# Norm Inequalities for One Dimensional Sobolev Hilbert Spaces (An Extension)

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**Abstract:** In this paper, we shall consider norm inequalities for one dimensional Sobolev Hilbert spaces by using the theory of reproducing kernels as fundamental inequalities and as an extension of [15].

**Key Words:** Sobolev Hilbert spaces of one dimensional Hilbert spaces, reproducing kernel, norm inequality, product of reproducing kernels, Green function, isoperimetric inequality, division by zero calculus, multiplicative operator.

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## 1 Introduction

In this paper, we shall consider norm inequalities for one dimensional Sobolev Hilbert spaces by using the theory of reproducing kernels as fundamental inequalities and as an extension of [15].

For the Bergman kernel and the Szegö kernel on a regular domain $D$ on the complex $z = x + iy$ plane, we have the basic and deep relation

$$K(z, \overline{u}) >> 4\pi \hat{K}(z, \overline{u})^2$$

– the left minus the right is a positive definite quadratic form function – which was given by D. A. Hejhal [22]. This profound result using the Riemann theta function was given on the long historical results as in

G.F.B. Riemann (1826-1866); F. Klein (1849-1925); S. Bergman; G. Szegö; Z. Nehari; M.M. Schiffer; P.R. Garabedian (1949 published); D.A. Hejhal (1972 published).

It seems that any elementary proof is impossible, however, the result will, in particular, mean a fairly simple and fundamental inequality:

For two functions $\varphi$ and $\psi$ of $H_2(D)$, analytic Hardy space, we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\int\int_D |\varphi(z)\psi(z)|^2 dxdy \leq \frac{1}{2\pi}\int_{\partial D}|\varphi(z)|^2|dz|\frac{1}{2\pi}\int_{\partial D}|\psi(z)|^2|dz|, \qquad (1.1)$$

and we can determine completely the case holding the equality here. In the thesis [42] of the author published in 1979 the result was given. The author realized the importance of the abstract and general theory of reproducing kernels by N. Aronszajn ([1]). In the paper, the core part was to determine the equality statement in the above inequality, surprisingly enough, some deep and general independing poof was appeared 26 years later in A. Yamada ([82]). A. Yamada was developed deeply equality problems for some general norm inequalities derived by the theory of reproducing kernels and it was published in the book appendix of [12]. Very recently his theory is developing much more in [17].

Of course, in the thesis we can find some fundamental idea for nonlinear transforms. In particular, for the special case $\varphi \equiv \psi \equiv 1$, for the plane measure $m(D)$ of $D$ and the length $\ell$ of the boundary we have the isoperimetric inequality

$$4\pi m(D) \leq \ell^2.$$

We have similar results and theory for the Sobolev spaces. For example, let $\rho$ be a positive continuous function on $(a, b)$ satisfying $\rho \in L_1(a, b)$. Let $f_j$ be complex valued- functions on $(a, b)$ satisfying $\lim_{x\to a-0} f_j(x) = 0$. Then, we have the inequality

$$\int_a^b |(f_1(x)f_2(x))'|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)\rho(x)}$$

$$\leq 2 \int_a^b |f_1'(x)|^2 \frac{dx}{\rho(x)} \int_a^b |f_2'(x)|^2 \frac{dx}{\rho(x)},$$

when the integrals in the last part are finite. Equality holds here if and only if each $f_j$ is expressible in the form $C_j K_\rho(x, x_2)$ for some constants $C_j$ and for some point $x_2 \in [a, b]$ which is independent of $j$. Here, $K_\rho(x, \cdot)$ is the reproducing kernel of the Sobolev space with the norm

$$\sqrt{\int_a^b |f_1'(x)|^2 \frac{dx}{\rho(x)}} < \infty$$

([7]).

# One basic meaning of the norm inequalities

Now, we note an important meaning or application of the inequality (1.1); that is, when we fix any member $\psi$ of $H_2(D)$, the multiplication operator

$$\varphi \longmapsto \varphi(z)\psi(z), \tag{1.2}$$

on $H_2(D)$ to the Bergman space is bounded. Therefore, by the general theory for general fractional functions, we can consider the generalized fractional functions: for any Bergman function $f(z)$ on the domain $D$

$$\frac{f(z)}{\psi(z)}, \tag{1.3}$$

at least in the sense of Tikhonov; that is, we can consider the best approximation problem for the functions $\psi(z)^{-1}f(z)$ by the functions $H_2(D)$. See [2, 3] for more detailed results. See also [3] for applications.

As a very special fraction, we can consider the division by zero and division by zero calculus. See [13, 14] for the details.

As an important contribution of the theory of reproducing kernel is on the following fact:

**For bounded linear operators on some reproducing kernel Hilbert spaces, we can give analytical and numerical solutions for the operator equations.** See [11, 12].

For the recent similar type norm inequalities on Hilbert Sobolev spaces of one dimensional by A. Yamada [18], we refer to the corresponding norm inequalities as basic results applying the theory of reproducing kernels, directly.

There exist some interesting differences in nature with his concrete and deep results.

## 2 The first order Sobolev spaces

We will consider the first order Sobolev Hilbert spaces $H(a, b; \mathbf{R}), \ (a, b > 0)$, as the basic reproducing kernel Hilbert space with finite norms

$$\sqrt{\int_{\mathbf{R}} (a^2|f'(x)|^2 + b^2|f(x)|^2)\, dx}$$

admitting the reproducing kernel $K_{H(a,b;\mathbf{R})}(x, x_1)$

$$K_{H(a,b;\mathbf{R})}(x, x_1) = \frac{1}{2ab} \exp\left(-\frac{b}{a}|x - x_1|\right).$$

See [12], pages10-18 for the related basic materials.

We will consider this space as in the Szegö space in (1.1). Note the identity

$$\begin{aligned} K_{H(a,b;\mathbf{R})}(x, x_1)^2 &= \frac{1}{ab}\frac{1}{2a(2b)} \exp\left(-\frac{(2b)}{a}|x - x_1|\right) \\ &= \frac{1}{ab} K_{H(a,2b;\mathbf{R})}(x, x_1). \end{aligned}$$

From the construction of the norms admitting the reproducing kernels corresponding to the product and multiplication of a positive number for reproducing kernels, we obtain the norm inequality as in (1.1).

$(A)$ For any $f, g \in H(a, b; \mathbf{R})$, we have the norm inequality

$$\begin{aligned} &\int_{\mathbf{R}} \left(a^2|(f(x)g(x))'|^2 + 4b^2|f(x)g(x)|^2\right) dx \\ \leq &\frac{1}{ab} \int_{\mathbf{R}} \left(a^2|f'(x)|^2 + b^2|f(x)|^2\right) dx \int_{\mathbf{R}} \left(a^2|g'(x)|^2 + b^2|g(x)|^2\right) dx. \end{aligned}$$

Of course, we have

($A'$) For any $f, g \in H(a, b/2; \mathbf{R})$, we have the norm inequality

$$\int_{\mathbf{R}} \left(a^2|(f(x)g(x))'|^2 + b^2|f(x)g(x)|^2\right) dx$$

$$\leq \frac{2}{ab} \int_{\mathbf{R}} \left(a^2|f'(x)|^2 + \frac{b^2}{4}|f(x)|^2\right) dx \int_{\mathbf{R}} \left(a^2|g'(x)|^2 + \frac{b^2}{4}|g(x)|^2\right) dx.$$

# 3 Finite interval cases

If we note that the kernel on an interval $[c, d], -\infty \leq c < d \leq +\infty$

$$K_{H(a,b;[c,d])}(x, x_1) = \frac{1}{2ab} \exp\left(-\frac{b}{a}|x - x_1|\right)$$

is the reproducing kernel on the Hilbert space $H(a, b; [c, d])$ with finite norms

$$\sqrt{\int_{[c,d]} (a^2|f'(x)|^2 + b^2|f(x)|^2)\, dx + ab(|f(c)|^2 + |f(d)|^2)} < \infty$$

as in the whole space case, the results in Section 2 are valid in the corresponding way. This fact may be confirmed directly by checking the reproducing property of the kernel as in [12], pages 11-12. Meanwhile, the kernel $K_{H(a,b;[c,d])}(x, x_1)$ is the restriction to the interval $[c, d]$ of the kernel $K_{H(a,b;\mathbf{R})}(x, x_1)$ and so by the general property of reproducing kernels, we see that any member $f(x)$ of $H(a, b; [c, d])$ is the restriction of a function $h(x)$ in $H(a, b; \mathbf{R})$ and its norm is given by

$$||f||_{H(a,b;[c,d])} = \min ||h||_{H(a,b;\mathbf{R})},$$

where the minimum is taken over all functions $h$ in $H(a, b; \mathbf{R})$ satisfying

$$f(x) = h(x) \quad on \quad [c, d].$$

See [12], pages 78-80. In particular, note that any member $f(x)$ of $H(a, b; [c, d])$ has a good property on the interval $[c, d]$.

## 4 Division by zero calculus

If $b = 0$, then, by the division by zero calculus

$$K_{H(a,0;\mathbf{R})}(x, x_1) = -\frac{1}{2a^2}|x - x_1|$$

and this is the reproducing kernel for the corresponding space $H(a, 0; \mathbf{R})$ equipped with the norm

$$\|f\|^2_{H(a,0;\mathbf{R})} = a^2 \int_0^a (f'(x)^2 dx.$$

See [13, 14] for the division by zero calculus. Note that it is the Green's function in one dimensional space on the whole space and the Green's function may be related to the reproducing kernel. See [12], pages 62-63.

Meanwhile, if $a = 0$, $K_{H(0,b;\mathbf{R})}(x, x_1) = 0$, then it is the trivial reproducing kernel for the zero function space.

However, from the representation

$$\frac{1}{2ab} \exp\left(-\frac{b}{a}|x - y|\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)} d\xi}{a^2\xi^2 + b^2},$$

for $a = 0$, we have the reasonable result

$$\frac{1}{b^2}\delta(x - y)$$

that may be considered as the reproducing kernel for the $L_2$ space. See Section 8.8 in [12].

## 5 Generalizations for the first order Sobolev Hilbert spaces

From the products of different type kernels, we shall consider the corresponding norm inequalities as generalizations.

First recall the result [12], page 16-17:

For the half-open interval $I = [a, b)$, we consider a positive continuous function $\rho : I \to (0, \infty)$, such that

$$\rho \in L^1[a, x] \quad (x \in I) \tag{5.1}$$

for all $x \in I$. Denote by $\mathrm{AC}(I)$ the set of all absolutely continuous functions on an interval $I$.

**Theorem**: *Let $r \geq 1$ be a real number and let a positive continuous function $\rho$ satisfy* (5.1). *Let us set*

$$W(t) \equiv \int_a^t \rho(\xi)\, d\xi \text{ and } K_\rho(s,t) \equiv \int_a^{s\wedge t} \rho(v)\, dv = W(\, s\wedge t\,) \quad (s,t \in I), \tag{5.2}$$

*where $s \wedge t \equiv \min(s,t)$. The reproducing kernel Hilbert space $H_{(K_\rho)^r}(I)$ and its norm are given by:*

$$H_{(K_\rho)^r}(I) \equiv \left\{ f \in \mathrm{AC}(I) \,:\, f(a)=0,\ f' \in L^2(I, W^{1-r}\rho^{-1}\, dt) \right\}, \tag{5.3}$$

*and*

$$\|f\|_{H_{(K_\rho)^r}(I)} \equiv \left( \frac{1}{r} \int_I |f'(t)|^2 W(t)^{1-r} \rho(t)^{-1}\, dt \right)^{\frac{1}{2}},$$

*respectively.*

(For [12], page 17, in (1.59) put the factor $\frac{1}{r}$.)

For any positive integers $m, n$

$$K_\rho(s,t)^{m+n} = K_\rho(s,t)^m K_\rho(s,t)^n,$$

and so we obtain the corresponding norm inequality

$$\int_a^b |(f_1(x) f_2(x))'|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)^{m+n-1} \rho(x)}$$

$$\leq \left(\frac{1}{m} + \frac{1}{n}\right) \int_a^b |f_1'(x)|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)^{m-1} \rho(x)} \int_a^b |f_2'(x)|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)^{n-1} \rho(x)}.$$

We note that

**Open problem:** *How will be the inequality for noninteger case $m, n$?*

Meanwhile, from the identity

$$K_{H(a,b;\mathbf{R})}(x,x_1) K_{H(a',b';\mathbf{R})}(x,x_1) = \frac{1}{2}\left(\frac{a}{b} + \frac{a'}{b'}\right) K_{H(aa', ab'+a'b;\mathbf{R})}(x,x_1),$$

we obtain the corresponding inequality

$$\int_{\mathbf{R}} \left((aa')^2|(f(x)g(x))'|^2 + (ab'+a'b)^2|f(x)g(x)|^2\right) dx$$
$$\leq \frac{1}{2}\left(\frac{a}{b}+\frac{a'}{b'}\right)\int_{\mathbf{R}} \left(a^2|f'(x)|^2 + b^2|f(x)|^2\right) dx \int_{\mathbf{R}} \left((a')^2|g'(x)|^2 + (b')^2|g(x)|^2\right) dx.$$

# 6 For infinite order Sobolev spaces

Note that the kernel

$$K(z,\overline{u};t) = \frac{1}{2\sqrt{2\pi t}}\exp\left(-\frac{1}{8t}(z-\overline{u})^2\right)$$

is the reproducing kernel for the Hilbert space with finite norms

$$\sqrt{\sum_{j=0}^{\infty}\frac{(2t)^j}{j!}\int_{\mathbb{R}}|\partial_x^j f(x)|^2 dx} = \sqrt{\frac{1}{\sqrt{2\pi t}}\iint_{\mathbb{C}}|f(x+iy)|^2\exp\left(-\frac{y^2}{2t}\right)dxdy}. \tag{6.1}$$

These mean that for the restriction to the real line, the Hilbert space is an infinite order Sobolev Hilbert space and on the complex plane the space is composed of entire functions ([12], pages 141-145). We thus have the identity

$$K(z,\overline{u};t/2) = 4\sqrt{\pi t}K(z,\overline{u};t)^2$$

and we have the corresponding norm inequality.

For the isometric inequality (1.1) for the Bergman and Szegö spaces, note their representations ([12], pages 146-147).

We write $S(r) \equiv \{z \in \mathbb{C} : 0 < \arg(z) < r\}$ for the open sector and its boundary $\partial S(r) \equiv \{z \in \mathbb{C} : z = 0 \text{ or } \arg(z) = \pm r\}$.

(Note that we defined as

$$\arg 0 = 0$$

as a result of the division by zero in [13].)

**Theorem:** *Let $r \in (0, \pi/2)$. For an analytic function $f$ on the open sector $S(r)$, we have the identity*

$$\iint_{S(r)} |f(x+iy)|^2 dxdy = \sin(2r) \sum_{j=0}^{\infty} \frac{(2\sin r)^{2j}}{(2j+1)!} \int_{\mathbb{R}} x^{2j+1} |f^{(j)}(x)|^2 dx. \quad (6.2)$$

*Conversely, if any $f \in C^{\infty}(\partial S(r))$ has a convergent sum in the right-hand side in* (6.2)*, then the function $f(x)$ can be extended analytically onto the sector $S(r)$ in the form $f(z)$ and the identity* (6.2) *is valid.*

In the Szegö space, we have the following formula:

**Theorem:** *Let $r \in (0, \pi/2)$. For any member $f$ in the Szegö space on the open sector $S(r)$, we have the identity*

$$\oint_{\partial S(r)} |f(z)|^2 |dz| = 2\cos r \sum_{j=0}^{\infty} \frac{(2\sin r)^{2j}}{(2j)!} \int_{\mathbb{R}} x^{2j} |f^{(j)}(x)|^2 dx, \quad (6.3)$$

*where $f(x)$ means the nontangential Fatou limit on $\partial S(r)$ for $x \in \mathbb{R}$ . Conversely, if any $f \in C^{\infty}(0, \infty)$ has a convergent sum in the right-hand side in* (6.3)*, then the function $f(x)$ extends analytically onto the open sector $S(r)$ and the identity* (6.3) *is valid.*

As a simple case, we shall refer to the Fischer space $\mathcal{F}_a(\mathbb{C})$ admitting the reproducing kernel, for any fixed $a > 0$

$$K_a(z, \overline{u}) = \exp(a^2 z \overline{u}) \quad (z, u \in \mathbb{C})$$

with finite norms for entire functions $f(z)$

$$\|f\|_{\mathcal{F}_a(\mathbb{C})} = a \sqrt{\frac{1}{\pi} \iint_{\mathbb{C}} |f(z)|^2 \exp(-a^2 |z|^2) \, dx \, dy}.$$

See [12], page 170. We thus have the relation for any positive $a, b$

$$K_a(z, \overline{u}) K_b(z, \overline{u}) = K_{\sqrt{a^2+b^2}}(z, \overline{u})$$

and the corresponding result.

Meanwhile, any positive definite Hermitian matrix may be considered as a reproducing kernel and so we can apply the theory of reproducing kernels to that of positive definite Hermitian matrices ([6, 8]). For the product of two positive definite Hermitian matrices $A, B$ with the same size, and for the Hadamard product $*$ and for the complex conjugate transpose $*$, we can state the results as in

$$\left(\mathbf{x}^{(1)} * \mathbf{x}^{(2)}\right)^{*} \left(A^{-1} * B^{-1}\right)^{-1} \left(\mathbf{x}^{(1)} * \mathbf{x}^{(2)}\right) \leq \left(\mathbf{x}^{(1)\,*} A \mathbf{x}^{(1)}\right) \left(\mathbf{x}^{(2)\,*} B \mathbf{x}^{(2)}\right)$$

and

$$\left(A^{-1} * B^{-1}\right)^{-1} \leq A * B,$$

symbolically ([6], page 128). Equality problems are all solved .

# 7 Open problems

Our norm inequalities will be very beautiful and fundamental, so we wonder their direct derivations apart from the theory of reproducing kernels. It seems that Yamada's results [18] and our results are different in nature for the similar norm inequalities for one dimensional Sobolev Hilbert spaces. What are the relations between our results?

Professor Yamada made a deep research for the equality problem for some general norm inequalities derived from the theory of reproducing kernels, however, he stated that the new equality problem in (A) is still an open problem on 23 August, 2022.

Our norm inequalities here have a similar form as in the Schwarz inequality, so we wonder does there exist $p, q$ $(1/p + 1/q = 1)$ versions as in the Hölder inequality.


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