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Proof of Legendre Conjecture

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Abstract:

The Legendre conjecture was proposed by the French mathematician Legendre (1752-1833) and has not been proved for nearly 200 years. The conjecture is that between any two adjacent perfect square numbers, there is at least one prime number. That is, for any positive integer n, there is a prime number p such that $n^2 < p < (n+1)^2$.

For the distribution of prime number is a distribution of deterministic random, problems related to prime numbers can be studied, analyzed and proved by probability statistics.

This paper proves the conjecture by the method of probability and statistics, and proves that the number of prime numbers in the interval from n^2 to $(n+1)^2$ is similar to the number of prime numbers smaller than the integer n.

Key words:

Legendre conjecture, Probability and statistics, Prime number theorem, Deterministic random distribution

The Legendre conjecture was proposed by the French mathematician Adrien Marie Legendre (1752-1833) and has not been proved for nearly 200 years. The conjecture is that between any two adjacent perfect square numbers, there is at least one prime number. That is, for any positive integer n, there is a prime number p such that $n^2 < p < (n+1)^2$.

This paper uses the probability and statistics method to prove the conjecture, and now the method is introduced as follows:

1. Proof method of Legendre conjecture

Since there is no mathematical model for prime numbers that can be completely and accurately represented [1][2]. The distribution of prime numbers is a distribution of deterministic random, so the problems related to prime numbers can be studied, analyzed and proved by means of probability and statistics [3][4][5].

1.1 Method A

Let n be any positive integer, then the number of integers between n^2 and $(n+1)^2$ is $(n+1)^2 - n^2-1 = 2n$.

According to the theorem of prime numbers, the probability that any integer in the region from n^2 to $(n+1)^2$ is a prime number is about $1/\ln((n+1)^2)$, so the number of prime numbers (cumulative probability) between n^2 and $(n+1)^2$ is about $2n/\ln((n+1)^2)$, which is simplified to $n/\ln(n+1)$. When n increases, $\ln(n+1)$ approaches the value of $\ln(n)$, so the above estimation can be simplified as $n/\ln(n)$.

This function is of the same type as the formula for calculating the number of prime numbers less than the integer n according to the prime number theorem. When n approaches infinity, it presents a divergent form. The value of the function, that is, the number of prime numbers, is infinity and is always greater than 1. This proves Legendre's conjecture, and proves that the number of prime numbers in the interval from n^2 to $(n+1)^2$ and the number of prime numbers less than the integer n are close to each other, which is $n/\ln(n)$.

1.2 Method B

Let n be any positive integer, according to the prime number theorem: the number of any integer that is a prime number in the $(n+1)^2$ area is about $(n+1)^2/\ln((n+1)^2)$; in the n^2 area the number of any integer that is prime is approximately $n^2/\ln(n^2)$.

So the number of prime numbers between n^2 and $(n+1)^2$ is about $(n+1)^2/\ln((n+1)^2) - n^2/\ln(n^2)$, which is $1/2 *((n+1)^2/\ln(n+1)-n^2/\ln(n))$ after simplification.

When n increases, $\ln(n+1)$ is close to $\ln(n)$, so it can be simplified to $(n+0.5)/\ln(n)$, and when n increases, 0.5 can be ignored, so the formula is $n/\ln(n)$.

Same as method A above, the function exhibits a divergent shape, and the function value, that is, the number of prime numbers, is always greater than 1, which proves Legendre's conjecture.

2.Experimental verification

See Table 1 for the predicted number and actual number of prime numbers in the region from n^2 to $(n+1)^2$.

Table 1 Number of prime numbers in the region from n^2 to $(n+1)^2$

integer n (≤)	prime number (cumulative probability) $[n^2, (n+1)^2]$	prime number actual number $[n^2, (n+1)^2]$
100	21.7	23
120	25.1	24
140	28.3	25
160	31.5	33
180	34.7	36
200	37.7	33

3. Conclusion

Between any two adjacent perfect square numbers, there is at least one prime number. That is, for any positive integer n, there is a prime number p such that $n^2 < p < (n+1)^2$. And the number of prime numbers in this interval is about $n/\ln(n)$.

4.References

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