The Zeta function as a particular case of The Euler-Maclaurin Formula

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Abstract

One of the most famous functions full of enigmas is the Riemann zeta function, since it is the basis of one of the most surprising hypotheses because of its relation with the Prime counting function. But in this paper, we will reveal some of those enigmas, using the Euler-Maclaurin formula.

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1 Introduction.

1.1 Definition of the Riemann Zeta Function.

The Riemann Zeta function is a function $\mathbb{C} \to \mathbb{C}$, defined as:

$$\zeta(s) := \sum_{m=1}^{\infty} \frac{1}{m^s} \tag{1}$$

where: $s \in \mathbb{C}$

This series converges when $Re[s] > 1$

Riemann managed to give it analytical continuity, obtaining the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

This function has a single pole in $\zeta(1)$.

1.2 Formulas for specific cases of the Riemann Zeta Function.

In addition to the functional equation, other formulas are known to obtain zeta:

Let: $s = 2k, k \in \mathbb{N}$

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!} \tag{3}$$

Let $s = -k, k \in \mathbb{N}$

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \tag{4}$$

Where: B_{2k} and B_{-k} are Bernoulli numbers.

2 Sums of Powers and the Bernoulli Numbers.

The Bernoulli numbers B_j are a set of successive rational numbers with relevant importance in number theory. They appear in combinatorics, in the expansion of tangent function and the hyperbolic tangent function by Taylor series, and the Euler Maclaurin equation.

One of the ways to obtain these numbers is through recursive addition:

$$B_j = -\frac{1}{1+j} \sum_{m=0}^{j-1} \binom{1+j}{m} B_m \tag{5}$$

Bernoulli numbers appear when we want to find a function to calculate sum of powers. Their formula was first obtained by Jakob Bernoulli, this formula is described below, but somewhat refined and with current nomenclature.

$$\sum_{m=1}^{n} m^k = S_k(n) = \sum_{p=1}^{1+k} A_p n^p \tag{6}$$

Where: $k \in \mathbb{N} \cup \{0\}$*, and* S_k *is a function of* n*.*

A_p is obtained by:

$$A_p = \frac{(-1)^{1+k-p}}{1+k} \binom{1+k}{p} B_{1+k-p} \tag{7}$$

3 Sum of powers as a sum of higher order derivatives.

In this section we will rearrange equation (6), and compare it to a sum of higher order derivatives of the function m^k.

Factoring $\frac{1}{1+k}$ and developing the summation of the equation (6) to find $S_k(n)$ we obtain:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^k(k+1)!}{1!k!}B_k n + \frac{(-1)^{k-1}(k+1)!}{2!(k-1)!}B_{k-1}n^2 + \frac{(-1)^{k-2}(k+1)!}{3!(k-2)!}B_{k-2}n^3 + \frac{(-1)^{k-3}(k+1)!}{4!(k-3)!}B_{k-3}n^4 + \cdots + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^0(k+1)!}{(k+1)!(0)!}B_0 n^{k+1}\right]$$

Now we will reverse the order of the summands, and simplify by reducing the factorials of each term:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^2(k+1)!}{(k-1)!2!}B_2 n^{k-1} + \cdots \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1 k!(k+1)}{k!1!}B_1 n^k + \frac{(-1)^2(k-1)!k(k+1)}{(k-1)!2!}B_2 n^{k-1} + \cdots \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0}{0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)}{1!}B_1 n^k + \frac{(-1)^2 k(k+1)}{2!}B_2 n^{k-1} + \cdots \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

It is possible to rewrite the formula as a summation of a production included:

$$\sum_{m=1}^{n} m^k = S_k(n) = \frac{1}{1+k}n^{k+1} + \sum_{p=2}^{1+k}\frac{(-1)^{p-1}B_{p-1}}{(1+k)(p-1)!}\prod_{m=1}^{p-1}(2+k-m)n^{2+k-p} \tag{8}$$

If we are observant, it is possible to obtain an equivalent formula with a summation where higher order derivatives of the function n^k are included as follows:

$$\sum_{m=1}^{n} m^k = S_k(n) = \int n^k dn + \sum_{p=1}^{k}\frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right) \tag{9}$$

Also:

$$\sum_{m=1}^{n}\frac{1}{m^{-k}} = S_{-k}(n) = \int \frac{1}{n^{-k}}dn + \sum_{p=1}^{k}\frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^{-k}}\right) \tag{10}$$

Therefore, we can say that equations (8), (9) and (10) are equivalent.

4 The Euler-Maclaurin Formula and the Riemann Zeta Function.

Given the well-known Euler-Maclaurin Formula [1], to approximate the sum of a function $f(m), \mathbb{R} \to \mathbb{R}$, and it is q times derivable:

$$\sum_{m=a+1}^{b} f(m) = \int_a^b f(x)dx + \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!}\left[f^{(p-1)}(b) - f^{(p-1)}(a)\right] + r_q \tag{11}$$

Where:

$$r_q = (-1)^q \int_0^n \frac{B_{q+1}(m - \lfloor m \rfloor)}{(q+1)!} f^{(q+1)}(m)dm \tag{12}$$

$f^{(p-1)}(b)$ and $f^{(p-1)}(a)$ are the derivative of order $p-1$ of the function $f(m)$ evaluated at b, and a respectively, and r_q is the residual error of the approximation to the q-th derivative. $B_j(x)$ is a Bernoulli polynomial obtained by the following formula:

$$B_j(x) = \sum_{m=0}^{j} (-1)^m \binom{j}{m} B_m x^{j-m} \tag{13}$$

If we define the limits of the summation of equation (11) at $a = 0$ and $b = n$, and define the function $f(m) = m^s$, f being a complex function, but of real variable, m^s and s^s, m^s being $f(m)$ q times derivable, we obtain:

$$\sum_{m=1}^{n} m^s = \int_0^n m^s dm + \sum_{p=1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dm^{p-1}} (m^s) \Big|_0^n + r_q \tag{14}$$

Where $s \in \mathbb{C}$.

If we add the initial limits of the integral and the derivatives, and accumulate them to the residual error, we can write:

$$\sum_{m=1}^{n} m^s = \int n^s dn + \sum_{p=1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} (n^s) + R_q \tag{15}$$

Where R_q is the total cumulative residual error.

In the same way we can obtain an expression for the sum of the function $f(m) = \frac{1}{m^s}$:

$$\sum_{m=1}^{n} \frac{1}{m^s} = \int \frac{1}{n^s} dn + \sum_{p=1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^s}\right) + R_q \tag{16}$$

Where $s \in \mathbb{C}$.

If $n \to \infty$, and if we consider R_q as a function of s then $R_q(s)$ can be written by clearing from equation (16):

$$R_q(s) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^s} - \int \frac{1}{n^s} dn - \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^s}\right) \right] \tag{17}$$

Where $s \in \mathbb{C}$.

Developing Equation (17) we obtain:

$$R_q(s) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^s} - \left(\frac{1}{1-s} n^{1-s} + \frac{(-1)^1 B_1}{1!} n^{-s} + \frac{(-1)^2 B_2}{2!} (-s) n^{-1-s} + \frac{(-1)^3 B_3}{3!} (-s)(-1-s) n^{-2-s} + \cdots \right) \right] \tag{18}$$

Where $s \in \mathbb{C}$.

We will now analyze 3 cases for equation (18)

Case I. Let $Re[s] > 1$, applying the limit we obtain:

$$R_q(s) = \left[\sum_{m=1}^{\infty} \frac{1}{m^s} - \left(\frac{1}{1-s} 0 + \frac{(-1)^1 B_1}{1!} 0 + \frac{(-1)^2 B_2}{2!} (-s) 0 + \frac{(-1)^3 B_3}{3!} (-s)(-1-s) 0 + \cdots \right) \right]$$

$$R_q(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \iff s > 1 \tag{19}$$

Therefore, it can be concluded that: when $Re[s] > 1$ the value of $R_q(s)$ is equal to $\zeta(s)$

$$R_q(s) = \zeta(s) \iff Re[s] > 1,\ s \in \mathbb{C} \tag{20}$$

Case II. Let $s = 1$ applying the limit we obtain:

$$R_q(1) = \left[\sum_{m=1}^{\infty} \frac{1}{m^1} - \left(\frac{1}{0} + \frac{(-1)^1 B_1}{1!}0 + \frac{(-1)^2 B_2}{2!}(-1)0 + \frac{(-1)^3 B_3}{3!}(-1)(-1-1)0 + \cdots\right)\right]$$

$$R_q(1) = \infty - (\infty + 0 + 0 + 0 + \cdots)$$
$$R_q(1) = indeterminate$$

Therefore it can be concluded that: when $s = 1$ the value of $R_q(1)$ is indeterminate, and it is equal to $\zeta(1)$:

$$R_q(1) = \zeta(1) \tag{21}$$

Case III. Consider $s = -k$ where $k \in \mathbb{N}$, equation (17) can be written separating the summation into two parts, the first with limits for $p = 1$ up to $p = k$, and the second summation from $p = k + 1$ up to $p = q$ leaving the expression as follows:

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \int n^k dn - \sum_{p=1}^{q=k} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right) - \sum_{p=k+1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right] \tag{22}$$

It can be seen that the integral with the first summation are equivalent to the equation (9), but with negative sign, therefore it can be written:

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \sum_{m=1}^{n} m^k - \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right]$$

The first two sums cancel out, leaving the expression as follows:

$$R_q(-k) = \lim_{n\to\infty} \left[-\sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right] \tag{23}$$

Then, extending the series:

$$R_q(-k) = \lim_{n\to\infty} \left[-\left(\frac{(-1)^{k+1} B_{k+1}}{(k+1)!}(k!)n^0 + \frac{(-1)^{k+2} B_{k+2}}{(k+2)!}(k)(k-1)\cdots n^{-1} + \frac{(-1)^{k+3} B_{k+3}}{(k+3)!}(k)(k-1)\cdots n^{-2} + \cdots\right)\right]$$

Applying the limit, then simplifying factorials:

$$R_q(-k) = -\frac{(-1)^{k+1} B_{k+1}}{(k+1)!}(k!)$$

$$R_q(-k) = \frac{(-1)^k B_{k+1}}{k+1}$$

Since $B_{2j+1} = 0$, $j \in \mathbb{N}$ we can write:

$$R_q(-k) = -\frac{B_{k+1}}{k+1} \iff k \in \mathbb{N} \tag{24}$$

If we compare equation (24) with equation (4), we can say that they are equivalent, therefore:

$$R_q(-k) = \zeta(-k) \iff k \in \mathbb{N} \tag{25}$$

We are ready to state the following theorem:

Theorem 1 *From equations (20), (21) and (25) by the principle of analytical continuity can be written:*

$$\zeta(s) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^s} - \int\frac{1}{n^s}dn - \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^s}\right)\right] \tag{26}$$

Where: $s \in \mathbb{C}$, and q is the number of times that the function can be derived $\frac{1}{n^s}$

5 Graphical interpretation of the Riemann Zeta Function.

Let us define the function $S^*_{-s}(n)$ as:

$$S^*_{-s}(n) := \int\frac{1}{n^s}dn + \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^s}\right) \tag{27}$$

When the points of the series of points $\sum_{m=1}^{n}\frac{1}{m^s}$ are plotted on the complex plane, we obtain a series of points arranged in a logarithmic spiral, the center of which is the value of $\zeta(s)$.

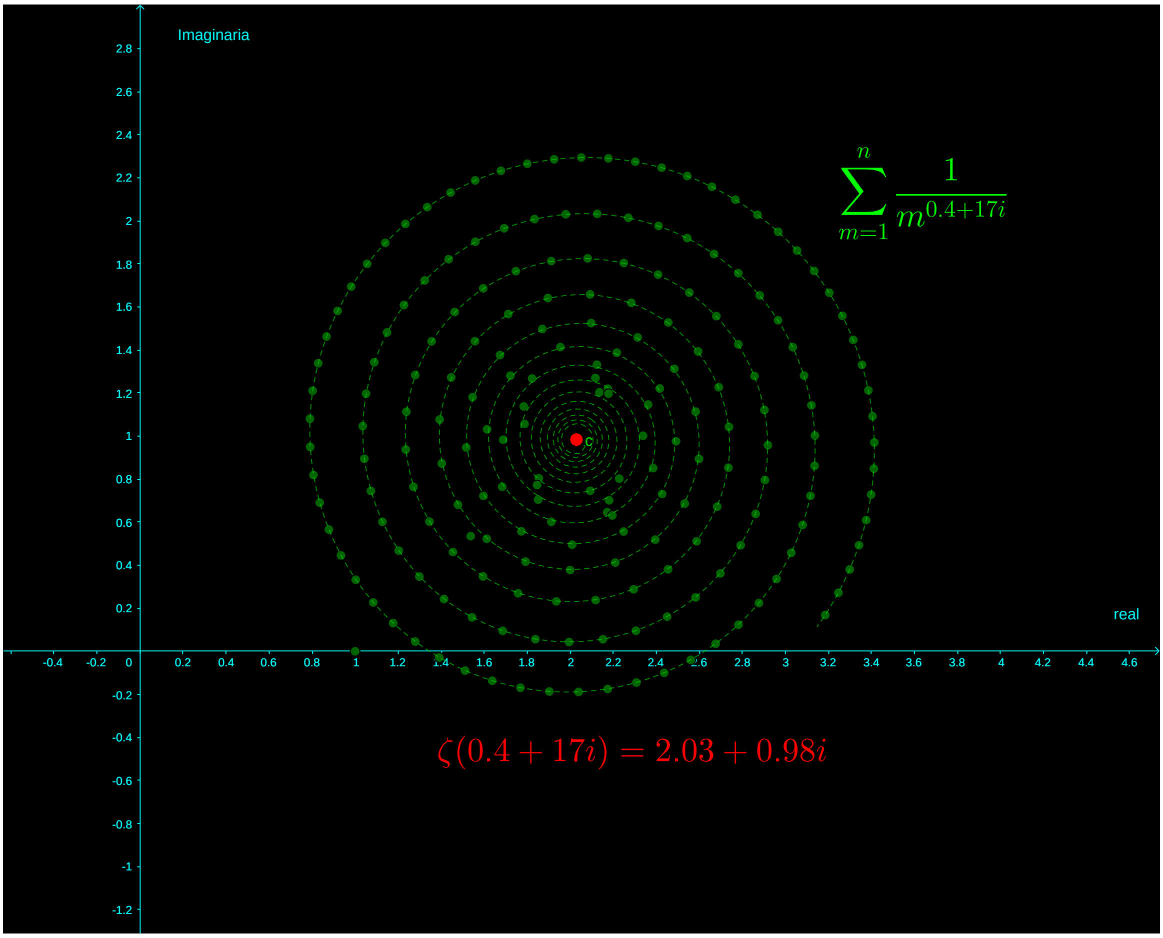

Figure 1: *Example of logarithmic spiral of the series* $\sum_{m=1}^{n}\frac{1}{m^s}$ for $s = 2 + 27i$ where $\zeta(s)$ is the center of the spiral.

When $Re[s] > 1$, the points of the spiral converge to its center, but if $Re[s] \leq 1$, the series of points diverges from its center. The function $S^*_{-s}(n)$, is a spiral with center at the origin of the complex plane, but by adding $\zeta(s)$, the spiral approaches the series of points, and, both spirals join when $n \to \infty$.

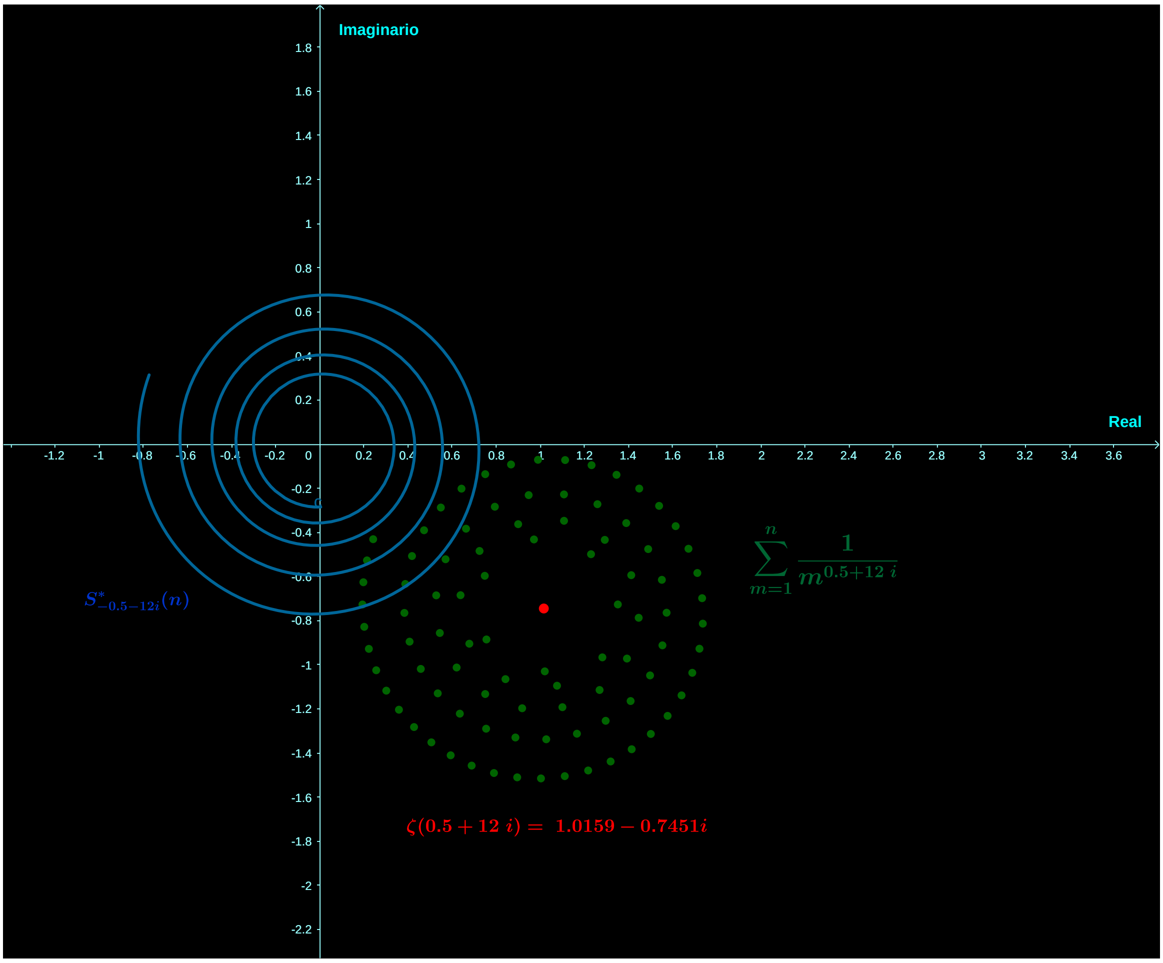

Figure 2: *Graph in the complex plane of the spirals given by the function $S^*_{-s}(n)$ (in blue color), and the series $\sum_{m=1}^{n} \frac{1}{m^s}$ (in green color) and the point $\zeta(s)$ (in red color).*

If we plot separately the real part and the imaginary part of the series of points and the function $S^*_{-s}(n)$, we plot periodic functions, of variable amplitude and period. The amplitude will be increasing when $Re[s] < 1$, and decreasing when $Re[s] > 1$, and the frequency will depend on the term $Im[s] * \ln(n)$. The axis or mean value of the periodic functions will be:

$$y_{Re} = Re[\zeta(s)]$$
$$y_{Im} = Im[\zeta(s)]$$

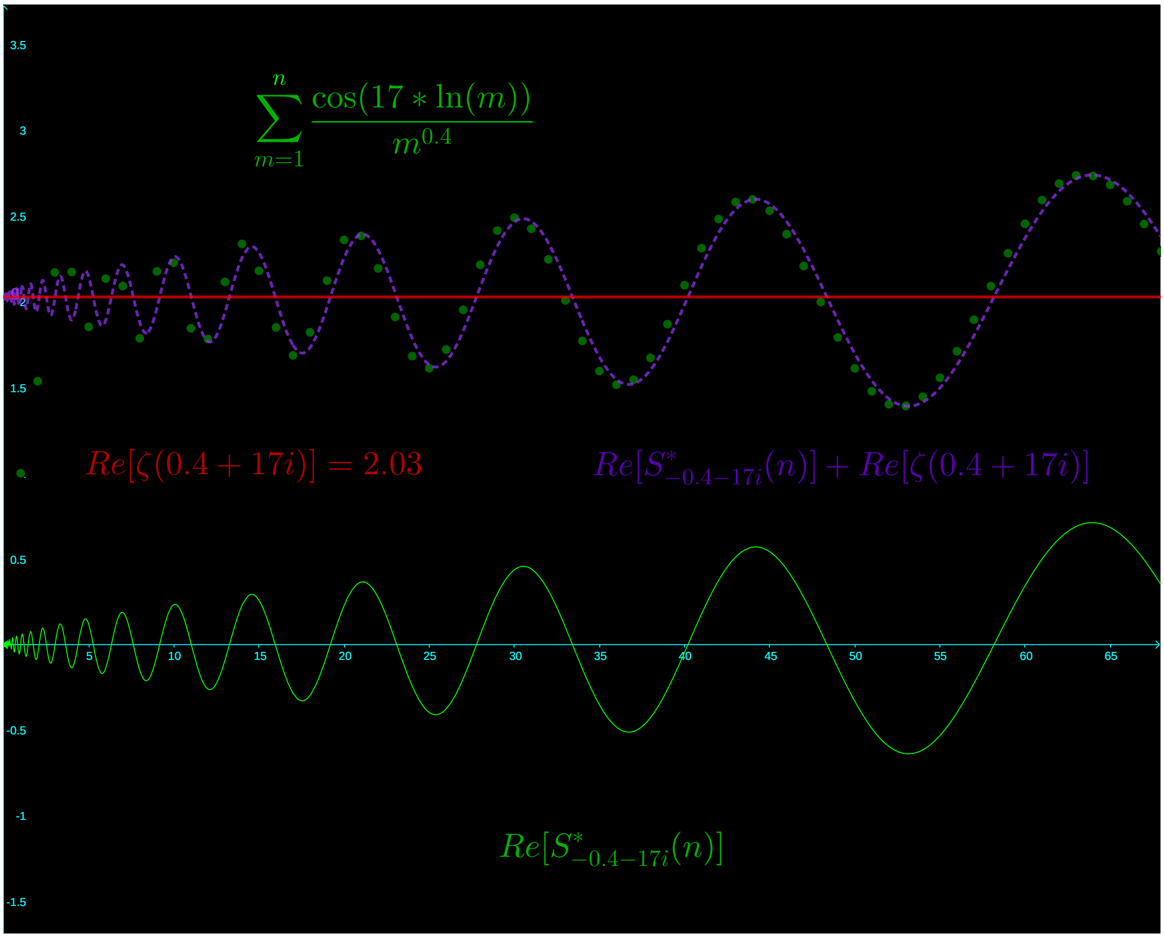

Figure 3: *Graphical representation of the functions* $Re\left[S^*_{-s}(n)\right]$ *and the serie* $\sum_{m=1}^{n} \frac{1}{m^{Re[s]}} \cos(Im[s] * \ln m)$ *that oscillates with respect to its mean value* $Re\left[\zeta(s)\right]$.

By adding $Re[\zeta(s)]$ to the function $Re[S^*_{-s}(n)]$, we manage to superimpose the $Re[S^*_{-s}(n)]$ with the series $\sum_{m=1}^{n} \frac{1}{m^{Re[s]}} cos(Im[s] * \ln m)$, when $s \to \infty$.

In the same way by adding $Im[\zeta(s)]$ to the function $Im[S^*_{-s}(n)]$, we succeed in superimposing $Im[S^*_{-s}(n)]$ with the series $-\sum_{m=1}^{n} \frac{1}{m^{Re[s]}} \sin(Im[s] * \ln m)$, when $s \to \infty$.

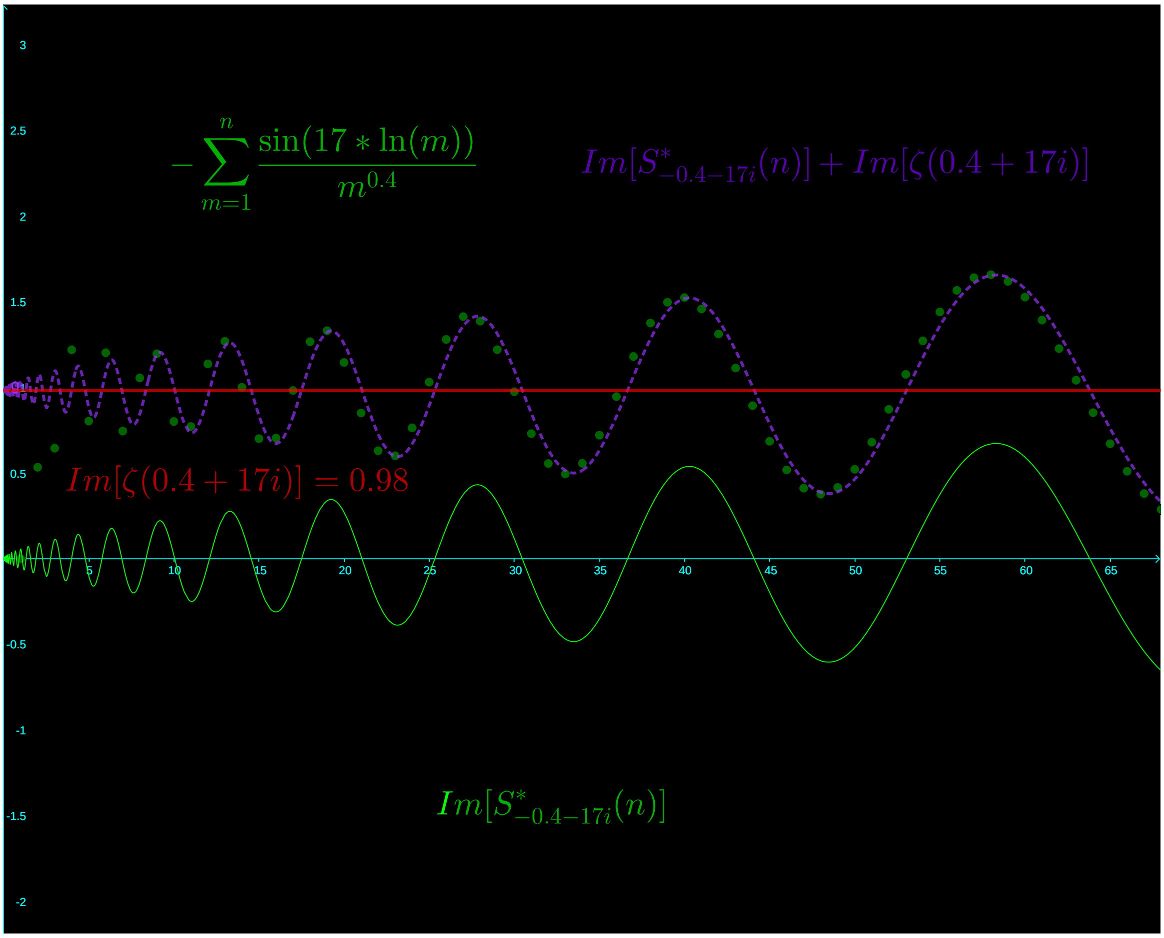

Figure 4: *Graphical representation of the functions* $Im\left[S^*_{-s}(n)\right]$ *and the serie* $\sum_{m=1}^{n}\frac{1}{m^a}\sin(-Im[s]*\ln n)$ *that oscillates with respect to its average value* $Im\left[\zeta(s)\right]$.

6 The Riemann Hypothesis.

6.1 Critical band of the Zeta Function.

With the Riemann functional equation (2), it is possible to obtain $\zeta(s)$ for values of $Re[s]<0$, since it depend on $\zeta(1-s)$, which is a convergent value. However, it is not possible to obtain $\zeta(s)$ when $0\le Re[s]\le 1$, since we would need to know $\zeta(1-s)$, it is also not convergent.

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{s\pi}{2}\right)\Gamma(1-s)\zeta(1-s)$$

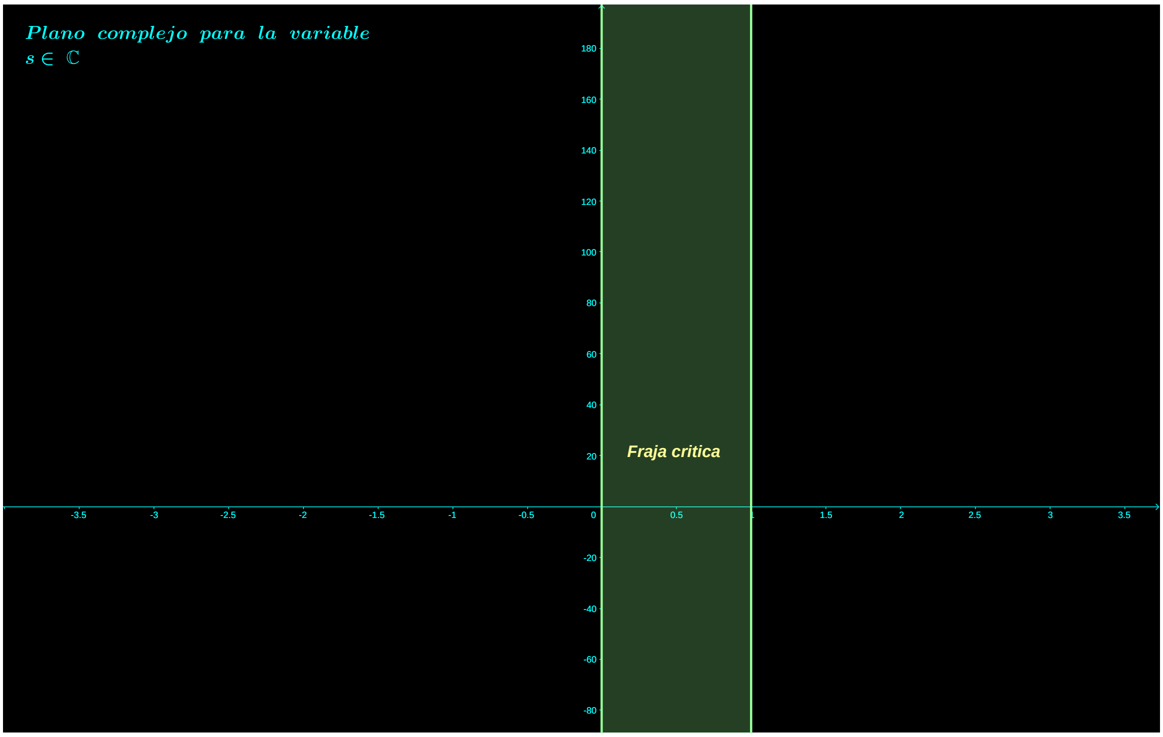

Figure 5: *Critical band in the complex plane, for the variable $s \in \mathbb{C}$, where it is not possible to determine $\zeta(s)$, with the Riemann Functional Equation.*

6.1.1 Trivial zeros

From the Riemann functional equation, it is easy to realize that for even negative values of s, $\zeta(s) = 0$, this is because the term $\sin(\frac{s\pi}{2})$ becomes zero, these zeros or roots of the function are called *trivial zeros*.

$$\zeta(-2k) = 2^{-2k}\pi^{-2k-1}\sin\left(\frac{-2k\pi}{2}\right)\Gamma(1+2k)\zeta(1+2k) = 0$$

Where $k \in \mathbb{N}$

When s is a positive even number, the function $\zeta(s)$ is nonzero, since the term $\sin(\frac{s\pi}{2})$ cancels with the poles of the gamma function $\Gamma(1-s)$.

6.1.2 Nontrivial zeros

There are other zeros of $\zeta(s)$, but it is shown that these zeros are inside the *critical band*. These zeros are called nontrivial zeros, which, being inside the critical band, can only be found by numerical methods.

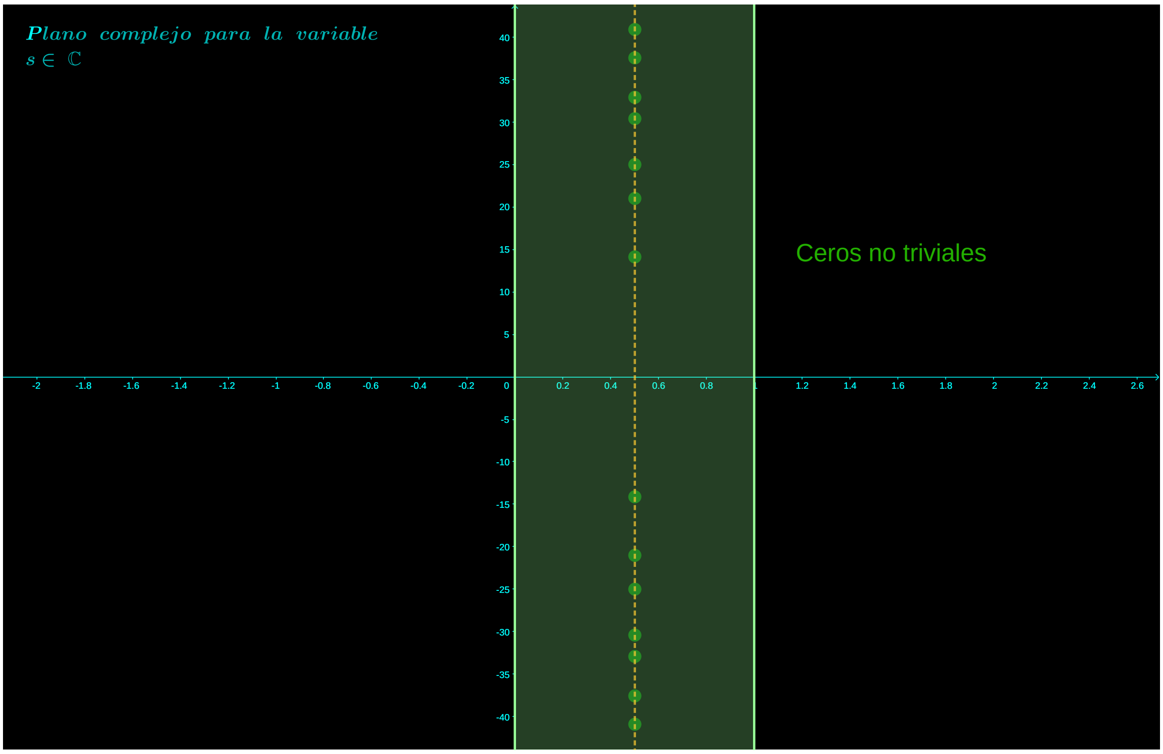

Figure 6: *Critical band in the complex plane, for the variable $s \in \mathbb{C}$, where it is not possible to determine $\zeta(s)$, with the Riemann Functional Equation.*

In 1859 the German mathematician Georg Friedrich Bernhard Riemann, in his doctoral thesis "On prime numbers less than a given magnitude" [2], in developing an explicit formula for calculating the number of prime numbers less than x, conjectured that: ***"The real part of every nontrivial zero of the Zeta Function $\zeta(s)$ is $\frac{1}{2}$"***. Riemann was sure of this statement, but could not prove it, leaving it as one of the most important hypotheses unproven for 163 years.

7 Demonstration of the hypothesis

Let $s \in \mathbb{C} : s = a + bi$, where $a, b \in \mathbb{R}$ and is a nontrivial zero so that $\zeta(s) = 0$.

By the Riemann Functional equation, if s is a nontrivial zero, the following must be true:

$$\zeta(a+bi) = \zeta(1-a-bi) = 0 \tag{28}$$

If we replace $a + bi$ in equation (26) and develop the expression:

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{a+bi}} - \int\frac{1}{n^{a+bi}}dn - \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^{a+bi}}\right)\right] \tag{29}$$

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{a+bi}} - \frac{1}{1-a-bi}n^{1-a-bi} - \frac{(-1)^1 B_1}{1!}n^{-a-bi} - \frac{(-1)^2 B_2}{2!}(-a-bi)n^{-1-a-bi}\cdots\right] \tag{30}$$

As $0 < a < 1$, applying the limit, the equation reduces to:

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{a+bi}} - \frac{1}{1-a-bi}n^{1-a-bi}\right] \tag{31}$$

Knowing that $\zeta(a+bi) = 0$, and taking it to its polar form we obtain:

$$0 = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{e^{-bi\ln(m)}}{m^a} - \frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}e^{i\left[\arctan\left(\frac{b}{1-a}\right)-b\ln(n)\right]}\right] \tag{32}$$

Then:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{e^{-bi\ln(m)}}{m^a}\right]=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}e^{i\left[\arctan\left(\frac{b}{1-a}\right)-b\ln(n)\right]}\right] \tag{33}$$

Applying Eulerian properties:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}-i\sum_{m=1}^{n}\frac{\sin(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\left\{\cos\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right]\right.\right.$$
$$\left.\left.+i\sin\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right]\right\}\right] \tag{34}$$

Similarly, for the nontrivial zero $s=1-a-bi$ we obtain:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}+i\sum_{m=1}^{n}\frac{\sin(b\ln m)}{m^{1-a}}\right]=\lim_{n\to\infty}\left[\frac{n^{a}}{\sqrt{a^2+b^2}}\left\{\cos\left[arctan\left(\frac{b}{a}\right)-b\ln n\right]\right.\right.$$
$$\left.\left.-i\sin\left[arctan\left(\frac{b}{a}\right)-b\ln n\right]\right\}\right] \tag{35}$$

Taking the real part of equations (34) and (35) we have:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\cos\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right]\right] \tag{36}$$

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}\right]=\lim_{n\to\infty}\left[\frac{n^{a}}{\sqrt{a^2+b^2}}\cos\left[arctan\left(\frac{b}{a}\right)-b\ln n\right]\right] \tag{37}$$

Equating the amplitudes to be able to compare equations (36) and (37), knowing that the axis of both functions is $Re[\zeta(a+bi)]=Re[\zeta(1-a-bi)]=0$:

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\cos\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right] \tag{38}$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{a^2+b^2}}{n^{a}}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}\right]=\lim_{n\to\infty}\cos\left[arctan\left(\frac{b}{a}\right)-b\ln n\right] \tag{39}$$

From equation (38) performing operations to reduce the equality to a simple periodic function:

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\cos\left[arctan\left(\frac{b}{1-a}\right)\right]\cos(b\ln n)+\sin\left[arctan\left(\frac{b}{1-a}\right)\right]\sin(b\ln n)\right]$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\cos(b\ln n)+\tan\left[arctan\left(\frac{b}{1-a}\right)\right]\sin(b\ln n)\right]$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{(1-a)}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\cos(b\ln n)+\frac{b}{1-a}\sin(b\ln n)\right]$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}-\cos(b\ln n)\right]=\lim_{n\to\infty}\frac{b}{1-a}\sin(b\ln n)$$

$$\lim_{n\to\infty}\left[\frac{(1-a)\sqrt{(1-a)^2+b^2}}{bn^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}-\frac{1-a}{b}\cos(b\ln n)\right]=\lim_{n\to\infty}\sin(b\ln n) \tag{40}$$

In the same way, we work with equation (39) and we obtain:

$$\lim_{n\to\infty}\left[\frac{a\sqrt{a^2+b^2}}{bn^a\cos\left[arctan\left(\frac{b}{a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}-\frac{a}{b}\cos(b\ln n)\right]=\lim_{n\to\infty}\sin(b\ln n) \tag{41}$$

Now we can equate (40) and (41):

$$\lim_{n\to\infty}\left[\frac{(1-a)\sqrt{(1-a)^2+b^2}}{bn^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}-\frac{1-a}{b}\cos(b\ln n)\right]=$$
$$\lim_{n\to\infty}\left[\frac{a\sqrt{a^2+b^2}}{bn^a\cos\left[arctan\left(\frac{b}{a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}-\frac{a}{b}\cos(b\ln n)\right] \tag{42}$$

Since $n\to\infty$, the equality must be satisfied for the last term of the sum:

$$\lim_{n\to\infty}\left[\frac{(1-a)\sqrt{(1-a)^2+b^2}}{bn^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\frac{\cos(b\ln n)}{n^a}\right]=\lim_{n\to\infty}\left[\frac{a\sqrt{a^2+b^2}}{bn^a\cos\left[arctan\left(\frac{b}{a}\right)\right]}\frac{\cos(b\ln n)}{n^{1-a}}\right] \tag{43}$$

Reducing terms:

$$\lim_{n\to\infty}\frac{(1-a)\sqrt{(1-a)^2+b^2}}{\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}=\lim_{n\to\infty}\frac{a\sqrt{a^2+b^2}}{\cos\left[arctan\left(\frac{b}{a}\right)\right]} \tag{44}$$

Removing the limit:

$$\frac{(1-a)\sqrt{(1-a)^2+b^2}}{\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}=\frac{a\sqrt{a^2+b^2}}{\cos\left[arctan\left(\frac{b}{a}\right)\right]} \tag{45}$$

Reducing terms:

$$\frac{(1-a)\sqrt{(1-a)^2+b^2}}{\frac{1}{\sqrt{\left(\frac{b}{1-a}\right)^2+1}}}=\frac{a\sqrt{a^2+b^2}}{\frac{1}{\sqrt{\left(\frac{b}{a}\right)^2+1}}} \tag{46}$$

$$(1-a)\sqrt{(1-a)^2+b^2}\sqrt{\left(\frac{b}{1-a}\right)^2+1}=a\sqrt{a^2+b^2}\sqrt{\left(\frac{b}{a}\right)^2+1} \tag{47}$$

$$\sqrt{[(1-a)^2+b^2]^2}=\sqrt{[a^2+b^2]^2} \tag{48}$$

$$(1-a)^2+b^2=a^2+b^2 \tag{49}$$

$$1-2a=0 \tag{50}$$

Therefore, we can say:

$$a=\frac{1}{2} \tag{51}$$

Finally we can conclude by indicating that $\exists\, b\in\mathbb{R}$ so that:

$$\zeta(a+bi)=\zeta(1-a-bi)=0\Rightarrow a=\frac{1}{2} \tag{52}$$

With which it is demonstrated that ***The Riemann Hypothesis is true***.

References

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Dedication.

"Call to me and I will answer you and tell you great and unsearchable things you do not know"
(Jeremiah 33:3)

I thank God for hearing my prayers, and showing me the way to the resolution of this problem. to him be the glory.

This work is dedicated to all my family who supported me at all times and in the most difficult moments of my life, especially my beloved wife Araceli, my two beautiful children Ocram and Arelys, and my parents who never lost faith in me.