariant derivative, we can no longer take g_i and g_k out of the differentiation, losing form invariance through the addition of new fields (fields in the physics sense) to the still present form invariant portions. Since the * isomorphism still holds at each **v** position, we can first resolve the g product, and then do the partial differentiation.

$\mathbf{E}(\mathbf{A}(\mathbf{v})) = \partial/\partial v_i\ [\ A_k\ g_i * g_k\ ] = \partial/\partial v_i\ [\ s_{ikm}\ A_k\ g_m\ ] = s_{ikm}\ g_m\ \partial/\partial v_i\ [A_k] + s_{ikm}\ A_k\ \partial/\partial v_i\ [g_m]$

The first term on the right side is the form invariant content, the same result as if g was independent of v. The last term provides the added physics type fields by scaling the undifferentiated A_k with $s_{ikm}\ \partial/\partial v_i\ [g_m]$. Expanding this scaling we have

$\partial/\partial v_i\, [g_m] = \partial/\partial v_i\, [T_{mn}]\, e_n = \partial^2/\partial v_i \partial v_m\, [u_n]\, e_n = \partial^2/\partial v_i \partial v_m\, [\mathbf{u}]$

The shift to differentiation of **u** comes from the fact that $T_{mn} = \partial/\partial v_m\, [u_n]$. For a local algebraic basis gauge transformation T, the Ensemble derivative is then

$$\mathbf{E}(\mathbf{A}(\mathbf{v})) = \nabla_{(v)} * \mathbf{A}(\mathbf{v}) + s_{ikm}\, A_k\, \partial/\partial v_i\, [T_{mn}]\, e_n = \nabla_{(v)} * \mathbf{A}(\mathbf{v}) + s_{ikm}\, A_k\, \partial^2/\partial v_i \partial v_m\, [\mathbf{u}]$$

Note the index m is not a free choice, it is determined by s_{ikm} in conjunction with summed indexes i and k. Below, we will define our one of 30 ways to enumerate Quaternion subalgebra triplet indexes such that the binary bit-wise exclusive-or (xor or the operator ^) of all three basis element indexes is zero. The scalar basis index is 0, and we can generally say for any indexes a and b, the product $e_a * e_b$ is within sign $e_{(a\wedge b)}$, so we have $m = i\wedge k$.

We can simplify the local gauge representation by defining a tensor like form expressing an Octonion differentiation "connection" Γ' specific to $\mathbb{O}$ local orthonormal algebraic basis gauge transformations given by

$$\Gamma'_{ik} = \partial/\partial v_i\, [g_{(i\wedge k)}]$$

$$\Gamma'^{n}_{ik} = \partial/\partial v_i\, [T_{(i\wedge k)n}]$$

$$\mathbf{E}(\mathbf{A}(\mathbf{v})) = s_{ik(i\wedge k)}\, \{\, g_{(i\wedge k)}\, \partial/\partial v_i\, [A_k] + \Gamma'_{ik}\, A_k \,\} \qquad \text{or}$$

$$\mathbf{E}(\mathbf{A}(\mathbf{v})) = s_{ik(i\wedge k)}\, \{\, g_{(i\wedge k)}\, \partial/\partial v_i\, [A_k] + \Gamma'^{n}_{ik}\, A_k\, e_n \,\}$$

The first **E** form is a bit slippery, since it is unclear what g basis element(s) $\Gamma'_{ik} A_k$ scales. This is clear cut for the other part of that sum. The second form gives a scaling on the intrinsic e basis, which we know how to map to the g basis set. Since the transformation T is orthonormal, we can also write $e_n = T_{pn}\, g_p$. Inserting this we will know how the connection scales the g basis elements:

$$\mathbf{E}(\mathbf{A}(\mathbf{v})) = s_{ik(i\wedge k)}\, \{\, g_{(i\wedge k)}\, \partial/\partial v_i\, [A_k] + g_p\, T_{pn}\, \Gamma'^{n}_{ik}\, A_k \,\}$$

This suggests an alternate connection form $\Gamma^{p}_{ik} = T_{pn}\, \Gamma'^{n}_{ik}$. Substituting in we have finally

$$\mathbf{E}(\mathbf{A}(\mathbf{v})) = s_{ik(i\wedge k)}\, \{\, g_{(i\wedge k)}\, \partial/\partial v_i\, [A_k] + g_p\, \Gamma^{p}_{ik}\, A_k \,\}$$

Octonion Algebra is not commutative, so we must define both right and left applications of the Ensemble Derivative. The forms above are the left-side application. We can write the general right-side application as

$$(\mathbf{A}(\mathbf{v}))\mathbf{E} = 1/J\, \partial/\partial v_i\, [\, C_{ij}\ T_{kl}\ A_k \,]\ e_l * e_j \qquad \text{equivalently in the g basis} = \partial/\partial v_i\, [\, A_k\, g_k * g_i \,]$$

Whether or not we do an algebraic basis gauge transformation, changing the Ensemble Derivative application side is equivalent to exchanging the order of intrinsic basis elements in the fundamental definition, e.g.: $e_j * e_l \rightarrow e_l * e_j$. This carries forward to the isomorphic g basis giving $g_i * g_k \rightarrow g_k * g_i$. We can implement this in the connection form for g basis results by simply exchanging i and k indexes in the structure constant only:

$$(\mathbf{A}(\mathbf{v}))\mathbf{E} = s_{ki(i\wedge k)}\, \{\, g_{(i\wedge k)}\, \partial/\partial v_i\, [A_k] + g_p\, \Gamma^{p}_{ik}\, A_k \,\}$$

Meaningful Octonion covariant mathematical physics will require multiple whole applications of the Ensemble Derivative. The second order forms will then differentiate the added zero order local gauge connection terms leading to a mix of first and second order partials on the functions of interest as well as undifferentiated functional components. Taking a path parallel to classical Electrodynamics, we could take **A** to be an 8D potential function algebraic element. Define (see ref. [1]) left and right (physics) fields as

$\mathbf{F}_L = \mathbf{E}(\mathbf{A}(\mathbf{v}))$ and $\mathbf{F}_R = (\mathbf{A}(\mathbf{v}))\mathbf{E}$

The proper form for the Octonion 8-current is found to be

$\mathbf{j} = ½ \{ \mathbf{E}(\mathbf{F}_L) + (\mathbf{F}_R)\mathbf{E} \}$

The proper form for the Octonion 8-work-force is found to be the content of the following that does not change when the Octonion Algebra orientation is changed up

$\mathbf{wf} = – ½ \{ \mathbf{j} * \mathbf{F}_R + \mathbf{F}_L * \mathbf{j} \}$

Expressing the Octonion equivalent of complex conjugation $e_0 \rightarrow e_0$ and $e_n \rightarrow –e_n$ for $n \neq 0$ with an underscore we can write

$\underline{\mathbf{E}}(\mathbf{A}(\mathbf{v})) = 1/J\ \partial/\partial v_i\ [\ C_{ij}\ \ T_{kl}\ \ A_k\ ]\ \ \underline{e}_j * e_l$ equivalently in the g basis $= \partial/\partial v_i\ [\ A_k\ \underline{g}_i * g_k\ ]$
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The proper form in any basis for the continuity equation expressing the conservation of 8-charge can be seen to be

Scalar $\underline{\mathbf{E}}(\mathbf{j}) = 0 =$ scalar $\mathbf{E}(\underline{\mathbf{j}}) =$ scalar $(\mathbf{j})\underline{\mathbf{E}} =$ scalar $(\underline{\mathbf{j}})\mathbf{E}$

For **E** and **j** represented in the intrinsic basis, these equivalent continuity equations hold identically, independent of any particular choice for the potential functions. We must require the continuity equation holds in any basis, it therefore must be a local algebraic basis gauge invariant.

In the intrinsic basis, the 8-current is seen to be form invariant for any and all Octonion Algebra orientation choices, it is an Octonion algebraic invariant. Since it is an observable, we must insist the 8-current is an algebraic invariant in any basis. It should be a local algebraic basis gauge invariant, at a minimum remaining an algebraic invariant if not fully a local algebraic basis gauge invariant.

The local algebraic basis gauge transformation is not form invariant with its global form, but could be put into form invariance with proper modifications to the potential functions. The modification requirements are of course dependent on the particular differential equation in play, and how soon in the progression of multiple applications of the Ensemble Derivative form invariance is first enforced. Consider these *induced functional gauge transformations*, since they transform the functions operated on.

It will be important now to establish some additional understanding and some motivations for such a process, to better understand it beyond the nice stuff just discussed. We will be required to cast our algebraic expressions in a way that is applicable to any and all orientations for the applied Octonion Algebra. Hopefully, the next few paragraphs will set the foundation.

Mathematics tells us that we need a sufficient number of independent variables (read dimensions) to span the problem at hand. If the math tells us the count is greater than the four dimensions our primary senses give us, so be it. Theoretical and experimental physics is not restricted to simply match the expectations provided by our limited senses. Our senses were refined genetically through natural selection, only by improving our chance of survival long enough to procreate, nothing deeper. The math connection is there to help us develop a deeper understanding of things than our senses can possibly provide. When math says more structure is needed, we should pay attention.

An early clue about the need for more than four dimensions was given by the mathematical treatment of

Electrodynamics, where the disparate nature of the magnetic and electric fields was revealed. This was uncovered when the seemingly free and arbitrary choice of coordinate system orientation was explored. Without being given reasonable cause to pick one orientation over the other, the mathematics was telling us proper physical theories needed to be structured such that the same result, say the physical direction a charged particle moving through a magnetic field is deflected, is independent of the orientation choice for the coordinate system.

This led to the more general notion of axial vectors (e.g., the magnetic field) and polar vectors (e.g., the electric field). The math was shouting to us that these are fundamentally different enough that they cannot simply be added or subtracted such that one type might be able to eliminate the other. They must be kept separate from each other at a fundamental level within any proper mathematical framework. The good and bad thing about mathematics is that it is robust enough to not always force us into a singular way to account for such intrinsic differences. Historically, the choice was made to stick with four fundamental dimensions (space-time) and place the six components for the magnetic and electric field in separate positions within the second rank combined field tensor. It is important to keep in mind this was a choice among alternatives, not a requirement. It works well, but issues consolidating Electrodynamics with Gravitation seem to be telling us not well enough.

A different choice would of course be to increase the number of fundamental mathematical spatial dimensions, two-fold at least to separately cover both 3D axial and 3D polar types fundamentally within a physical xyz framework. We could then stick within the knitting of a suitable dimension base algebra, rather than achieving the required additional structure through tensor algebra rank increases or the like.

This algebra must be true to the vector multiplication rules for axial like and polar like vectors. The vector product of two axial types is another axial type. In other words, the multiplication rules for the three basis elements partitioning axial type vectors must be closed. The vector product of two polar types is an axial type, so the multiplication rules for the three basis elements partitioning the polar types cannot be closed. We do however, find product order permutations on one axial and two polar components *is* closed. The open polar type product rules are seen then to be appropriately defined by three additional closed basis triplet product rules, one for each included axial component. Closed basis triplet product rules show up in the seven $\mathbb{O}$ Quaternion subalgebras.

The general concept of 3D polar and axial types charaterized by coordinate system orientation concerns should be supplanted with basis element triplet sets open and closed for multiplication respectively. When we shift to a higher dimension vector space and include a prescribed algebra defining multiplication of algebraic elements spanning this vector space, the simplistic right-handed/left-handed choice invariance to mathematical physics results must be supplanted by result invariance to any and all possible orientation choices defined by said algebra. For Octonion Algebra, I have called this *The Law of Octonion Algebraic Invariance*, (refs. [1], [2], [5]) stating the Octonion mathematical physics cover of any experimentally observable must be invariant across all possible $\mathbb{O}$ algebra orientation changes.

To this end, we must carry the impact of orientation alternatives within the structure of our mathematical physics expressions, for only then we will not fall into the trap of developing theoretical results that might adversely change if the orientation of the algebra is changed up, or perhaps worse remain oblivious to the impact of orientation changes. We will be required to carry this structure below when methods to produce algebraic basis gauge transformations are developed, so it is important to fully understand how to do this before jumping in.

One of three required and fundamental rules defining an algebra tells us we can only combine coefficients that are attached to the same basis element when algebraic elements are added. When we concern ourselves with Algebraic Variance/Invariance, we find this is not good enough. We must additionally only add coefficients scaling identical basis elements if they have identical variance/invariance classification.

The set of orientation options for any order 2^n chain of hypercomplex algebras of order four and up are fully and exclusively specified by the free choice of two possible orientations for each Quaternion algebra/subalgebra. The non-Quaternion triplet basis products, effectively real and complex subalgebra limited products, are unmodified for any orientation change since real and complex algebras are singularly oriented. We can therefore minimally classify our $\mathbb{O}$ orientation modified coefficients by attaching the algebraic structure constants defining the orientation choices for Quaternion subalgebra triplet product rules. The first step is to enumerate these seven Quaternion subalgebra non-scalar basis element triplet sets.

The Quaternion subalgebra triplet enumeration scheme used here is the vastly superior one of 30 possible ways to do it, where the binary logic bit-wise exclusive-or (operator ^) of all three basis element indexes is zero (see ref [4]). Their set partitioning with optimal Q index enumeration is the following:

$Q_1 = \{e_2\ e_4\ e_6\}$ $Q_2 = \{e_1\ e_4\ e_5\}$ $Q_3 = \{e_3\ e_4\ e_7\}$
$Q_4 = \{e_1\ e_2\ e_3\}$ $Q_5 = \{e_2\ e_5\ e_7\}$ $Q_6 = \{e_1\ e_6\ e_7\}$ $Q_7 = \{e_3\ e_5\ e_6\}$

I call this enumeration index n on Q_n optimal because the three Q triplets any single non-scalar basis element e_n will appear in are indexed by the indexes of the three basis element members of Q_n. As an example, intrinsic basis element e_4 is found within Q_1, Q_2 and Q_3 and the content of Q_4 is $\{e_1\ e_2\ e_3\}$. Additionally, the single basis element intersection for Q_n and Q_m will also be found in $Q_{n\wedge m}$.

The { } forms are simply set designations, not specifying product rules. The Quaternion products of different non-scalar basis elements were given to us by W. R. Hamilton. They are commonly expressed by what is called an *ordered permutation triplet product rule* $(e_a\ e_b\ e_c)$.

The paired cyclic right gives all +results, and paired cyclic left gives all –results rule sets the orientation choice for any particular Quaternion subalgebra by defining its six non-scalar different basis element products as follows:

$(e_a\ e_b\ e_c)$ implies
$e_a * e_b = +e_c$
$e_b * e_c = +e_a$
$e_c * e_a = +e_b$
$e_c * e_b = -e_a$
$e_b * e_a = -e_c$
$e_a * e_c = -e_b$

From this, clearly $(e_a\ e_b\ e_c)$, $(e_b\ e_c\ e_a)$ and $(e_c\ e_a\ e_b)$ represent the same rule set. The orientation change for any ordered permutation triplet product rule is any odd count of transpositions of any two of three basis elements, each of which equivalently changes the resultant signs of all six products, thus negating the rule. This triplet negation clearly is an involution, and the two choices fully cover the orientation options for a given Quaternion algebra. If we include the real and complex subalgebra rules $e_0 * e_0 = e_0$, $e_0 * e_n = e_n * e_0 = e_n$, and $e_n * e_n = -e_0$ for $n \neq 0$ with orientation selections for all seven Quaternion subalgebra triplets, we cover all 8^2 basis element product combinations, fully defining the Octonion Algebra through the orientation choices on the Quaternion subalgebras.

For Octonion Algebra, the 128 possible orientation choices for its seven Q_n Quaternion subalgebras result in 16 proper Octonion Algebra orientations, and 112 that I have called Broctonion (Broken Octonion) forms, which are one Quaternion subalgebra orientation off of a proper Octonion form (see ref [7]). The 16 proper Octonion orientations partition chirally into two structurally different sets of eight: Right Octonion and Left Octonion.

Two n dimensional algebras are considered isomorphic *if and only if* their basis element multiplication tables are equivalent. One may exchange rows and columns of any multiplication table without changing any product rule, and any names given to the basis elements have no fundamental structural importance, they just need to be distinct. This tells us the map between any two isomorphic algebras is a permutation of basis elements. So we can rightfully call PSL(2,7) the automorphism group of any Octonion Algebra defined within a single enumeration of its seven Quaternion subalgebra triplets. It gives us the full group of consistent basis element permutations, hence the full complement of permutation maps between equivalent Octonion multiplication tables, hence the full set of Octonion Algebra orientation automorphisms.

Every basis element permutation created by members of PSL(2,7) will map Right Octonion to Right Octonion, and Left Octonion to Left Octonion. No basis element permutation exists that will map between Right and Left, their basis element multiplication tables are not equivalent and hence they should not be strictly considered isomorphic algebras even though all 16 are proper Octonion normed composition division algebras.

One could map between Right and Left Octonion by negating an odd number of basis elements, then absorbing these –1 values into the algebra structure constants, but it would be both foolish and incorrect to assume this does not change the structure of the Octonion Algebra fundamentally in an identifiable manner (see refs, [3],[5],[7]). In terms of our algebraic basis gauge transformation, the map between Right and Left would require the lower block diagonal portion of orthogonal matrix M to have determinant –1, not our +1 restriction that will keep things within the confines of the group PSL(2,7) which we will use below.

This is not the case for Quaternion Algebra. Negating one or three non-scalar basis element yields a determinant –1 for this transformation, but it is equivalent to a permutation exchanging any two basis elements. All Quaternion multiplication tables are therefore equivalent. The difference might be because each non-scalar basis element appears in three separate Quaternion subalgebra triplets that partially define Octonion orientation, or perhaps because Octonion Algebra has an orientable subalgebra whereas Quaternion Algebra does not.

Understand here that we seek an algebraic basis gauge transformation that has precisely the same multiplication table the intrinsic basis element basis set has been given, not simply an equivalent one. Our bijective transformation is $e_n \rightarrow g_n$, without index permutation.

We can now define our 16 different Octonion orientations. Octonion Algebra **R0** is defined by the following ordered triplet orientations:

$(e_6\ e_4\ e_2)$, $(e_5\ e_4\ e_1)$, $(e_7\ e_4\ e_3)$, $(e_1\ e_2\ e_3)$, $(e_5\ e_7\ e_2)$, $(e_7\ e_6\ e_1)$, $(e_6\ e_5\ e_3)$

These set the +1 valued structure constants for **R0** as

$s_{642} = s_{541} = s_{743} = s_{123} = s_{572} = s_{761} = s_{653} = +1$

The optimal enumeration (ref. [1] et. al.) for the remaining seven Right $\mathbb{O}$ orientations **Rn** for n = 1 through 7 negates the four **R0** orientation triplets that do not include e_n. The anti-automorphism map between Right and Left $\mathbb{O}$ **Rm** ↔ **Lm** negates all seven triplet orientations.

It is optimal to pick a single proper Octonion Algebra orientation and always use it within any Octonion mathematical expression. There is no loss of generality doing this if we carry, when needed, Quaternion subalgebra triplet structure constants with indexes ordered in the +1 orientation for the chosen full algebra orientation. If we choose **R0** as defined above, we would prefer to specify the product $c\ e_2 * d\ e_1$ as $-s_{123}\ cd\ e_3$ instead of $c\ e_2 * d\ e_1 = +s_{213}\ cd\ e_3$ although both are correct as written and the latter is in line with the fundamental definition of the structure constants. This simplifies the task of evaluating products of structure constants. Clearly, if we simply wrote $c\ e_2 * d\ e_1 = -\ cd\ e_3$, this would be incorrect for some other Octonion orientation.

Since any $\mathbb{O}$ s_{abc} is either +1 or –1, we have $s_{abc}\ s_{abc} = +1$. When we form the product of two different ordered Quaternion subalgebra structure constants, they will always share a single common index, and their product result will be within sign the third structure constant sharing that common index. Example for **R0** defined +1 index order we have $s_{572}\ s_{653} = s_{541}$ (see ref [5]). Always using the structure constants in the +1 index order for a select Octonion orientation obviates the need to track four separate index order possibilities for a product of two. These sign changes are instead processed identically as non-oriented scalars are, through products of their attached signs.

This is the essence of what I have called the in-place Octonion Variance Sieve. Each formed product term carries not just its resultant basis element and attached signed scalar (math field) coefficient, but also the orientation choice variance as a characteristic that is updated each subsequent product throughout the product history. The update is done at the time each successive product is processed (an in-place computation). Reductions like trig identities and cancellation by otherwise sum of equal but opposite sign coefficients scaling the same basis element, can only be performed if the variance characteristics are the same in all product terms used.

This gives results that naturally partition into 16 different possible variance categories: odd/even parity times eight from seven triplet orientations plus one not defined by any triplet orientation (e.g., $s_{abc}\ s_{abc} = +1$, $e_0 * e_n$, $e_n * e_n$). The two parity choices are an odd or even count of applied oriented basis element products throughout the product term's full product history. This accounts for the anti-automorphism map between Right and Left Octonion Algebra orientations, where odd parity will yield a sign change and even parity will not.

Separately maintaining the odd/even parity, a single numeric value of zero can represent no orientation, or the appropriate Q index can be used for oriented products. All products of variance classification are then represented by a simple exclusive-or of the two variance/invariance indexes involved. This is because our Q index enumeration tells us the basis element in common with Q_a and Q_b is also found in $Q_{a\wedge b}$. This is the rule above for Quaternion structure constant products.

Any calculation can be performed this way within a single chosen Octonion orientation, and the final result can be mapped to what it would be if some other orientation was used, by simply negating product terms whose variance triplet changes sign from that of the chosen algebra, mindful of parity considerations.

The set of product terms with even parity and no triplet designation are Octonion Algebraic Invariants, they will not change sign across all 16 possible Octonion orientation choices. The set of product terms with odd parity and no triplet designation will be invariant within every Right or within every Left Octonion, but will change sign with the anti-automorphism map between Right and Left. The remaining 14 sets of product terms may or may not change sign when specific Octonion orientation changes are made, but it is important to realize every product term within any variance set will change sign or not, in like fashion.

We could assign a value of zero to the sum of all signed product terms in each of the variance sets. Doing so would yield a result that is fully an Octonion algebraic invariant, since $+0 = -0$. I call these homogeneous equations of algebraic constraint. This methodology is important to an Octonion cover of physics, since observables must be algebraic invariants, and notions like confinement tell us some things may be arguably present but not directly observable.

We can partition the eight Octonion basis elements into two equal size subspaces. One subspace holds the four basis elements of a Quaternion subalgebra. The remaining four basis elements do not form an algebra for numerous reasons, but their basis element product definitions can be used to generate the whole of the particular Octonion orientation. This set of four basis elements is commonly referred to as a "basic quad" for this reason. The product of any two basic quad elements will always be within sign a member of the Quaternion triplet defining it.

It is possible to perform manipulations on a basic quad set only, then legitimately and consistently complete the full new isomorphic algebra using products between two modified basic quad set members. With a proper automorphically transformed basic quad set in hand, we can complete the full algebra definition by forming three separate products of two transformed basic quad basis elements. These products will induce both algebraic variance and parity modifications to the results, since all are oriented products. The induced variance/parity modifications are not part of the fundamental definition for the resultant basis element. They must be backed out by appending the product's variance triplet to the basic quad pair product to cancel it, then flipping the odd/even parity in all result product terms. Define this process as *basic quad algebraic completion*. We will make good use of this below.

With just a little more background, we will finally be able to rip into the construction and utility of particular algebraic basis gauge transformations. The notion of time clearly has a dimensional home partitioned at least by e_0, the Octonion scalar basis element. We need to double up on the three physical xyz dimensions, but have seven, not six non-scalar basis elements. This can be remedied by selecting one non-scalar basis element to be non-spatial in the physical 3D xyz sense. I submit that this is a free choice within a seven-fold symmetry, but once made, the die is cast so to speak. My choice is e_4.

Having arbitrarily chosen e_4 to not be part of the spatial xyz scene, we set our four triplets required to cover axial and polar type product rules to be Q_4, Q_5, Q_6 and Q_7, none of which include e_4. We cannot determine which one of the four to associate with axial types, any one will do. Just like the apparent free choice of non-spatial non-scalar basis element, this remains a symmetry of the algebra. Maybe we need to do a little more work instead of simply picking one. If we can devise an algebraic basis gauge transformation that would map any single spatial triplet choice to any one of the other three, we might be able to "gauge out" this symmetry given to us by the fundamental structure of Octonion Algebra.

We are given clues on how to do this within the group PSL(2,7), the automorphism group for the Fano Plane. The members of this group can be represented by 7x7 orthonormal permutation matrices where each row and column have a single +1 entry with remaining entries 0. Their determinants then are always +1. When we apply them to permute the set of seven non-scalar Octonion basis elements, this group gives us the full complement of basis element permutations that do not violate our triplet enumerations Q_n, nor any attributes that qualify the algebra as proper Octonion. From the covariant derivative form invariance to global gauge transformations analyzed above, the ability to use any of these orthonormal +1 determinant constant permutation matrices as a global algebraic basis gauge transformation in the Ensemble Derivative matrix T helps to validate its proper covariance for any Octonion orientation, form invariance is maintained by any choice.

PSL(2,7) has 14 order 24 subgroups which are isomorphic to the symmetric group S_4, the group of permutations on four objects. Seven of these, label them N_x preserve basis element e_x, one group for each of the seven non-scalar basis elements. The other seven, label them T_x preserve the set of basis elements within triplet Q_x, one group for each of the seven triplets. For the moment we will focus on N_4, the group of all basis element permutations that leave our selected non-spatial e_4 alone.

Both groups N_x and T_x have similar normal subgroups isomorphic to the Klein 4-group where non-identity members include two separate transpositions of basis elements. The product of transposed basis elements in one transposition is within sign the product of basis elements in the other paired transposition. This common basis element product is the same for all three double transpositions within each N_x definition and is the preserved e_x. We have for group N_4 the normal subgroup A_n defining the following permutation cycles where the product of each transposed element pair is $\pm e_4$:

$A_0 = [I]$ (identity) $A_1 = [e_1\ e_5]\ [e_2\ e_6]$ $A_2 = [e_1\ e_5]\ [e_3\ e_7]$ $A_3 = [e_2\ e_6]\ [e_3\ e_7]$

These transform Q_4, Q_5, Q_6 and Q_7 paired with the set of remaining non-scalar basis elements excluding e_4 as follows

A_0	A_1	A_2	A_3
$\{e_1\, e_2\, e_3\} :: \{e_5\, e_6\, e_7\} \rightarrow$	$\{e_5\, e_6\, e_3\} :: \{e_1\, e_2\, e_7\}$	$\{e_5\, e_2\, e_7\} :: \{e_1\, e_6\, e_3\}$	$\{e_1\, e_6\, e_7\} :: \{e_5\, e_2\, e_3\}$
$\{e_5\, e_6\, e_3\} :: \{e_1\, e_2\, e_7\} \rightarrow$	$\{e_1\, e_2\, e_3\} :: \{e_5\, e_6\, e_7\}$	$\{e_1\, e_6\, e_7\} :: \{e_5\, e_2\, e_3\}$	$\{e_5\, e_2\, e_7\} :: \{e_1\, e_6\, e_3\}$
$\{e_5\, e_2\, e_7\} :: \{e_1\, e_6\, e_3\} \rightarrow$	$\{e_1\, e_6\, e_7\} :: \{e_5\, e_2\, e_3\}$	$\{e_1\, e_2\, e_3\} :: \{e_5\, e_6\, e_7\}$	$\{e_5\, e_6\, e_3\} :: \{e_1\, e_2\, e_7\}$
$\{e_1\, e_6\, e_7\} :: \{e_5\, e_2\, e_3\} \rightarrow$	$\{e_5\, e_2\, e_7\} :: \{e_1\, e_6\, e_3\}$	$\{e_5\, e_6\, e_3\} :: \{e_1\, e_2\, e_7\}$	$\{e_1\, e_2\, e_3\} :: \{e_5\, e_6\, e_7\}$

This normal subgroup of N_4 is seen to map any of our four declared pure spatial Q triplets to each of the other three. We are also shown how to correlate the pairing of basis elements for x, y and z physical dimensions. The two basis elements excluding e_4 within each of Q_1, Q_2 and Q_3 define a pair of basis elements associated with one and the same physical x, y or z. In this way the product of e_4 with any other non-scalar basis element reveals its pairing. Likewise, the indexes of any two basis elements paired to the same physical x, y or z will exclusive-or to 4, the index of the non-spatial choice.

We seek not replacements as done with these transpositions, but smooth continuous transformations on the intrinsic Octonion e basis set for our algebraic basis gauge transformation. To accomplish this, we will do equivalent oriented angle rotations about e_4 within both of the planes defined by the pair of transposed basis elements, a different angle for each of the three cycles shown.

Start with the A_1 smooth map, requiring rotations about e_4 in the $e_1\, e_5$ and $e_2\, e_6$ planes by the same angle β_3 with $\mathbb{O}$ algebra specific orientations as indicated, maintaining $\{e_0\, e_3\, e_4\, e_7\}$

$e'_1 = e_1 \cos(\beta_3) - s_{541}\, e_5 \sin(\beta_3)$
$e'_5 = e_5 \cos(\beta_3) + s_{541}\, e_1 \sin(\beta_3)$
$e'_2 = e_2 \cos(\beta_3) + s_{642}\, e_6 \sin(\beta_3)$
$e'_6 = e_6 \cos(\beta_3) - s_{642}\, e_2 \sin(\beta_3)$
$e'_0 = e_0$
$e'_3 = e_3$
$e'_4 = e_4$
$e'_7 = e_7$

Next do the A_2 smooth map, rotations about e_4 in the $e_1\, e_5$ and $e_3\, e_7$ planes by the same angle β_2 but with $\mathbb{O}$ algebra specific orientations as indicated, maintaining $\{e'_0\, e'_2\, e'_4\, e'_6\}$

$e''_1 = e'_1 \cos(\beta_2) + s_{541}\, e'_5 \sin(\beta_2)$
$e''_5 = e'_5 \cos(\beta_2) - s_{541}\, e'_1 \sin(\beta_2)$
$e''_3 = e'_3 \cos(\beta_2) - s_{743}\, e'_7 \sin(\beta_2)$
$e''_7 = e'_7 \cos(\beta_2) + s_{743}\, e'_3 \sin(\beta_2)$
$e''_0 = e'_0$
$e''_2 = e'_2$
$e''_4 = e'_4$
$e''_6 = e'_6$

Writing all e'_a in terms of the intrinsic e basis we have

$e''_0 = e_0$
$e''_1 = e_1 \{\cos(\beta_2) \cos(\beta_3) + \sin(\beta_2) \sin(\beta_3)\} + s_{541}\, e_5 \{\sin(\beta_2) \cos(\beta_3) - \cos(\beta_2) \sin(\beta_3)\}$
$e''_2 = e_2 \cos(\beta_3) + s_{642}\, e_6 \sin(\beta_3)$
$e''_3 = e_3 \cos(\beta_2) - s_{743}\, e_7 \sin(\beta_2)$
$e''_4 = e_4$
$e''_5 = e_5 \{\cos(\beta_2) \cos(\beta_3) + \sin(\beta_2) \sin(\beta_3)\} - s_{541}\, e_1 \{\sin(\beta_2) \cos(\beta_3) - \cos(\beta_2) \sin(\beta_3)\}$
$e''_6 = e_6 \cos(\beta_3) - s_{642}\, e_2 \sin(\beta_3)$
$e''_7 = e_7 \cos(\beta_2) + s_{743}\, e_3 \sin(\beta_2)$

Finally, do the A_3 smooth map, rotations about e_4 in the e_2 e_6 and e_3 e_7 planes by the same angle β_1 but with $\mathbb{O}$ algebra specific orientation as indicated, , maintaining $\{e''_0\ e''_1\ e''_4\ e''_5\}$

$g_2 = e''_2 \cos(\beta_1) - s_{642}\ e''_6 \sin(\beta_1)$
$g_6 = e''_6 \cos(\beta_1) + s_{642}\ e''_2 \sin(\beta_1)$
$g_3 = e''_3 \cos(\beta_1) + s_{743}\ e''_7 \sin(\beta_1)$
$g_7 = e''_7 \cos(\beta_1) - s_{743}\ e''_3 \sin(\beta_1)$
$g_0 = e''_0$
$g_1 = e''_1$
$g_4 = e''_4$
$g_5 = e''_5$

Write e''_a in terms of the intrinsic basis e_b to form the following definitions.

$g_0 = e_0$
$g_1 = e_1\ \{\cos(\beta_2) \cos(\beta_3) + \sin(\beta_2) \sin(\beta_3)\} + s_{541}\ e_5\ \{\sin(\beta_2) \cos(\beta_3) - \cos(\beta_2) \sin(\beta_3)\}$
$g_2 = e_2\ \{\cos(\beta_3) \cos(\beta_1) + \sin(\beta_3) \sin(\beta_1)\} + s_{642}\ e_6\ \{\sin(\beta_3) \cos(\beta_1) - \cos(\beta_3) \sin(\beta_1)\}$
$g_3 = e_3\ \{\cos(\beta_1) \cos(\beta_2) + \sin(\beta_1) \sin(\beta_2)\} + s_{743}\ e_7\ \{\sin(\beta_1) \cos(\beta_2) - \cos(\beta_1) \sin(\beta_2)\}$
$g_4 = e_4$
$g_5 = e_5\ \{\cos(\beta_2) \cos(\beta_3) + \sin(\beta_2) \sin(\beta_3)\} - s_{541}\ e_1\ \{\sin(\beta_2) \cos(\beta_3) - \cos(\beta_2) \sin(\beta_3)\}$
$g_6 = e_6\ \{\cos(\beta_3) \cos(\beta_1) + \sin(\beta_3) \sin(\beta_1)\} - s_{642}\ e_2\ \{\sin(\beta_3) \cos(\beta_1) - \cos(\beta_3) \sin(\beta_1)\}$
$g_7 = e_7\ \{\cos(\beta_1) \cos(\beta_2) + \sin(\beta_1) \sin(\beta_2)\} - s_{743}\ e_3\ \{\sin(\beta_1) \cos(\beta_2) - \cos(\beta_1) \sin(\beta_2)\}$

Make the angle assignments

$\zeta_1 = \beta_2 - \beta_3$
$\zeta_2 = \beta_3 - \beta_1$
$\zeta_3 = \beta_1 - \beta_2$

Our definitions for the N_4 g basis algebraic gauge transformation may then be written as

$g_0 = e_0$
$g_1 = e_1 \cos(\zeta_1) + s_{541}\ e_5 \sin(\zeta_1)$
$g_2 = e_2 \cos(\zeta_2) + s_{642}\ e_6 \sin(\zeta_2)$
$g_3 = e_3 \cos(\zeta_3) + s_{743}\ e_7 \sin(\zeta_3)$
$g_4 = e_4$
$g_5 = e_5 \cos(\zeta_1) - s_{541}\ e_1 \sin(\zeta_1)$
$g_6 = e_6 \cos(\zeta_2) - s_{642}\ e_2 \sin(\zeta_2)$
$g_7 = e_7 \cos(\zeta_3) - s_{743}\ e_3 \sin(\zeta_3)$

The transformation matrix defined here is seen to be orthonormal with +1 determinant, and given any Octonion orientation $e_a * e_b = s_{abc}\ e_c$, our g results will show $g_a * g_b = s_{abc}\ g_c$. The smooth map $e_n \rightarrow g_m$ is a like index algebraic isomorphism. Note that from our definitions we have $\zeta_1 + \zeta_2 + \zeta_3 = 0$ identically, but actually 0 mod 2π will do. This identity will be required for trigonometric reductions when demonstrating the g basis algebraic isomorphism with any initial intrinsic e basis Octonion Algebra orientation choice when using the ζ angle form of g in which the β angle difference is only implicit. The restriction is not required when using the β angle explicit form, simple trig reductions will do. We will find below each of the β angles parametrize a free choice of any three points on the unit circle within the $e_0 - e_4$ plane, and there is no unique β angle choice that could be called the identity map.

If we choose $\zeta_1 = \zeta_2 = \zeta_3 = 0$, the map $e_n \rightarrow g_m$ is the identity map $e_n = g_n$.

For $\zeta_1 = \pi\ \ \zeta_2 = -\pi/2\ \ \zeta_3 = -\pi/2$ and $\mathbb{O}$ algebra **R0**

$(g_1\ g_2\ g_3) = (-e_1\ -e_6\ -e_7) = (e_7\ e_6\ e_1)$ $\quad$ $(g_7\ g_6\ g_1) = (e_3\ e_2\ -e_1) = (e_1\ e_2\ e_3)$
$(g_5\ g_7\ g_2) = (-e_5\ e_3\ -e_6) = (e_6\ e_5\ e_3)$ $\quad$ $(g_6\ g_5\ g_3) = (e_2\ -e_5\ -e_7) = (e_5\ e_7\ e_2)$
$(g_5\ g_4\ g_1) = (-e_5\ e_4\ -e_1) = (e_5\ e_4\ e_1)$ $\quad$ $(g_6\ g_4\ g_2) = (e_2\ e_4\ -e_6) = (e_6\ e_4\ e_2)$
$(g_7\ g_4\ g_3) = (e_3\ e_4\ -e_7) = (e_7\ e_4\ e_3)$

For $\zeta_1 = -\pi/2\ \ \zeta_2 = \pi\ \ \zeta_3 = -\pi/2$ and $\mathbb{O}$ algebra **R0**

$(g_1\ g_2\ g_3) = (-e_5\ -e_2\ -e_7) = (e_5\ e_7\ e_2)$ $\quad$ $(g_7\ g_6\ g_1) = (e_3\ -e_6\ -e_5) = (e_6\ e_5\ e_3)$
$(g_5\ g_7\ g_2) = (e_1\ e_3\ -e_2) = (e_1\ e_2\ e_3)$ $\quad$ $(g_6\ g_5\ g_3) = (-e_6\ e_1\ -e_7) = (e_7\ e_6\ e_1)$
$(g_5\ g_4\ g_1) = (e_1\ e_4\ -e_5) = (e_5\ e_4\ e_1)$ $\quad$ $(g_6\ g_4\ g_2) = (-e_6\ e_4\ -e_2) = (e_6\ e_4\ e_2)$
$(g_7\ g_4\ g_3) = (e_3\ e_4\ -e_7) = (e_7\ e_4\ e_3)$

For $\zeta_1 = -\pi/2\ \ \zeta_2 = -\pi/2\ \ \zeta_3 = \pi$ and $\mathbb{O}$ algebra **R0**

$(g_1\ g_2\ g_3) = (-e_5\ -e_6\ -e_3) = (e_6\ e_5\ e_3)$ $\quad$ $(g_7\ g_6\ g_1) = (-e_7\ e_2\ -e_5) = (e_5\ e_7\ e_2)$
$(g_5\ g_7\ g_2) = (e_1\ -e_7\ -e_6) = (e_7\ e_6\ e_1)$ $\quad$ $(g_6\ g_5\ g_3) = (e_2\ e_1\ -e_3) = (e_1\ e_2\ e_3)$
$(g_5\ g_4\ g_1) = (e_1\ e_4\ -e_5) = (e_5\ e_4\ e_1)$ $\quad$ $(g_6\ g_4\ g_2) = (e_2\ e_4\ -e_6) = (e_6\ e_4\ e_2)$
$(g_7\ g_4\ g_3) = (-e_7\ e_4\ -e_3) = (e_7\ e_4\ e_3)$

We can see for these particular angle selections meeting the sum to zero restriction, we map each of the three Quaternion subalgebra triplets that include e_4 to themselves without orientation change, and map each of the four spatial only Quaternion subalgebra triplets excluding e_4 to any one of the other three, and their resultant orientations stay within the **R0** definition. Since this map $e \rightarrow g$ is an algebraic isomorphism, demonstrating that each of the four Quaternion subalgebra triplets not including e_4 uniquely map to each of the other three one to one and onto for $\mathbb{O}$ algebra **R0**, this mapping holds for every $\mathbb{O}$ orientation.

Holding ζ_1, ζ_2, and ζ_3 fixed over all of 8-space makes this a global algebraic basis gauge transformation. The covariant derivative analysis above for any global algebraic gauge transformation assures this global g basis will be form invariant with the intrinsic e basis form. Additional usefulness of the gauge transformation will come into play when ζ_1, ζ_2, and ζ_3 can vary over 8-space, becoming a local basis gauge transformation.

With this algebraic basis gauge transformation creating an algebraic basis isomorphism, we are now free to assign without preference nor privilege $\{g_1\ g_2\ g_3\}$ to be the 3D physical space axial basis triplet where the magnetic field lives (part of a single Quaternion subalgebra), and $\{g_5\ g_6\ g_7\}$ to be the 3D physical space polar basis triplet where the electric field lives (not part a single Quaternion subalgebra).

As it works out, there is an additional central force living in the same Quaternion subalgebra we placed the magnetic field in, the g basis subspace $\{g_1\ g_2\ g_3\}$. This is algebraically distinct from the charge/electric field central force living in the open set polar type basis defined by the g basis subspace $\{g_5\ g_6\ g_7\}$. This fact is independent of whether or not we even do a gauge transformation. My money is on this being Gravitation using the classical potential function approach instead of space-time curvature, cleanly integrated with Electrodynamics. One may claim the “magnetic monopole” is actually Gravitation. The fact that Gravitation lives in a closed basis set likely accounts for a lack of gravitational induction paralleling EM induction. The 8-current mentioned above is a bit more than charge current, it would also include mass momentum. Just an aside.

There is an alternate construction method that will simplify as well as illuminate things going forward.

For N_4, the appropriate Quaternion subalgebra partition is $\{e_0\ e_1\ e_2\ e_3\}$. Its basic quad is $\{e_4\ e_5\ e_6\ e_7\}$.

We can recreate the N_4 g basis gauge transformation just presented with the following three points on the circle group in the $e_0\ e_4$ plane, noticing for this type of construction e_4 is one member of our basic quad set. Because it is, we will have to exclude e_4 from like modification just below since the result would include terms in basis element e_0. As shown above, having scalar content kills any chance of a basis automorphism. Define these three different parametrizations of the same unit circle as

$p_5 = \cos(\zeta_1)\, e_0 + \sin(\zeta_1)\, e_4$ odd variance parity on sine
$p_6 = \cos(\zeta_2)\, e_0 + \sin(\zeta_2)\, e_4$ odd variance parity on sine
$p_7 = \cos(\zeta_3)\, e_0 + \sin(\zeta_3)\, e_4$ odd variance parity on sine

Once again, we require $\zeta_1 + \zeta_2 + \zeta_3 = 0$, and this notably gives us $p_5 * p_6 * p_7 = +1$. The assignment of odd parity on the sine functions is required to match the non-product assigned rotations above now using the products below that will have algebraic variances inducing a parity change. This is analogous to the result parity flip used in the basic quad algebraic completion process, just done automatically by this pre-assignment.

We can now map the remaining three basic quad Octonion intrinsic basis elements excluding e_4 to the same index gauge basis g_m using the p_m by forming the product $g_m = p_m * e_m$, in a manner of speaking, "fibering" over individual basis subspaces with different but relational cross sections of the same circle group. The result is:

$g_5 = p_5 * e_5 = \cos(\zeta_1)\, e_5 - s_{541} \sin(\zeta_1)\, e_1$
$g_6 = p_6 * e_6 = \cos(\zeta_2)\, e_6 - s_{642} \sin(\zeta_2)\, e_2$
$g_7 = p_7 * e_7 = \cos(\zeta_3)\, e_7 - s_{743} \sin(\zeta_3)\, e_3$

These are identical to the gauge basis mappings $\{g_5\ g_6\ g_7\}$ above using the first approach. We can now use basic quad algebraic completion to generate the proper automorphism forms for the non-scalar basis element set $\{g_1\ g_2\ g_3\}$ using products of pairs in the set $\{g_5\ g_6\ g_7\}$ as follows undoing the induced parity changes and using the restriction $\zeta_1 + \zeta_2 + \zeta_3 = 0$:

$g_1 = s_{761}\, g_7 * g_6 = \cos(\zeta_1)\, e_1 + s_{541} \sin(\zeta_1)\, e_5$
$g_2 = s_{572}\, g_5 * g_7 = \cos(\zeta_2)\, e_2 + s_{642} \sin(\zeta_2)\, e_6$
$g_3 = s_{653}\, g_6 * g_5 = \cos(\zeta_3)\, e_3 + s_{743} \sin(\zeta_3)\, e_7$

The process does not define g_0 and g_4 so leaving these equal to their same index intrinsic basis elements we reproduce the whole of our N_4 group g algebraic basis gauge transformation developed above.

We finish up now on N_x type automorphisms with their general requirements. We have found these types take three different parametrizations of the same circle group using the complex subalgebra including one of the four basic quads, then scales the other three basic quad elements uniquely pairing one circle group with each. Taking the three simply as different complex numbers instead, working again with N_4, define $U = u_0\, e_0 + u_4\, e_4$, $V = v_0\, e_0 + v_4\, e_4$, and $W = w_0\, e_0 + w_4\, e_4$. Next form the general automorphic forms as above: $g_5 = U * e_5$, $g_6 = V * e_6$ and $g_7 = W * e_7$. Using basic quad algebraic completion, form g_1, g_2 and g_3 leaving $g_0 = e_0$ and $g_4 = e_4$. We can then form equations of constraint on U, V and W in order to have a proper automorphism from all unique solutions to equations given by

$g_a * g_b - s_{abc}\, g_c = \mathbf{0}$ the null Octonion.

All are satisfied by the following restrictions on the u, v and w coefficients:

$u_0^2 + u_4^2 = 1 \quad v_0^2 + v_4^2 = 1 \quad w_0^2 + w_4^2 = 1$
$-u_4 = v_0\, w_4 + v_4\, w_0 \quad -v_4 = u_0\, w_4 + u_4\, w_0 \quad -w_4 = u_0\, v_4 + u_4\, v_0$
$u_0 = v_0\, w_0 - v_4\, w_4 \quad v_0 = u_0\, w_0 - u_4\, w_4 \quad w_0 = u_0\, v_0 - u_4\, v_4$

The first row is satisfied with

$U = \cos(\theta)\, e_0 + \sin(\theta)\, e_4$, $V = \cos(\varphi)\, e_0 + \sin(\varphi)\, e_4$ $W = \cos(\gamma)\, e_0 + \sin(\gamma)\, e_4$

Inserting into the next two rows we have the two groupings

$-\sin(\theta) = \cos(\varphi) \sin(\gamma) + \sin(\varphi) \cos(\gamma) = \sin(\varphi + \gamma) \rightarrow (\varphi + \gamma) = -(\theta)$
$-\sin(\varphi) = \cos(\gamma) \sin(\theta) + \sin(\gamma) \cos(\theta) = \sin(\gamma + \theta) \rightarrow (\gamma + \theta) = -(\varphi)$
$-\sin(\gamma) = \cos(\theta) \sin(\varphi) + \sin(\theta) \cos(\varphi) = \sin(\theta + \varphi) \rightarrow (\theta + \varphi) = -(\gamma)$

$\cos(\theta) = \cos(\varphi) \cos(\gamma) - \sin(\varphi) \sin(\gamma) = \cos(\varphi + \gamma) \rightarrow (\varphi + \gamma) = \pm(\theta)$
$\cos(\varphi) = \cos(\gamma) \cos(\theta) - \sin(\gamma) \sin(\theta) = \cos(\gamma + \theta) \rightarrow (\gamma + \theta) = \pm(\varphi)$
$\cos(\gamma) = \cos(\theta) \cos(\varphi) - \sin(\theta) \sin(\varphi) = \cos(\theta + \varphi) \rightarrow (\theta + \varphi) = \pm(\gamma)$

These last two groupings are satisfied by the restriction $(\theta + \varphi + \gamma) = 0$. This is comparable to what we found above. The extensive nature of the restrictions might make it difficult to form a different style of N_4 solution for the u, v and w coefficients. We perhaps only have flexibility in angle choices within the restriction that they sum to 0 mod 2π.

The N_4 identity algebraic basis gauge transformation requires $\zeta_1 = \zeta_2 = \zeta_3 = 0$. Replaying their source

$\zeta_1 = \beta_2 - \beta_3$
$\zeta_2 = \beta_3 - \beta_1$
$\zeta_3 = \beta_1 - \beta_2$

We see there is no preferred identity transformation choice for β_n, they are only required to be equal.

Let's move on now to the other seven order 24 subgroups of PSL(2,7) which preserve Quaternion subalgebra triplet sets. Their order 4 normal subgroup non-identity members are also characterized by two basis element transpositions, now exclusively utilizing pairs of the basic quad set associated with the preserved Quaternion triplet. The group T_n is enumerated by the index n associated with the preserved triplet Q_n as defined above. The product of the two basis elements in each of the paired transpositions is bijectively within sign one of the basis elements in the preserved triplet. T_4 is intimately related to N_4 just covered in some detail, so we will proceed with it. Group T_4 preserves the triplet set $Q_4 = \{e_1\ e_2\ e_3\}$. Its basic quad set is $\{e_4\ e_5\ e_6\ e_7\}$. Its Klein 4-group normal subgroup is:

$A_0 = [I]$ (identity) $A_1 = [e_4\ e_5]\ [e_6\ e_7]$ $A_2 = [e_4\ e_6]\ [e_5\ e_7]$ $A_3 = [e_4\ e_7]\ [e_5\ e_6]$

To generate smooth maps instead of basis element exchanges we now follow the same path of rotations about the basis element given by the product of the two transposed basis elements, in the planes defined by them. For these A_n, n not 0 the product of transposed basis elements in each paired transposition is within sign e_n. Unlike N_x these rotation circle groups will lie exclusively within the Quaternion subalgebra of the preserved triplet rather than using any member of the basic quad partition. We can again fiber over the basic quad subspace, but now since the fibers are external to the basic quad, we can take the whole basic quad set as the subspace fibered over, since we will not produce any content scaling an e_0 basis preventing an automorphism result. Rather than follow the laborious path of different circle group scalings, we can cut to the chase so to speak by examining the general requirements to create this type of algebraic basis gauge automorphism similar to the general considerations above with the group N_4.

Since our fiber fully resides within the Q_4 triplet Quaternion subalgebra, specify a generic simple Quaternion for it. Let $F = f_0\, e_0 + f_1\, e_1 + f_2\, e_2 + f_3\, e_3$. Fiber over the basic quad subspace with F to form the automorphic products $g_n = F^*e_n$ for n: 4 to 7. Next do the basic quad algebraic completion to generate g_1, g_2 and g_3 while leaving $g_0 = e_0$ as required. We require F to generate an algebraic automorphism so we must once again insist on the following which will generate equations of constraint on F:

$g_a * g_b – s_{abc}\, g_c = \mathbf{0}$ the null Octonion.

Doing the math, we will find all equations are satisfied by simply requiring the norm of F := |F| =1. Any, but clearly not all of the f coefficients may be zero.

All four basic quad elements appear in each of our three normal subgroup dual transpositions, and the three dual transpositions indicate using smooth maps about each of the three basis elements of Q_4. This suggests scaling all four basic quads by three circle groups defined in separate $\{e_0\ \ e_n\}$ complex subalgebras for n: 1 to 3, which we will define as rotations that are oriented as the preserved Quaternion subalgebra is:

$c_1 = \cos(\alpha_1/2)\, e_0 – s_{123} \sin(\alpha_1/2)\, e_1$ all sine functions are also odd variance parity
$c_2 = \cos(\alpha_2/2)\, e_0 – s_{123} \sin(\alpha_2/2)\, e_2$
$c_3 = \cos(\alpha_3/2)\, e_0 – s_{123} \sin(\alpha_3/2)\, e_3$

The use of half angles will be justified shortly. We actually could fiber over the basic quad subspace with any of these individually, any product of two of them, or products of all three since each of these will be unity norm. The resultant basic quad gauge transformation basis set $\{g_4\ g_5\ g_6\ g_7\}$ is then used within basic quad algebraic completion to form the Quaternion subalgebra set $\{g_1\ g_2\ g_3\}$. Form the product of all three as $R_4 = c_1 * c_2 * c_3$ and force all non-scalar results to odd parity to undo the parity flip induced by its fiber product. This is a convenient choice, not a requirement. The result is:

$R_4 =$
$+\cos(\alpha_1/2) \cos(\alpha_2/2) \cos(\alpha_3/2)\, e_0$
$+\sin(\alpha_1/2) \sin(\alpha_2/2) \sin(\alpha_3/2)\, e_0$
$–s_{123} \sin(\alpha_1/2) \cos(\alpha_2/2) \cos(\alpha_3/2)\, e_1$ all non-scalar terms forced to odd variance parity
$+s_{123} \cos(\alpha_1/2) \sin(\alpha_2/2) \sin(\alpha_3/2)\, e_1$
$–s_{123} \cos(\alpha_1/2) \sin(\alpha_2/2) \cos(\alpha_3/2)\, e_2$
$–s_{123} \sin(\alpha_1/2) \cos(\alpha_2/2) \sin(\alpha_3/2)\, e_2$
$–s_{123} \cos(\alpha_1/2) \cos(\alpha_2/2) \sin(\alpha_3/2)\, e_3$
$+s_{123} \sin(\alpha_1/2) \sin(\alpha_2/2) \cos(\alpha_3/2)\, e_3$

Unrestricted, R_4 can be seen to be an 8-fold cover of the 3-sphere. This can easily be seen by examination of the antipodal points on the 3-sphere $\pm e_0$, $\pm e_1$, $\pm e_2$, $\pm e_3$, where each will have multiple $\{\alpha_1/2\ \alpha_1/2\ \alpha_1/2\}$ solution sets. This multiple cover cannot be fully reduced, since the 3-sphere is topologically distinct from the product (Quaternion or cartesian product) of three circles. R_4 is instead a Quaternion 3-torus. Set this aside for now. Fibering over the $\{e_4\ e_5\ e_6\ e_7\}$ basic quad subspace with R_4 creates the basic quad gauge transformation basis set $\{g_4\ g_5\ g_6\ g_7\}$ given by

$g_4 =$
$+\cos(\alpha_1/2) \cos(\alpha_2/2) \cos(\alpha_3/2)\, e_4$
$+\sin(\alpha_1/2) \sin(\alpha_2/2) \sin(\alpha_3/2)\, e_4$
$+s_{761} \sin(\alpha_1/2) \cos(\alpha_2/2) \cos(\alpha_3/2)\, e_5$
$–s_{761} \cos(\alpha_1/2) \sin(\alpha_2/2) \sin(\alpha_3/2)\, e_5$
$+s_{572} \cos(\alpha_1/2) \sin(\alpha_2/2) \cos(\alpha_3/2)\, e_6$
$+s_{572} \sin(\alpha_1/2) \cos(\alpha_2/2) \sin(\alpha_3/2)\, e_6$
$+s_{653} \cos(\alpha_1/2) \cos(\alpha_2/2) \sin(\alpha_3/2)\, e_7$
$–s_{653} \sin(\alpha_1/2) \sin(\alpha_2/2) \cos(\alpha_3/2)\, e_7$

$g_5 =$
$–s_{761} \sin(\alpha_1/2) \cos(\alpha_2/2) \cos(\alpha_3/2)\, e_4$
$+s_{761} \cos(\alpha_1/2) \sin(\alpha_2/2) \sin(\alpha_3/2)\, e_4$
$+\cos(\alpha_1/2) \cos(\alpha_2/2) \cos(\alpha_3/2)\, e_5$

$+\sin(\alpha_1/2)\sin(\alpha_2/2)\sin(\alpha_3/2)\, e_5$
$-s_{743}\sin(\alpha_1/2)\sin(\alpha_2/2)\cos(\alpha_3/2)\, e_6$
$+s_{743}\cos(\alpha_1/2)\cos(\alpha_2/2)\sin(\alpha_3/2)\, e_6$
$-s_{642}\sin(\alpha_1/2)\cos(\alpha_2/2)\sin(\alpha_3/2)\, e_7$
$-s_{642}\cos(\alpha_1/2)\sin(\alpha_2/2)\cos(\alpha_3/2)\, e_7$

$g_6 =$
$-s_{572}\cos(\alpha_1/2)\sin(\alpha_2/2)\cos(\alpha_3/2)\, e_4$
$-s_{572}\sin(\alpha_1/2)\cos(\alpha_2/2)\sin(\alpha_3/2)\, e_4$
$+s_{743}\sin(\alpha_1/2)\sin(\alpha_2/2)\cos(\alpha_3/2)\, e_5$
$-s_{743}\cos(\alpha_1/2)\cos(\alpha_2/2)\sin(\alpha_3/2)\, e_5$
$+\cos(\alpha_1/2)\cos(\alpha_2/2)\cos(\alpha_3/2)\, e_6$
$+\sin(\alpha_1/2)\sin(\alpha_2/2)\sin(\alpha_3/2)\, e_6$
$-s_{541}\cos(\alpha_1/2)\sin(\alpha_2/2)\sin(\alpha_3/2)\, e_7$
$+s_{541}\sin(\alpha_1/2)\cos(\alpha_2/2)\cos(\alpha_3/2)\, e_7$

$g_7 =$
$-s_{653}\cos(\alpha_1/2)\cos(\alpha_2/2)\sin(\alpha_3/2)\, e_4$
$+s_{653}\sin(\alpha_1/2)\sin(\alpha_2/2)\cos(\alpha_3/2)\, e_4$
$+s_{642}\sin(\alpha_1/2)\cos(\alpha_2/2)\sin(\alpha_3/2)\, e_5$
$+s_{642}\cos(\alpha_1/2)\sin(\alpha_2/2)\cos(\alpha_3/2)\, e_5$
$+s_{541}\cos(\alpha_1/2)\sin(\alpha_2/2)\sin(\alpha_3/2)\, e_6$
$-s_{541}\sin(\alpha_1/2)\cos(\alpha_2/2)\cos(\alpha_3/2)\, e_6$
$+\cos(\alpha_1/2)\cos(\alpha_2/2)\cos(\alpha_3/2)\, e_7$
$+\sin(\alpha_1/2)\sin(\alpha_2/2)\sin(\alpha_3/2)\, e_7$

All results above have even parity due to the odd parity force on R_4. Next use these four basic quads within the basic quad algebraic completion to form the Quaternion gauge transformation subalgebra set $\{g_1\ g_2\ g_3\}$. The results including the trivial g_0 map are:

$g_0 = e_0$

$g_1 =$
$+\cos(\alpha_2)\cos(\alpha_3)\, e_1$
$+\cos(\alpha_2)\sin(\alpha_3)\, e_2$
$-\sin(\alpha_2)\, e_3$

$g_2 =$
$+\sin(\alpha_1)\sin(\alpha_2)\cos(\alpha_3)\, e_1$
$-\sin(\alpha_3)\cos(\alpha_1)\, e_1$
$+\cos(\alpha_1)\cos(\alpha_3)\, e_2$
$+\sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)\, e_2$
$+\sin(\alpha_1)\cos(\alpha_2)\, e_3$

$g_3 =$
$+\sin(\alpha_1)\sin(\alpha_3)\, e_1$
$+\sin(\alpha_2)\cos(\alpha_1)\cos(\alpha_3)\, e_1$
$+\cos(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)\, e_2$
$-\sin(\alpha_1)\cos(\alpha_3)\, e_2$
$+\cos(\alpha_1)\cos(\alpha_2)\, e_3$

Notice for the set $\{g_1\ g_2\ g_3\}$, we have converted all half angles to full angles. These forms are a representation of an algebraic invariant Euler Angle basis for the Quaternion subalgebra defined by the preserved triplet. If our initial definitions for c_1, c_2 and c_3 were not oriented by the structure constant s_{123} this would not be the case, particular portions of the Euler Angles would indicate orientations. Either way, the transformation matrix for this g basis is seen to be orthonormal as required.

If we used a different product order for the creation of $R_4 = c_1 * c_2 * c_3$, the basic quad g forms will have some sign changes and we will shuffle the representations of Euler Angles. All basic quads will remain in terms of half-angles, and all Euler Angle bases will remain in terms of full angles. Every full gauge basis transformation representation will be an isomorphism with the chosen intrinsic e basis Octonion orientation. The different Euler Angle representations are the Octonion equivalent of the three-dimensional fact that three rotations performed on an ordinary vector will not result in the same outcome if the order of rotation is changed. This is the genesis of the different known forms for common cartesian Euler Angle transformations. Different generating rotation order and directions lead to different forms, but all should be generally considered proper Euler Angle representations.

Our algebraic basis gauge transformations have been presented as transformations directly on the intrinsic basis set e producing algebraic automorphisms/isomorphisms. As such, call them *primary algebraic automorphisms*. If we have two primary algebraic automorphisms defined as $a_i = A_{ij}\ e_j$ and $b_i = B_{ij}\ e_j$ we can form a composition of these by either replacing all e_n in a_m with b_n or by replacing all e_n in b_m with a_n. We can see both will result in another algebraic automorphism by writing out each replacement:

$a'_i = A_{ij}\ B_{jk}\ e_k = C_{ik}\ e_k$ where C is the matrix product AB

$b'_i = B_{ij}\ A_{jk}\ e_k = D_{ik}\ e_k$ where D is the matrix product BA

One requirement on A, B, C and D forming an algebraic automorphism is they must be orthonormal matrices. Matrices A and B are given to be orthonormal, and C and D will also be orthonormal since the matrix product of two orthonormal matrices will always be orthonormal. We must additionally prove we still have a basis isomorphism.

From our fundamental gauge transformation $g_r = T_{rs}\ e_s$ we have both

$g_u * g_v = T_{ur}\ e_r * T_{vs}\ e_s = s_{rs(r\hat{}s)}\ T_{ur}\ T_{vs}\ e_{(r\hat{}s)}$

$g_u * g_v = s_{uvw}\ g_w = s_{uvw}\ T_{wy}\ e_y$

Like basis index y scalings for these two must be an equality. Equate them by limiting the otherwise free sums over r and s to r^s = y for each y individually. Remove e_y from both sides. This fixes y in addition to fixed u,v. and w

$\Sigma_{r\hat{}s=y}\ s_{rsy}\ T_{ur}\ T_{vs} = s_{uvw}\ T_{wy}$

Restricting to r^s=y gives r^s^y= 0. Since y is fixed, we can replace s with r^y. Summation convention returns to operational status, specifying fixed u,v,w,y and summed r. Replace dependent w with w = u^v

$s_{uv(u\hat{}v)}\ T_{(u\hat{}v)y} = s_{r(r\hat{}y)y}\ T_{ur}\ T_{v(r\hat{}y)}$ (eq. 1)

Individually for fixed u, v, and y summing r (eq. 1) is another requirement on any T for it to produce an algebraic basis gauge transformation.

Assume our matrix product C = AB is a proper algebraic basis gauge transformation. Replace T with C in (eq. 1).

$s_{uv(u\wedge v)}\, C_{(u\wedge v)y} = s_{r(r\wedge y)y}\, C_{ur}\, C_{v(r\wedge y)}$

Insert the representative matrix products:
$C_{(u\wedge v)y} = A_{(u\wedge v)i}\, B_{iy}$
$C_{ur} = A_{uj}\, B_{jr}$
$C_{v(r\wedge y)} = A_{vk}\, B_{k(r\wedge y)}$

This gives for summed r,i,j,k and fixed u,v,y

$s_{uv(u\wedge v)}\, A_{(u\wedge v)i}\, B_{iy} = s_{r(r\wedge y)y}\, A_{uj}\, B_{jr}\ A_{vk}\, B_{k(r\wedge y)}$ (eq. 2)

Our new transformation requirement (eq. 1) on given proper B is $s_{uv(u\wedge v)}\, B_{(u\wedge v)y} = s_{r(r\wedge y)y}\, B_{ur}\, B_{v(r\wedge y)}$. Replace dummy fixed indexes u,v with u→j and v→k.

$s_{uv(u\wedge v)}\, B_{(u\wedge v)y} = s_{r(r\wedge y)y}\, B_{ur}\, B_{v(r\wedge y)} \rightarrow s_{jk(j\wedge k)}\, B_{(j\wedge k)y} = s_{r(r\wedge y)y}\, B_{jr}\, B_{k(r\wedge y)}$ (eq. 3)

The right side of (eq. 3) is found in (eq. 2). Substitute in:

$s_{uv(u\wedge v)}\, A_{(u\wedge v)i}\, B_{iy} = A_{uj}\, A_{vk}\, s_{jk(j\wedge k)}\, B_{(j\wedge k)y}$ (eq. 4)

Multiply both sides of (eq. 4) by B_{yp} sum over y

$s_{uv(u\wedge v)}\, A_{(u\wedge v)i}\, B_{iy}\, B_{yp} = A_{uj}\, A_{vk}\, s_{jk(j\wedge k)}\, B_{(j\wedge k)y}\, B_{yp} \rightarrow s_{uv(u\wedge v)}\, A_{(u\wedge v)i}\, \delta_{ip} = A_{uj}\, A_{vk}\, s_{jk(j\wedge k)}\, \delta_{(j\wedge k)p}$ This yields

$s_{uv(u\wedge v)}\, A_{(u\wedge v)p} = s_{jkp}\, A_{uj}\, A_{vk}$ (eq. 5)

Rename dummy indexes in (eq.5): p→y, j→r, k→r^y.

$s_{uv(u\wedge v)}\, A_{(u\wedge v)y} = s_{r(r\wedge y)y}\, A_{ur}\, A_{v(r\wedge y)}$ This our general requirement (eq. 1) replacing T with A, given proper.

This proves our composition or equivalently the matrix product of two algebraic basis gauge transformation matrices will always produce another algebraic basis gauge transformation.

The two composition replacement directions will not generally have the same result since matrix products generally do not commute. Call the composition of two primary algebraic automorphisms a *secondary algebraic automorphism*. For completeness defining terms used below, take the next step and define a *tertiary algebraic automorphism* as the composition of a primary and a secondary algebraic automorphism. Clearly this composition process can be repeated ad nauseum without any limitation on the two being composed other than both representing proper algebraic automorphisms.

Algebraic basis gauge transformation matrices form a group with goup operation matrix multiplication. Equivalently algebraic basis gauge transformations form an isomorphic group under composition. Algebraic basis gauge transformation group closure is assured by the proof above. This group is the automorphism group of all Octonion algebraic basis gauge transformations.

Applying this composition on our N_4 algebraic basis gauge transformation, define two such gauge transformations, g as above, and a new one called h where the angle set maintains index but changes the base angle as follows:

$g_0 = e_0$
$g_1 = e_1 \cos(\zeta_1) + s_{541}\, e_5 \sin(\zeta_1)$
$g_2 = e_2 \cos(\zeta_2) + s_{642}\, e_6 \sin(\zeta_2)$
$g_3 = e_3 \cos(\zeta_3) + s_{743}\, e_7 \sin(\zeta_3)$
$g_4 = e_4$

$g_5 = e_5 \cos(\zeta_1) - s_{541}\, e_1 \sin(\zeta_1)$
$g_6 = e_6 \cos(\zeta_2) - s_{642}\, e_2 \sin(\zeta_2)$
$g_7 = e_7 \cos(\zeta_3) - s_{743}\, e_3 \sin(\zeta_3)$

$h_0 = e_0$
$h_1 = e_1 \cos(\varphi_1) + s_{541}\, e_5 \sin(\varphi_1)$
$h_2 = e_2 \cos(\varphi_2) + s_{642}\, e_6 \sin(\varphi_2)$
$h_3 = e_3 \cos(\varphi_3) + s_{743}\, e_7 \sin(\varphi_3)$
$h_4 = e_4$
$h_5 = e_5 \cos(\varphi_1) - s_{541}\, e_1 \sin(\varphi_1)$
$h_6 = e_6 \cos(\varphi_2) - s_{642}\, e_2 \sin(\varphi_2)$
$h_7 = e_7 \cos(\varphi_3) - s_{743}\, e_3 \sin(\varphi_3)$

If we now form the composition of these by replacing the e_n in g with h_n the result, call gh is the following:

$gh_0 = e_0$
$gh_1 = e_1 \cos(\varphi_1 + \zeta_1) + s_{541}\, e_5 \sin(\varphi_1 + \zeta_1)$
$gh_2 = e_2 \cos(\varphi_2 + \zeta_2) + s_{642}\, e_6 \sin(\varphi_2 + \zeta_2)$
$gh_3 = e_3 \cos(\varphi_3 + \zeta_3) + s_{743}\, e_7 \sin(\varphi_3 + \zeta_3)$
$gh_4 = e_4$
$gh_5 = e_5 \cos(\varphi_1 + \zeta_1) - s_{541}\, e_1 \sin(\varphi_1 + \zeta_1)$
$gh_6 = e_6 \cos(\varphi_2 + \zeta_2) - s_{642}\, e_2 \sin(\varphi_2 + \zeta_2)$
$gh_7 = e_7 \cos(\varphi_3 + \zeta_3) - s_{743}\, e_3 \sin(\varphi_3 + \zeta_3)$

We find the composition of two different N_x gauge transformations, same x, yields a result that is sine/cosine/variance form invariant, with angular results the sum of respective angles for the two composed transformations. The composition of two N_x algebraic basis gauge transformations will commute; it does not matter which is inserted into the other. N_x algebraic basis gauge transformations form an abelian group for each x. Clearly the sum to zero mod 2π angle restriction is maintained by the composition. This gives the N_x composition the look of the direct product U(1)xU(1)xU(1), but within the angle sum restriction.

The flexibility afforded by the norm +1 T_n type subgroups also bolts up nicely to the composition process just outlined. To see this, let's take the T_4 group consideration one circle group at a time rather than subspace fibering with the triple product $R_4 = c_1 * c_2 * c_3$. Repeating the definitions for our three unit circles we have:

$c_1 = \cos(\alpha_1/2)\, e_0 - s_{123} \sin(\alpha_1/2)\, e_1$ all sine coefficients are odd variance parity
$c_2 = \cos(\alpha_2/2)\, e_0 - s_{123} \sin(\alpha_2/2)\, e_2$
$c_3 = \cos(\alpha_3/2)\, e_0 - s_{123} \sin(\alpha_3/2)\, e_3$

Taking these one at a time, fibering with each over the full intrinsic basis basic quad subspace then doing the basic quad algebraic completion, the results are as follows:

$c_1g_0 = e_0$
$c_1g_1 = e_1$
$c_1g_2 = \cos(\alpha_1)\, e_2 + \sin(\alpha_1)\, e_3$
$c_1g_3 = \cos(\alpha_1)\, e_3 - \sin(\alpha_1)\, e_2$
$c_1g_4 = \cos(\alpha_1/2)\, e_4 + s_{761} \sin(\alpha_1/2)\, e_5$
$c_1g_5 = \cos(\alpha_1/2)\, e_5 - s_{761} \sin(\alpha_1/2)\, e_4$
$c_1g_6 = \cos(\alpha_1/2)\, e_6 + s_{541} \sin(\alpha_1/2)\, e_7$
$c_1g_7 = \cos(\alpha_1/2)\, e_7 - s_{541} \sin(\alpha_1/2)\, e_6$

$c_2g_0 = e_0$

$c_2g_1 = \cos(\alpha_2)\ e_1 - \sin(\alpha_2)\ e_3$
$c_2g_2 = e_2$
$c_2g_3 = \cos(\alpha_2)\ e_3 + \sin(\alpha_2)\ e_1$
$c_2g_4 = \cos(\alpha_2/2)\ e_4 + s_{572} \sin(\alpha_2/2)\ e_6$
$c_2g_5 = \cos(\alpha_2/2)\ e_5 - s_{642} \sin(\alpha_2/2)\ e_7$
$c_2g_6 = \cos(\alpha_2/2)\ e_6 - s_{572} \sin(\alpha_2/2)\ e_4$
$c_2g_7 = \cos(\alpha_2/2)\ e_7 + s_{642} \sin(\alpha_2/2)\ e_5$

$c_3g_0 = e_0$
$c_3g_1 = \cos(\alpha_3)\ e_1 + \sin(\alpha_3)\ e_2$
$c_3g_2 = \cos(\alpha_3)\ e_2 - \sin(\alpha_3)\ e_1$
$c_3g_3 = e_3$
$c_3g_4 = \cos(\alpha_3/2)\ e_4 + s_{653} \sin(\alpha_3/2)\ e_7$
$c_3g_5 = \cos(\alpha_3/2)\ e_5 + s_{743} \sin(\alpha_3/2)\ e_6$
$c_3g_6 = \cos(\alpha_3/2)\ e_6 - s_{743} \sin(\alpha_3/2)\ e_5$
$c_3g_7 = \cos(\alpha_3/2)\ e_7 - s_{653} \sin(\alpha_3/2)\ e_4$

Just as with the subspace fibration using R_4 we see the basic quad half angles are converted to whole angles in the algebraic invariant preserved Quaternion subalgebra components. All three can be seen to be algebraic isomorphisms with any chosen intrinsic basis element algebra. Algebraic basis gauge transformation c_ng is seen to be a rotation by full angle about e_n within the plane defined by the other two triplet members of the preserved Quaternion subalgebra Q_4. This gauge transformation also includes two rotations by half angle about e_n in the two planes orthogonal to e_n defined by the pairs of basic quad members for Q_4 whose products are both within sign e_n.

Now create a secondary automorphism by replacing each e_n in c_2g with c_3g_n. Call the result $c_{23}g$ which follows:

$c_{23}g_0 = e_0$
$c_{23}g_1 = +\cos(\alpha_2)\cos(\alpha_3)\ e_1 + \cos(\alpha_2)\sin(\alpha_3)\ e_2 - \sin(\alpha_2)\ e_3$
$c_{23}g_2 = -\sin(\alpha_3)\ e_1 + \cos(\alpha_3)\ e_2$
$c_{23}g_3 = +\sin(\alpha_2)\cos(\alpha_3)\ e_1 + \sin(\alpha_2)\sin(\alpha_3)\ e_2 + \cos(\alpha_2)\ e_3$
$c_{23}g_4 =$
$+\cos(\alpha_2/2)\cos(\alpha_3/2)\ e_4 - s_{761} \sin(\alpha_2/2)\sin(\alpha_3/2)\ e_5 + s_{572} \sin(\alpha_2/2)\cos(\alpha_3/2)\ e_6 + s_{653} \cos(\alpha_2/2)\sin(\alpha_3/2)\ e_7$
$c_{23}g_5 =$
$+s_{761} \sin(\alpha_2/2)\sin(\alpha_3/2)\ e_4 + \cos(\alpha_2/2)\cos(\alpha_3/2)\ e_5 + s_{743} \cos(\alpha_2/2)\sin(\alpha_3/2)\ e_6 - s_{642} \sin(\alpha_2/2)\cos(\alpha_3/2)\ e_7$
$c_{23}g_6 =$
$-s_{572} \sin(\alpha_2/2)\cos(\alpha_3/2)\ e_4 - s_{743} \cos(\alpha_2/2)\sin(\alpha_3/2)\ e_5 + \cos(\alpha_2/2)\cos(\alpha_3/2)\ e_6 - s_{541} \sin(\alpha_2/2)\sin(\alpha_3/2)\ e_7$
$c_{23}g_7 =$
$-s_{653} \cos(\alpha_2/2)\sin(\alpha_3/2)\ e_4 + s_{642} \sin(\alpha_2/2)\cos(\alpha_3/2)\ e_5 + s_{541} \sin(\alpha_2/2)\sin(\alpha_3/2)\ e_6 + \cos(\alpha_2/2)\cos(\alpha_3/2)\ e_7$

Now form the product of these two circle groups $c_2 * c_3$, call $= c_{23}$, we have the following:

$c_{23} =$ (odd variance parity forced on non-scalar terms for the usual reason)
$+\cos(\alpha_2/2)\cos(\alpha_3/2)\ e_0 + s_{123} \sin(\alpha_2/2)\sin(\alpha_3/2)\ e_1 - s_{123} \sin(\alpha_2/2)\cos(\alpha_3/2)\ e_2 - s_{123} \cos(\alpha_2/2)\sin(\alpha_3/2)\ e_3$

If we now fiber over the subspace $(e_4 + e_5 + e_6 + e_7)$ with c_{23} we will find $c_{23} * e_n = c_{23}g_n$ above for n: 4,5,6,7. As we did for fibering with R_4, do the basic quad algebraic completion for n: 1,2,3. Doing so reproduces the totality of $c_{23}g$ we originally derived as a secondary algebraic automorphism composition. c_{23} can be seen to be a Quaternion 2-torus, the Quaternion product of two circles.

Notice that $c_{23}g_1$, $c_{23}g_2$ and $c_{23}g_3$ are an algebraic invariant representation of a classical spherical-polar orthonormal (θ, φ, r) basis respectively, indeed a standard 3D basis representation of the 2-sphere for $r = 1$.

Thus, we have an algebraic method to embed the Quaternion 2-torus in 4D to the 2-sphere in a 3D representation, all within an Octonion Algebra framework. Keep in mind the use of half angles in the circle groups. If we use the full circles in both circles of the 2-torus, α_n ranges from 0 to 4π which does more than a double cover of the 2-sphere.

This Octonion representation of a spherical-polar orthonormal (θ, φ, r) basis embedded in the Quaternion subalgebra is extremely interesting and important. When a 3D cartesian xyz basis is mapped to a spherical-polar basis, or equivalently restricting the covariant Ensemble Derivative to a Quaternion subalgebra with a similar transformation, the Jacobian of the transformation is $r^2 \sin(\theta)$ which is obviously zero for r=0 or $\sin(\theta)=0$. This is problematic, and the typical approach is to simply turn a blind eye to it. When we cast classical spherical-polar coordinates as a Quaternion subalgebra of an Octonion Algebra algebraic basis gauge transformation, the Jacobian is identically +1 or c = the speed of light, independent of any angle or radius. 20-20 vision eyes wide open, no problem in sight.

If we now multiply c_{23} on the left by c_1 the result will be R_4 from above. We might expect, and indeed it is true that forming a circle group tertiary automorphism by the composition replacing all e_n in c_1g with the secondary automorphism $c_{23}g_n$ reproduces the T_4 group algebraic basis gauge transformation g above. So, we see this g is reduceable, it is the composition of three circle group automorphisms just as R_4 is the Quaternion triple product of the same circle group representations.

Following the logic above for an algebraic method for embedding the Quaternion 4D 2-torus into the 3D 2-sphere, we can say that the T_4 group basis gauge transformation g_1, g_2 and g_3 are also an algebraic embedding. As mentioned above, R_4 appears Quaternion 3-torus. This is embedded into a 3D representation as a doubled angle Euler Angle representation. Again, with the use of half angles in the three circle groups, using the full circle causes α_n to range from 0 to 4π, which does more than a double cover of the Euler angle representation. The prudent thing to do is probably limiting the range of the angles meaningful for single covers of spherical-polar or Euler Angle bases, and thus restricting the range of the circle group parametrizations whose products source the fibers over the basic quad g subspace.

Setting $\alpha_1 = 0$ in this particular Euler angle representation clearly will reproduce the $c_{23}g$ algebraic basis element gauge transformation appropriately covering a spherical-polar orthonormal basis within the preserved Quaternion subalgebra triplet. One could say these Euler Angle and spherical-polar forms are compatible or mutually appropriate.

In conclusion, the basic quad subspace fibration with basic quad algebraic completion method provides a beautiful and general method to create algebraic basis gauge transformations. The composition presented is a simple method to meaningfully combine two known algebraic basis gauge transformations to form another.

The N_x group gauge transformations can either be explicitly carried in the Octonion mathematical physics, or more simply its ability to gauge out the symmetries that give four equivalent choices to place 3D physical entity types within closed set multiplication rules, justifies this assignment being a free choice along with the free choice of non-spatial basis element. Inclusion is obviously required if this gauge is desired to be a local gauge, and not required in the case of a global gauge where form invariance makes it a moot point.

The T_4 group gauge transformations do not require a non-spatial basis element choice. Instead, the physical spatial-temporal space is a Quaternion subalgebra, and its basic quad set in its entirety may be considered required extra-spatial. The common 3D Euler Angle and spherical-polar basis representations are perhaps more meaningful when embedded within the direct physical Quaternion subalgebra of a full Octonion Algebra mathematical physics representation. Pathological issues caused by zero valued transformation Jacobians leading to x/0 coefficients are avoided since being a proper algebraic basis gauge transformation, the full Octonion transformation has a Jacobian that is always an extremely nice non-zero +constant value, +1 or +c.

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