Paradox in General Relativity for a Charged Sphere

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Abstract

For a static charged sphere with small mass and charge we make gauge and coordinate transformations so that the electromagnetic vector potential has a time component of one and zero space components at all points. We consider Einstein field equations with this potential. There is a solution that has no electromagnetic field. This is a contradiction since we began, before the transformations, with a nonzero electromagnetic field.

1 Gauge transformation and Einstein field equations

Let $A_\mu(x)$, $g_{\mu\nu}(x)$, and $A^\mu(x)$ be the electromagnetic potential, metric tensor, and electromagnetic vector potential, respectively where $x = (x^0, x^1, x^2, x^3)$ is a point of $\mathbb{R}^4$. The electromagnetic field is

$$F_{\mu\nu}(x) = A_{\nu,\mu}(x) - A_{\mu,\nu}(x) \tag{1}$$

A gauge transformation

$$A_\mu(x) \to A_\mu(x) + \phi_{,\mu}(x) \tag{2}$$

where $\phi(x)$ is a function on $\mathbb{R}^4$ leaves $F_{\mu\nu}(x)$ unchanged. Let $g_{\mu\nu}(x)$ satisfy the Einstein field equations with electromagnetic and matter energy-momentum tensor

$$G_{\mu\nu} = 8\pi\left(g^{\sigma\tau}F_{\mu\sigma}F_{\nu\tau} - \frac{1}{4}g_{\mu\nu}g^{\sigma\alpha}g^{\tau\beta}F_{\sigma\tau}F_{\alpha\beta}\right) + 8\pi T_{\mu\nu} \tag{3}$$

where $T^{\mu\nu}(x)$ is the energy-momentum tensor of matter. After making a gauge transformation (2) we have $g_{\mu\nu}(x)$ will still be a solution of (3). Define

$$\hat{A}^\mu(x) = A^\mu(x) + g^{\mu\alpha}(x)\phi_{,\alpha}(x) \tag{4}$$

By (1) and (4)

$$\begin{aligned} F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu,\mu} - A_{\mu,\nu} + \phi_{,\nu\mu} - \phi_{,\mu\nu} = (A_\nu + \phi_{,\nu})_{,\mu} - (A_\mu + \phi_{,\mu})_{,\nu} \\ &= (g_{\nu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\mu} - (g_{\mu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\nu} = (g_{\nu\alpha}\hat{A}^\alpha)_{,\mu} - (g_{\mu\alpha}\hat{A}^\alpha)_{,\nu} \end{aligned} \tag{5}$$

For the static sphere where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$

$$A_\mu(r) \qquad T_{\mu\nu}(r) \qquad g_{\mu\nu}(r) \qquad A_k(r) = g_{0k}(r) = g^{0k}(r) = 0 \qquad k = 1,2,3 \tag{6}$$

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2 Coordinate transformation

Let

$$\phi(x) = -t \tag{7}$$

hence by (4), (6), and (7)

$$\hat{A}^0(r) = A^0(r) - g^{00}(r) \qquad \hat{A}^k(r) = 0 \qquad k = 1,2,3 \tag{8}$$

Let the sphere have small mass and charge. Define $h_{\mu\nu}(r) = g_{\mu\nu}(r) - \eta_{\mu\nu}$. We then have $h_{\mu\nu}(r)$ and $A_\mu(r)$ are small hence $\hat{A}^0(r)$ is approximately one for all r. Consider the coordinate transformation

$$x'^0 = \frac{x^0}{\hat{A}^0(r)} \qquad x'^1 = x^1 \qquad x'^2 = x^2 \qquad x'^3 = x^3 \tag{9}$$

We have by (8) and (9) that

$$\hat{A}'^0(t', r') = 1 \qquad \hat{A}'^k(t', r') = 0 \qquad k = 1,2,3 \tag{10}$$

By (5) and (10) then

$$F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu} = (g'_{\nu\alpha}\hat{A}'^\alpha)_{,\mu} - (g'_{\mu\alpha}\hat{A}'^\alpha)_{,\nu} = g'_{\nu 0},_{\mu} - g'_{\mu 0},_{\nu} \tag{11}$$

Require $A_\mu(r) \to 0$ and $h_{\mu\nu}(r) \to 0$ as $r \to \infty$. We then have $h'_{\mu\nu}(t', r') \to 0$ as $r' \to \infty$.

3 Contradiction

We can use (11) and transform (3) to x' coordinates and write

$$G'_{\mu\nu} = 8\pi g'^{\sigma\tau}[h'_{\sigma 0,\mu} - h'_{\mu 0,\sigma}][h'_{\tau 0,\nu} - h'_{\nu 0,\tau}] - 2\pi g'_{\mu\nu} g'^{\alpha\sigma} g'^{\beta\tau}[h'_{\tau 0,\sigma} - h'_{\sigma 0,\tau}][h'_{\beta 0,\alpha} - h'_{\alpha 0,\beta}] + 8\pi T'_{\mu\nu} \tag{12}$$

Now instead begin with an equation of form (12) having $h'_{\mu\nu}(t', r') \to 0$ as $r' \to \infty$. Linear theory gives

$$G'^{(1)}_{\mu\nu}(t', r') = 8\pi T'_{\mu\nu}(t', r') \tag{13}$$

where $G'^{(1)}_{\mu\nu}(t', r')$ is the first order in $h'(t', r')$ Einstein tensor. From (13) we can conclude the Einstein tensor is zero outside the mass. Consequently the energy-momentum tensor and hence the electric field are zero outside the mass. However a charged sphere has nonzero electric field outside the sphere. We have a contradiction.

References

[1] K. De Paepe, *Physics Essays*, September 2007