# The expression of binomial formula $(a+b)^n$ when n is a prime

Nguyen Van Quang
Hue - Vietnam, 10-2022

**Abstract**

*We give the expression of binomial formula $(a+b)^n$ when $n$ is a prime number.*

The binomial formula :
$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ... + b^n$

If n is a prime number,$n = p > 3$, we obtain:

$$(a+b)^p = a^p + b^p + pab(a+b)(a^2+ab+b^2)M_{a,b}$$

$M_{a,b}$ : Polynomial(a,b) of degree $p-5$

For p = 5:

$(a+b)^5 = a^5 + b^5 + 5ab(a+b)(a^2+ab+b^2)$

For p = 7:

$(a+b)^7 = a^7+b^7+7ab(a+b)(a^2+ab+b^2)(a^2+ab+b^2) = a^7+b^7+5ab(a+b)(a^2+ab+b^2)^2$

For p = 11:

$(a+b)^{11} = a^{11}+b^{11}+11ab(a+b)(a^2+ab+b^2)(a^6+3a^5b+7a^4b^2+9a^3b^3+7a^2b^4+3ab^5+b^6)$

$(a+b)^{11} = a^{11}+b^{11}+11ab(a+b)(a^2+ab+b^2)((a^2+ab+b^2)^3+a^2b^2(a+b)^2)$

$(a^2+ab+b^2)^3+a^2b^2(a+b)^2 = (a^2-ab+b^2+2ab)^3+a^2b^2(a^2-ab+b^2+2ab)$

$= (a^2-ab+b^2)^3+6(a^2-ab+b^2)^2ab+12(a^2-ab+b^2)a^2b^2+8a^3b^3+a^2b^2(a^2-ab+b^2)+3a^3b^3$
$= (a^2-ab+b^2)[(a^2-ab+b^2)^2+6(a^2-ab+b^2)ab+13a^2b^2]+11a^3b^3$
$= (a^2-ab+b^2)[(a^2-ab+b^2+3ab)^2+4a^2b^2]+11a^3b^3$
$= (a^2-ab+b^2)[(a+b)^4+4a^2b^2]+11a^3b^3$
is not divisible by 11 ( since 11 = 3n + 2, 4m + 3 )

For p = 13:

$(a+b)^{13} = a^{13}+b^{13}+13ab(a+b)(a^2+ab+b^2)(a^8+4a^7b+12a^6b^2+22a^5b^3+27a^4b^4+22a^3b^5+22a^2b^6+4ab^7+b^8)$

$(a+b)^{13} = a^{13}+b^{13}+13ab(a+b)(a^2+ab+b^2)^2(a^6+3a^5b+8a^4b^2+11a^3b^3+8a^2b^4+3ab^5+b^6)$

For p = 17:

$(a+b)^{17} = a^{17}+b^{17}+17ab[a^{15}+b^{15}+8(a^{14}b+ab^{14})+40(a^{13}b^2+a^2b^{13})+140(a^{12}b^3+a^3b^{12})+28.13(a^{11}b^4+a^4b^{11})+28.26(a^{10}b^5+a^5b^{10})+26.44(a^9b^6+a^6b^9)+13.11.10(a^8b^7+a^7b^8)]$

$(a+b)^{17} = a^{17}+b^{17}+17ab(a+b)(a^2+ab+b^2)[a^{12}+b^{12}+6(a^{11}b+ab^{11})+26(a^{10}b^2+a^2b^{10})+75(a^9b^3+a^3b^9)+156(a^8b^4+a^4b^8)+240(a^7b^5+a^5b^7)+277a^6b^6]$

Proof:

We have:
$(a+b)^5 = a^5+b^5+5ab(a+b)(a^2+ab+b^2)$
In other words $a^2+ab+b^2$ is a factor of $(a+b)^5-(a^5+b^5)$

1. $p = 3m+1$

$p = 3m+1 \Leftrightarrow p = 6k+7$
$(a+b)^p = (a+b)^{2(3k+1)}(a+b)^5 = (a^2+2ab+b^2)^{3k+1}(a+b)^5$
$= (a^2+ab+b^2+ab)^{3k+1}(a+b)^5 = [(a^2+ab+b^2)N + a^{3k+1}b^{3k+1}](a+b)^5 = (a^2+ab+b^2)N(a+b)^5 + a^{3k+1}b^{3k+1}(a+b)^5$
and
$a^p+b^p = a^{6k+7}+b^{6k+7} = (a^{3k+6)}-b^{3k+6)})(a^{3k+1}-b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)$

$a^p+b^p = (a^{3(k+2)}-b^{3(k+2)})(a^{3k+1}-b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)$
$a^{3(k+2)}-b^{3(k+2)} = (a^3-b^3)L$
$\Rightarrow a^p+b^p = (a^3-b^3)L(a^{3k+1}-b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)$
$\Rightarrow (a+b)^p-(a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 + a^{3k+1}b^{3k+1}(a+b)^5 - [(a^3-b^3)L(a^{3k+1}-b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)]$
$\Rightarrow (a+b)^p-(a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{3k+1}-b^{3k+1}) + a^{3k+1}b^{3k+1}(a+b)^5 - a^{3k+1}b^{3k+1}(a^5+b^5)$
$\Rightarrow (a+b)^p-(a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{3k+1}-b^{3k+1}) + a^{3k+1}b^{3k+1}[(a+b)^5-(a^5+b^5)]$

$(a^2+ab+b^2)N(a+b)^5, (a^3-b^3)L(a^{3k+1}-b^{3k+1}), a^{3k+1}b^{3k+1}[(a+b)^5-(a^5+b^5)]$ have a common factor $a^2+ab+b^2$, hence $(a+b)^p-(a^p+b^p)$ has the factor $a^2+ab+b^2$

2. $p = 3n+2$

$p = 3n+2 \Rightarrow p = 6k+5$
$(a+b)^p = (a+b)^{6k}(a+b)^5 = (a^2+2ab+b^2)^{3k}(a+b)^5$
$= (a^2+ab+b^2+ab)^{3k}(a+b)^5 = [(a^2+ab+b^2)N + a^{3k}b^{3k}](a+b)^5 = (a^2+ab+b^2)N(a+b)^5 + a^{3k}b^{3k}(a+b)^5$
and
$a^p+b^p = a^{6k+5}+b^{6k+5} = (a^{3k+5}-b^{3k+5})(a^{3k}-b^{3k}) + a^{3k}b^{3k}(a^5+b^5)$

$a^p+b^p = (a^{3k+5}-b^{3k+5})(a^{3k}-b^{3k}) + a^{3k}b^{3k}(a^5+b^5)$
$a^{3k}-b^{3k} = (a^3-b^3)L$
$\Rightarrow a^p+b^p = (a^3-b^3)L(a^{3k+5}-b^{3k+5}) + a^{3k}b^{3k}(a^5+b^5)$

$\Rightarrow (a+b)^p - (a^p + b^p) = (a^2+ab+b^2)N(a+b)^5 + a^{3k}b^{3k}(a+b)^5 - [(a^3-b^3)L(a^{3k+5} - b^{3k+5}) + a^{3k}b^{3k}(a^5+b^5)]$

$\Rightarrow (a+b)^p - (a^p + b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{3k+5} - b^{3k+5}) + a^{3k}b^{3k}(a+b)^5 - a^{3k}b^{3k}(a^5+b^5)]$

$\Rightarrow (a+b)^p - (a^p + b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{3k+5} - b^{3k+5}) + a^{3k}b^{3k}[(a+b)^5 - (a^5+b^5)]$

$(a^2+ab+b^2)N(a+b)^5, (a^3-b^3)L(a^{3k+5}-b^{3k+5}), a^{3k}b^{3k}[(a+b)^5-(a^5+b^5)]$ have a common factor $a^2+ab+b^2$, hence $(a+b)^p - (a^p+b^p)$ has the factor $a^2+ab+b^2$

It is easy to show that p, a, b and (a + b) are the factors of $(a+b)^p - (a^p+b^p)$, finally we obtain:

$(a+b)^p = a^p + b^p + pab(a+b)(a^2+ab+b^2)M_{a,b}$

Note that if $a, b = 1$ then $a, b, (a+b), (a^2+ab+b^2) = 1$

We also get:

$a^p + b^p = (a+b)^p - pab(a+b)(a^2+ab+b^2)M_{a,b}$

$$a^p + b^p = (a+b)[(a+b)^{p-1} - pab(a^2+ab+b^2)M_{a,b}]$$

$$a^{p-1} - a^{p-2}b + a^{p-3}b^2 - ... + b^{p-1} = (a+b)^{p-1} - pab(a^2+ab+b^2)M_{a,b}$$

# References


[1] Quang N V, A new solvable quintic equation of the Bring - Jerrard form $x^5 + ax + b = 0$, Vixra: 2108.0060 (AL)

[2] Quang N V, A new solvable quintic equation of the shape $x^5 + a^2x + b = 0$, Vixra: 2104.0008 (AL)

[3] Quang N V, A proof of the four color theorem by induction Vixra: 1601.0247 (CO)cited by IPJO 2020, London, UK and others - Semanticscholar.org:124682326



Email:
nguyenvquang67@gmail.com
quangnhu67@yahoo.com.vn