Hypergeometric relations , Catalan constant

Edgar Valdebenito

September 3, 2022

Abstract

In this note we give some formulas related to Catalan constant

Introduction

The number Pi (Pi $\equiv \pi$) is defined by

$$\pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right) = 3.141592\ldots$$

The Catalan constant is defined by

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \ldots = 0.915965\ldots$$

The Gauss hypergeometric function is defined by

$${}_2F_1(a, b, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n \cdot (b)_n}{(c)_n \; n!} x^n \qquad , |x| < 1$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, $(a)_0 = 1$.

In this note we give some series related to G .

Formulas

Entry 1. for $0 < p \leq 1$ we have

$$\frac{\pi}{2}\ln p + G = \sum_{n=0}^{\infty} \frac{(-1)^n p^{2n+1}}{(2n+1)^2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(\frac{1-p}{1+p}\right)^{2n+1} {}_2F_1\left(2n+1, 1, n+\frac{3}{2}, \frac{1}{1+p}\right)$$

Entry 2. for $p > 1$ we have

$$\frac{\pi}{2}\ln p + G = \sum_{n=0}^{\infty} \frac{2^{-2n}}{(2n+1)^2} \binom{2n}{n} \left(\frac{p^2}{1+p^2}\right)^{n+\frac{1}{2}} {}_2F_1\left(n+\frac{1}{2}, 1, n+\frac{3}{2}, \frac{p^2}{1+p^2}\right)$$
$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(\frac{p-1}{p+1}\right)^{2n+1} {}_2F_1\left(2n+1, 1, n+\frac{3}{2}, \frac{1}{1+p}\right)$$

Entry 3. for $0 < p \le 1$ we have

$$\frac{\pi}{2}\ln p + G = \sum_{n=0}^{\infty} \frac{(-1)^n p^{2n+1}}{(2n+1)^2}$$
$$- \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(\frac{1-p}{1+p}\right)^{2n+1} {}_2F_1\left(2n+1, 1, n+\frac{3}{2}, \frac{1}{1+p}\right)$$

Entry 4. for $p > 1$ we have

$$\frac{\pi}{2}\ln p + G = \sum_{n=0}^{\infty} \frac{2^{-2n}}{(2n+1)^2} \binom{2n}{n} \left(\frac{p^2}{1+p^2}\right)^{n+\frac{1}{2}} {}_2F_1\left(n+\frac{1}{2}, 1, n+\frac{3}{2}, \frac{p^2}{1+p^2}\right)$$
$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(\frac{p-1}{p}\right)^{2n+1} {}_2F_1\left(2n+1, n+\frac{1}{2}, n+\frac{3}{2}, -\frac{1}{p}\right)$$

Entry 5. for $p > 1$ we have

$$\frac{\pi}{2}\ln p + G = \sum_{n=0}^{\infty} \frac{2^{-2n}}{(2n+1)^2} \binom{2n}{n} \left(\frac{p^2}{1+p^2}\right)^{n+\frac{1}{2}} {}_2F_1\left(n+\frac{1}{2}, 1, n+\frac{3}{2}, \frac{p^2}{1+p^2}\right)$$
$$+ \left(\frac{1+p}{p}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(\frac{p-1}{p+1}\right)^{2n+1} {}_2F_1\left(1, -n+\frac{1}{2}, n+\frac{3}{2}, -\frac{1}{p}\right)$$

Entry 6. for $p > 1$ we have

$$\frac{\pi}{2}\ln p + G = \sum_{n=0}^{\infty} \frac{2^{-2n}}{(2n+1)^2} \binom{2n}{n} \left(\frac{p^2}{1+p^2}\right)^{n+\frac{1}{2}} {}_2F_1\left(n+\frac{1}{2}, 1, n+\frac{3}{2}, \frac{p^2}{1+p^2}\right)$$
$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(\frac{p-1}{\sqrt{p(p+1)}}\right)^{2n+1} {}_2F_1\left(-n+\frac{1}{2}, n+\frac{1}{2}, n+\frac{3}{2}, \frac{1}{1+p}\right)$$

Entry 7.

$$\frac{\pi}{2}\ln\left(\sqrt{2}-1\right) + G = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{2}-1\right)^{2n+1}}{(2n+1)^2} \left(1 - {}_2F_1\left(2n+1, 1, n+\frac{3}{2}, \frac{1}{\sqrt{2}}\right)\right)$$
$$\frac{\pi}{2}\ln\left(\frac{1}{2}\right) + G = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n-1}}{(2n+1)^2} \left(1 - \left(\frac{2}{3}\right)^{2n+1} {}_2F_1\left(2n+1, 1, n+\frac{3}{2}, \frac{2}{3}\right)\right)$$

Entry 8. for $p > 1$ we have

$$\frac{\pi}{2}\ln p + G = \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{p^2}{1+p^2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{n} \frac{2^{-2k}}{2k+1} \binom{2k}{k}$$
$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left(\frac{p-1}{p+1}\right)^{2n+1} {}_2F_1\left(2n+1, 1, n+\frac{3}{2}, \frac{1}{1+p}\right)$$

remark: $\sum_{k=0}^{n}\frac{2^{-2k}}{2k+1}\binom{2k}{k}=\frac{\pi}{2}-\frac{4^{-n-1}}{2n+3}\binom{2n+2}{n+1}\,{}_3F_2\left(1,n+\frac{3}{2},n+\frac{3}{2};n+2,n+\frac{5}{2};1\right)$

Entry 9.

$$\frac{\pi}{2}\ln\left(1+\sqrt{2}\right)+G=$$
$$\sqrt{\frac{2+\sqrt{2}}{4}}\sum_{n=0}^{\infty}\frac{2^{-2n}}{(2n+1)^2}\binom{2n}{n}\left(\frac{2+\sqrt{2}}{4}\right)^n {}_2F_1\left(n+\frac{1}{2},1,n+\frac{3}{2},\frac{2+\sqrt{2}}{4}\right)$$
$$+\sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\sqrt{2}-1\right)^{2n+1}{}_2F_1\left(1,-n+\frac{1}{2},n+\frac{3}{2},-\left(\sqrt{2}-1\right)\right)$$

Entry 10.

$$\frac{\pi}{2}\ln\left(1+\sqrt{2}\right)+G=\sqrt{\frac{2+\sqrt{2}}{4}}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{2+\sqrt{2}}{4}\right)^n\sum_{k=0}^{n}\frac{2^{-2k}}{2k+1}\binom{2k}{k}$$
$$+\sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\sqrt{2}-1\right)^{2n+1}{}_2F_1\left(1,-n+\frac{1}{2},n+\frac{3}{2},-\left(\sqrt{2}-1\right)\right)$$

Endnote

$$G=\int_0^1\left(\frac{1}{4+(1-x)^2}\ln\left(\frac{6-2x}{1-x^2}\right)+\frac{1}{4+(1+x)^2}\ln\left(\frac{6+2x}{1-x^2}\right)\right)\mathrm{d}x$$

$$G=\frac{\pi}{8}\ln(2)+\int_0^1\left(\frac{1}{4+x^2}\ln\left(\frac{2+x}{x(2-x)}\right)+\frac{1}{4+(2-x)^2}\ln\left(\frac{4-x}{x(2-x)}\right)\right)\mathrm{d}x$$

References

[1] D. H. Bailey and J. M. Borwein, Experimental Mathematics: examples, methods and implications. Notices Amer. Math. Soc., 52(5): 502-514.

[2] J. M. Borwein and D. H. Bailey, Mathematics by Experiment: Plausible Reasoning in the 21st century. AK Peters Ltd, Natick, MA, 2003.