Tiling the Plane with k-Gons

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Abstract

We present a way to tile the plane by k-gons for a fixed k. We use usual regular 6-gons by putting some in a row and fill them with k-gons. We use only one or two or four different k-gons.

1 Introduction

It is a widespread opinion that one can tile the plane $\mathbb{R}^2$ only with triangles, squares and regular 6-gons. This is wrong. A further possibility is to put regular 6-gons in a row.
We think that it is useful to repeat the definition of a *simple polygon*.
A simple polygon with k vertices consists of k points $(x_1,y_1),(x_2,y_2),\dots(x_{k-1},y_{k-1}),(x_k,y_k)$ called *vertices*, and the straight lines between the vertices, where $k>2$. It is homeomorphic to a circle. We demand that there are no three consecutive collinear points $(x_i,y_i),(x_{i+1},y_{i+1}),(x_{i+2},y_{i+2})$ for $1\le i\le k-2$. Also we demand that the three points $(x_k,y_k),(x_1,y_1),(x_2,y_2)$ and $(x_{k-1},y_{k-1}),(x_k,y_k),(x_1,y_1)$ are not collinear.
We call this just described simple polygon a *k-gon*.

Definition 1. Let t be any natural number. We call a simple polygon a $\mathsf{t\ row\ 6-gon}$, if t regular 6-gons are put in a row.

See the example in Figure 2. There we show a $\mathsf{5\ row\ 6-gon}$.
Note that a $1\ \mathsf{row\ 6-gon}$ is just a regular 6-gon.

Proposition 1. One can tile the plane with $\mathsf{t\ row\ 6-gon}s$ for all fixed t.

Proof. Trivial. □

Proposition 2. A $\mathsf{t\ row\ 6-gon}$ has $2+4\cdot\mathsf{t}$ vertices.

Proof. Easy. □

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2 Tiling

Theorem 1. Let k be a natural number larger than 2. There exists for all k a tiling of $\mathbb{R}^2$ with k-gons.

Proof. For $k = 3$ and $k = 4$ and $k = 6$ the theorem is well-known. For $k = 5$ please see Figure 1. We also can take a regular 6-gon instead of a rectangle. We cut it into congruent halves.
Now let k be a natural number larger than 6.

Lemma 1. It holds $k - 2 \equiv p \bmod 4$, where $p \in \{0, 1, 2, 3\}$.

Proof. Well-known. □

We discuss the four possibilities.

• Possibility 1: $p = 0$. In this easy case we take the polygon t row $6-$gon as a k-gon. We get t from the equation $k - 2 = 4 \cdot \mathsf{t}$.
The sequence of the numbers of k is $10, 14, 18, \ldots$. By Proposition 2 the number of vertices of a t row $6-$gon is $2 + 4 \cdot \mathsf{t}$. This is k.

• Possibility 2: $p = 1$. In this case we had to calculate. We take a $(4 \cdot \mathsf{t} + 1)$ row $6-$gon. It is filled with four k-gons. We use the vertices of the $(4 \cdot \mathsf{t} + 1)$ row $6-$gon as vertices of the four k-gons. See Figure 2. Note that the four k-gons have three edges in common. Therefore we have to subtract 6 from the number of the vertices.
We get t from the equation $k - 2 = 4 \cdot \mathsf{t} + 1$.
The number of vertices both for a $(4 \cdot \mathsf{t} + 1)$ row $6-$gon and 4 k-gons $- 6$ is $16 \cdot \mathsf{t} + 6$.
The sequence of the numbers of k is $7, 11, 15, 19, \ldots$.

• Possibility 3: $p = 2$. We take a $(2 \cdot \mathsf{t} + 1)$ row $6-$gon. It is filled with two k-gons. We get t from $k - 2 = 4 \cdot \mathsf{t} + 2$.
The sequence of the numbers of k is $8, 12, 16, 20, \ldots$. Two k-gons alltogether have $8 + 8 \cdot \mathsf{t}$ vertices. See Figure 3. Note that if two k-gons tile a polygon a pair of vertices is canceled, since the k-gons have a common edge.Therefore they have $6 + 8 \cdot \mathsf{t}$ vertices. This is also the number of vertices of a $(2 \cdot \mathsf{t} + 1)$ row $6-$gon.

• Possibility 4: $p = 3$. We take a $(4 \cdot \mathsf{t} + 3)$ row $6-$gon. It is filled with four k-gons.
We get t from $k - 2 = 4 \cdot \mathsf{t} + 3$.
The common number of vertices is $16 \cdot \mathsf{t} + 14$.
The sequence of the numbers of k is $9, 13, 17, 21, \ldots$.

The theorem is proved. □

It follows three figures.

Figure 1:

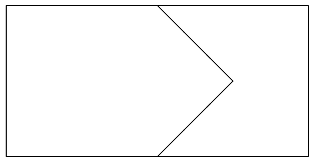


Figure 2:

See below a 5 row 6 − gon, which is subdivided in four 7-gons.

We see also three edges. Each is a common edge of two 7-gons.

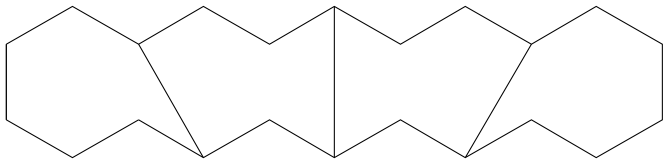
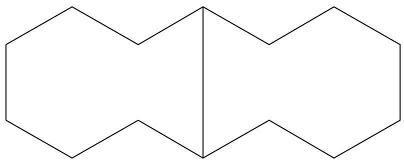


Figure 3:

On the right hand we see

a 3 row 6 − gon.

It consists of two 8-gons.



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