Validity of the Riemann series theorem

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A classic example of the Riemann series theorem use the alternating harmonic series $\sum_{n=1}^{k}\frac{(-1)^{n+1}}{n}$ that converges to $\ln(2)$ when $k \to \infty$ and when rearranged to $\sum_{n=1}^{k}(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n})$ converges now to a different value $\frac{1}{2}\ln(2)$ when $k \to \infty$ But here I demonstrate that the rearranged series always exclude terms that constitute a third series which itself converges to exactly the difference between the original series and the rearranged one and thus explains why the rearranged series do not converge toward the original value. Eventually, it demonstrates also that the theorem is certainly false.

If we compare the terms of the tow series, the original and the rearranged, from n=1 to 4k so $\sum_{n=1}^{4k}\frac{(-1)^{n+1}}{n}$ we found for k=1 :

$\sum_{n=1}^{4}\frac{(-1)^{n+1}}{n}$	1	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{4}$
$\sum_{n=1}^{1}(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n})$	1	$-\frac{1}{2}$		$-\frac{1}{4}$
Term not taken in account in the rearranged series			$+\frac{1}{3}$	

for k=2 :

$\sum_{n=1}^{8}\frac{(-1)^{n+1}}{n}$	1	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{4}$	$+\frac{1}{5}$	$-\frac{1}{6}$	$+\frac{1}{7}$	$-\frac{1}{8}$
$\sum_{n=1}^{2}(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n})$	1	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{4}$		$-\frac{1}{6}$		$-\frac{1}{8}$
Terms not taken in account in the rearranged series					$+\frac{1}{5}$		$+\frac{1}{7}$	

for k=3 :

$\sum_{n=1}^{12}\frac{(-1)^{n+1}}{n}$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}$
$\sum_1^3(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n})$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\quad-\frac{1}{8}\quad-\frac{1}{10}\quad-\frac{1}{12}$
Terms not taken in account in the rearranged series	$+\frac{1}{7}\quad+\frac{1}{9}\quad+\frac{1}{11}$

And so on, with **<u>always</u>** a number of terms not taken in account in the rearranged series increasing with k, whose sum is exactly equal to the sum of inverses of odds numbers going from $\frac{1}{2k+1}$ to $\frac{1}{4k-1}$ so (thanks to Carlo [1]) to the series:

$$\sum_{p=k}^{2k-1}\frac{1}{2p+1}$$

Which can be approximated by an integral over p equals to $\frac{1}{2}\ln\left(\frac{4k-1}{2k+1}\right)$ that converges to $\frac{1}{2}\ln(2)$ when $k\ \rightarrow\ \infty$

And finally:

$$\begin{array}{lcccc}
\sum_{n=1}^{k}(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}) & = & \sum_{n=1}^{4k}\frac{(-1)^{n+1}}{n} & - & \sum_{p=k}^{2k-1}\frac{1}{2p+1} \\
\text{Rearranged series} & = & \text{original series} & - & \text{Terms not taken in account} \\
\frac{1}{2}\ln(2) & = & \ln(2) & - & \frac{1}{2}\ln(2)
\end{array}$$

So we see here that the rearranged series exclude terms that constitute a third series which itself converge to exactly the difference between the original series and the rearranged one and thus explain why the rearranged series do not converge toward the original value. Moreover, it demonstrate that the theorem is false which is finally a good news because else the commutativity of the addition should have been taken in question which is very unlikely.

[1] : https://mathoverflow.net/questions/412658/sum-of-changing-termes-serie