SIEVE OF ERATOSTHENES AND WHEEL FACTORIZATION

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Abstract

This paper presents a refinement of the Sieve method of Eratosthenes in conjunction with wheel factorization. The result is to use less memory.

Sieve Wheel

With the sieve of Eratosthenes[1] algorithm in the Boolean vector $SIEVE$ of size n initially all set to $true$ all elements associated with multiples of primes p can be set to $false$ using this pseudocode:

```
for (p=2; p<sqrt(n); p++)
   if ( SIEVE[p] )
      for (m=p*p; m<n; m+=p)
         SIEVE[m]=false;
```

An improvement can be made by using the Wheel factorization[2] which can be associated with modular arithmetic[3].

Given an integer bW , called modulus, two integers p and q are congruent modulo bW $p \equiv q (mod\, bW)$ if bW is a divisor of their difference $p-q$.

We therefore consider the modulo operator $p\ mod\ bW$ which denotes the unique integer r such that $0 \le r < bW$ and $r \equiv p\ (mod\ bW)$

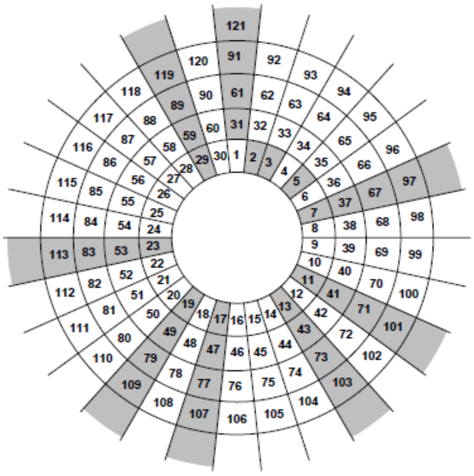
then $p = r + k \cdot bW$ where r is the remainder of p when divided by bW .

In modular arithmetic the set of integers $\{0, 1, 2, \ldots, bW-1\}$ is called the least residue system modulo bW so let's take a specific one residue system modulo bW set of $\varphi(bW)$ integers, where $\varphi(bW)$ is Euler's totient function[4], that are relatively prime to bW and mutually incongruent under modulus bW , called a reduced residue system modulo bW .

To make a sieve wheel for the purpose to find prime numbers less than n we choose $bW < \sqrt{n}$ and bW divisible by a set of prime numbers $\{p_1, p_2, \ldots, p_s\}$ then we choose an appropriate reduced residue system modulo bW stored in the vector RW of length $nR = \varphi(bW)$.

In this way we only store the numbers belonging to the congruence class or residue in RW and therefore multiples of $\{p_1, p_2, \ldots, p_s\}$ are automatically excluded.

Example if $bW=30$ then $\varphi(30)=8$ and $RW=[-23,-19,-17,-13,-11,-7,-1,1]$ is a reduced residue system modulo bW .



7	37	67	97	
11	41	71	101	
13	43	73	103	
17	47	77	107	
19	49	79	109	
23	53	83	113	
29	59	89	119	
31	61	91	121	

Analyzing the graph to describe the wheel factorization, it can be seen that, excluding 2 , 3 and 5 , if $p=RW[i]+30\cdot k$ with $k>0$ we can only store the useful areas in gray equal to a matrix with a number of rows equal to $nR=\varphi(30)=8$.

To find prime numbers different from $\{p_1,\ p_2,\ldots,\ p_s\}$ then we use a Boolean array $SIEVE$ of size $nR\ *\ \lceil n/bW\rceil$ in order to associate the possible residue in RW to each row of the array and so all elements associated with multiples of the prime numbers $\{p_1,\ p_2,\ldots,\ p_s\}$ are not stored.

So we want to get after the sieve that $p=RW[i]+bW\cdot j$ is prime if $SIEVE[i,j]\ ==\ true$.

Example in the case of $bW=6$ it's used a Boolean array $2\ *\ \lceil n/6\rceil$ or two Boolean vectors of size $\lceil n/6\rceil$.

In the second for loop of the pseudocode of the sieve of Eratosthenes for set to $false$ all elements associated multiples of p the initial index is $m_{min}=p\cdot p$ so now we have $p=r+k\cdot bW$ and $p\cdot p$ must be replaced by $(r+bW\cdot k)\cdot(s+bW\cdot k)$

where s is a remainder such that $(r\cdot s)\ \%\ bW=t$ and the residue t is the one associated with the row we are using, then

$$(r+bW\cdot k)\cdot(s+bW\cdot k)=(r\cdot s)\ \%\ bW\ +bW\cdot(bW\cdot k\cdot k+k\cdot r+k\cdot s+\lfloor(r\cdot s)/bW\rfloor)$$

and so for the row associated with remainder t for multiples of $p=r+k\cdot bW$ we use

$$m_{min}=bW\cdot k\cdot k+k\cdot r+k\cdot s+\lfloor(r\cdot s)/bW\rfloor$$

Example *bW = 6*

for *p= -1+6*k*

in the row *0* corresponding to the remainder *-1* : s= *1 r=-1 r*s=-1* m_{min}*= 6*k*k*

in the row *1* corresponding to the remainder *1* : s=*-1 r=-1 r*s= 1* m_{min}*= 6*k*k-2**k

for *p= 1+6*k*

in the row *0* corresponding to the remainder *-1 : s=-1 r=1 r*s=-1* m_{min}*= 6*k*k*

in the row *1* corresponding to the remainder *1 : s= 1 r=1 r*s= 1* m_{min}*= 6*k*k+2*k*

Then in the Boolean array *SIEVE* of size *2*(n/6+1)* initially all set to *true* and elements associated with multiples of primes *-1+6*k* and *1+6*k* can be set to *false* using this pseudocode:

```
for (k=1; k<=sqrt(n)/6; k++){
    if (SIEVE[0,k]){
        for (m=6*k*k; m<n/6+1; m+=-1+6*k)
            SIEVE[0,m]=false;
        for (m=6*k*k-2*k; m<n/6+1; m+=-1+6*k)
            SIEVE[1,m]=false;}
    if (SIEVE[1,k]){
        for (m=6*k*k; m<n/6+1; m+=1+6*k)
            SIEVE[0,m]=false;
        for (m=6*k*k+2*k; m<n/6+1; m+=1+6*k)
            SIEVE[1,m]=false;}
}
```

In general if $p=RW[j]+bW\cdot k$ (for convenience we consider $RW[j]\leq 1$ and $k>0$) and if $s=RW[x]$ we have:

$$(RW[x]+bW\cdot k)\cdot(RW[j]+bW\cdot k)=(RW[x]\cdot RW[j])+bW\cdot(bW\cdot k\cdot k+k\cdot RW[x]+k\cdot RW[j])=$$

$$=(RW[x]\cdot RW[j])\ \%\ bW+bW\cdot(bW\cdot k\cdot k+k\cdot RW[x]+k\cdot RW[j]+\lfloor(RW[x]\cdot RW[j])/bW\rfloor)$$

and $m_{min}=bW\cdot k\cdot k+k\cdot(RW[x]+RW[j])+\lfloor(RW[x]\cdot RW[j])/bW\rfloor$

or if $(RW[x]\cdot RW[j])\ \%\ bW>1$ adding and subtracting bW becomes

$$m_{min}=bW\cdot k\cdot k+k\cdot(RW[x]+RW[j])+\lfloor(RW[x]\cdot RW[j])/bW\rfloor+1$$

we build two array of size $nR * nR$ for the coefficients C_1 and C_2 then for each $RW[i]$

and for each $RW[j]$ finding $RW[x]$ such that $(RW[x]\cdot RW[j])\ \%\ bW=RW[i]$

then if $(RW[x]\cdot RW[j])\ \%\ bW=RW[i]$ we have $C_1[i,j]=RW[x]+RW[j]$

and if $RW[i]=1$ or $RW[j]=1$ or $RW[i]=RW[j]$ then $C_2[i,j]=\lfloor(RW[x]+RW[j])/bW\rfloor$ otherwise $C_2[i,j]=1+\lfloor(RW[x]+RW[j])/bW\rfloor$

In the row corresponding to the residue $RW[i]$ if $p=RW[j]+bW\cdot k$ then

$$m_{min}=bW\cdot k\cdot k+k\cdot C_1[i,j]+C_2[i,j]$$

Example *bW = 30*

n*R=8 and RW=[-23, -19, -17, -13, -11, -7, -1, 1]*

C1=

-22, -32, -28, -32, -28, -8, -8, -22
-30, -18, -30, -30, -12, -30, -12, -18
-34, -26, -16, -14, -34, -26, -14, -16
-42, -42, -18, -12, -18, -18, -18, -12
-46, -20, -34, -26, -10, -14, -20, -10
-24, -36, -36, -24, -24, -6, -24, -6
-36, -30, -24, -36, -30, -24, 0, 0
-40, -38, -40, -20, -22, -20, -2, 2

C2=

0, 9, 7, 9, 7, 1, 1 , 0
6, 0, 8, 8, 1, 6, 1, 0
9, 5, 0, 1, 9, 5, 1, 0
15, 15, 1, 0, 3, 3, 1, 0
18, 1, 10, 6, 0, 2, 1, 0
1, 11, 11, 5, 5, 0, 1, 0
10, 7, 4, 10, 7, 4, 0, 0
13, 12, 13, 3, 4, 3, 0, 0

In the Boolean array $SIEVE$ of size $nR * \lceil n/bW \rceil$ initially all set to $true$ all elements associated with multiples of primes $p=RW[j]+bW\cdot k$ can be set to $false$ using this pseudocode:

```
for (k=1; k<=sqrt(n)/bW; k++)
    for (j=0; j<nR ; j++)
        if( SIEVE[j,k] )
            for (i=0; i<nR ; i++)
            {
                m_min=bW*k*k + k*C1[i,j] + C2[i,j];
                for (m=m_min; m<n/bW+1; m+=RW[j]+bW*k)
                    SIEVE[i,m]=false;
            }
```

An improvement obtained is to have numbers smaller than n/bW and the use of a memory equal to $\varphi(bW)\cdot n/bW$.

In addition the possibility of making a segmented version using a bit space $\varphi(bW)\cdot\sqrt{n}/bW$, an example is shown below with the possible choice of the value of the wheel modulus.

Other sieves generally use $\sqrt{n}$ as memory for segmentation instead this wheel sieve uses the product of the prime numbers following the basis $\{p_1, p_2, \ldots, p_s\}$ with $p_1=2$ and $p_1<p_2<\ldots<p_s$ so that a pre-sieving can be done, in this way the memory used is slightly higher than $\varphi(bW)\cdot\sqrt{n}/bW$ but is always less than $\sqrt{n}$.

Segmented bit Wheel Sieve

Below is shown the C++ code of a segmented bit wheel sieve with adjustable modulus:

```
///   This is an implementation of the bit wheel segmented sieve
///   with max modulus wheel choice 30,  210, 2310

#include <iostream>
#include <cmath>
#include <algorithm>
#include <vector>
#include <cstdlib>
#include <stdint.h>


const int64_t PrimesBase[5]={2,3,5,7,11};
const int64_t n_PB_max = 5;

const int64_t del_bit[8] =
{
  ~(1 << 0),~(1 << 1),~(1 << 2),~(1 << 3),
  ~(1 << 4),~(1 << 5),~(1 << 6),~(1 << 7)
};

const int64_t bit_count[256] =
{
  0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4,
  1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5,
  1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5,
  2, 3, 3, 4, 3, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 6,
  1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5,
  2, 3, 3, 4, 3, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 6,
  2, 3, 3, 4, 3, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 6,
  3, 4, 4, 5, 4, 5, 5, 6, 4, 5, 5, 6, 5, 6, 6, 7,
  1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5,
  2, 3, 3, 4, 3, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 6,
  2, 3, 3, 4, 3, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 6,
  3, 4, 4, 5, 4, 5, 5, 6, 4, 5, 5, 6, 5, 6, 6, 7,
  2, 3, 3, 4, 3, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 6,
  3, 4, 4, 5, 4, 5, 5, 6, 4, 5, 5, 6, 5, 6, 6, 7,
  3, 4, 4, 5, 4, 5, 5, 6, 4, 5, 5, 6, 5, 6, 6, 7,
  4, 5, 5, 6, 5, 6, 6, 7, 5, 6, 6, 7, 6, 7, 7, 8
};
```

```
int64_t Euclidean_Diophantine( int64_t coeff_a, int64_t  coeff_b)
{
  // return y in Diophantine equation coeff_a x + coeff_b y  = 1
  int64_t k=1;
  std::vector<int64_t> div_t;
  std::vector<int64_t> rem_t;
  std::vector<int64_t> coeff_t;
  div_t.push_back(coeff_a);
  rem_t.push_back(coeff_b);
  coeff_t.push_back((int64_t)0);
  div_t.push_back((int64_t)div_t[0]/rem_t[0]);
  rem_t.push_back((int64_t)div_t[0]%rem_t[0]);
  coeff_t.push_back((int64_t)0);
  while (rem_t[k]>1)
  {
    k=k+1;
    div_t.push_back((int64_t)rem_t[k-2]/rem_t[k-1]);
    rem_t.push_back((int64_t)rem_t[k-2]%rem_t[k-1]);
    coeff_t.push_back((int64_t)0);
  }
  k=k-1;
  coeff_t[k]=-div_t[k+1];
  if (k>0)
    coeff_t[k-1]=(int64_t)1;
  while (k > 1)
  {
    k=k-1;
    coeff_t[k-1]=coeff_t[k+1];
    coeff_t[k]+=(int64_t)(coeff_t[k+1]*(-div_t[k+1]));
  }
  if (k==1)
    return (int64_t)(coeff_t[k-1]+coeff_t[k]*(-div_t[k]));
  else
    return (int64_t)(coeff_t[0]);
}
```

```
int64_t segmented_bit_sieve_wheel(uint64_t n, int64_t max_bW)
{

   int64_t sqrt_n = (int64_t) std::sqrt(n);

   int64_t  count_p=(int64_t)0;

   int64_t n_PB=(int64_t)3;
   int64_t bW=(int64_t)30;
   //get bW modulus equal to p1*p2*...*pn <=max_bW  with n=n_PB
   while(n_PB<n_PB_max && (bW*PrimesBase[n_PB]<=std::min(max_bW,sqrt_n)))
   {
      bW*=PrimesBase[n_PB];
      n_PB++;
   }
   for (int64_t i=0; i< n_PB; i++)
      if (n>PrimesBase[i])
         count_p++;

   if (n>1+PrimesBase[n_PB-1]){

      int64_t k_end = (n < bW) ? (int64_t)2 :(int64_t) (n/(uint64_t)bW+1);
      int64_t k_sqrt = (int64_t) std::sqrt(k_end/bW)+1;

      //find reduced residue system modulo bW
      std::vector<char> Remainder_t(bW, true);
      for (int64_t i=0; i< n_PB; i++)
         for (int64_t j=PrimesBase[i]; j< bW; j+=PrimesBase[i])
            Remainder_t[j]=false;
      std::vector<int64_t> RW;
      for (int64_t j=2; j< bW; j++)
         if (Remainder_t[j]==true)
            RW.push_back(-bW+j);
      RW.push_back(1);
      int64_t  nR=RW.size();   // nR=phi(bW)
```

```
std::vector<int64_t> C1(nR*nR);
std::vector<int64_t> C2(nR*nR);
for (int64_t j=0; j<nR-2; j++)
{
   int64_t rW_t,rW_t1;
   rW_t1=Euclidean_Diophantine(bW,-RW[j]);
   for (int64_t i=0; i<nR; i++)
   {
      if (i==j)
      {
         C2[nR*i+j]=0;
         C1[nR*i+j]=RW[j]+1;
      }
      else if(i==nR-3-j )
      {
         C2[nR*i+j]=1;
         C1[nR*i+j]=RW[j]-1;
      }
      else
      {
         rW_t=(int64_t)(rW_t1*(-RW[i]))%bW;
         if (rW_t>1)
            rW_t-=bW;
         C1[nR*i+j]=rW_t+RW[j];
         C2[nR*i+j]=(int64_t)(rW_t*RW[j])/bW+1;
         if (i==nR-1)
            C2[nR*i+j]-=1;
      }
   }
   C2[nR*j+nR-2]=(int64_t)1;
   C1[nR*j+nR-2]=-(bW+RW[j])-1;
   C1[nR*j+nR-1]=RW[j]+1;
   C2[nR*j+nR-1]=(int64_t)0;
}
for (int64_t i=nR-2; i<nR; i++)
{
   C2[nR*i+nR-2]=(int64_t)0;
   C1[nR*i+nR-2]=-RW[i]-1;
   C1[nR*i+nR-1]=RW[i]+1;
   C2[nR*i+nR-1]=(int64_t)0;
}
```

```
int64_t  nB=nR/8;
int64_t segment_size=1;
int64_t p_mask_i=(int64_t)4;
for (int64_t i=0; i<p_mask_i;i++)
   segment_size*=(bW+RW[i]);      // if bW=30  =7*11*13*17
while (segment_size<k_sqrt && p_mask_i<7)
{
   segment_size*=(bW+RW[p_mask_i]);      // if bW=30 max value =7*11*13*17*19*23*29
   p_mask_i++;
}

int64_t segment_size_b=nB*segment_size;
std::vector<uint8_t> Primes(nB+segment_size_b, 0xff);
std::vector<uint8_t> Segment_i(nB+segment_size_b, 0xff);
 int64_t  pb,mb,mmin,ib,i,jb,j,k,kb;
int64_t kmax = (int64_t) std::sqrt(segment_size/bW)+(int64_t)1;
for (k =(int64_t)1; k  <= kmax; k++)
{
   kb=k*nB;
   for (jb = 0; jb<nB; jb++)
   {
      for (j = 0; j<8; j++)
      {
         if(Primes[kb+jb] & (1 << j))
         {
            for (ib = 0; ib<nB; ib++)
            {
               for (i = 0; i<8; i++)
               {
                  pb=nB*(bW*k+RW[j+jb*8]);
                  mmin=nB*(bW*k*k + k*C1[(i+ib*8)*nR+j+jb*8] + C2[(i+ib*8)*nR+j+jb*8]);
                  for (mb =mmin; mb <= segment_size_b && mb>=(int64_t)0; mb +=pb )
                     Primes[mb+ib] &= del_bit[i];
                  if (pb<nB*(bW+RW[p_mask_i]) && k_end>segment_size)
                  {
                     mb-=segment_size_b;
                     while (mb<(int8_t)0)
                        mb+=pb;
                     for (; mb <= segment_size_b; mb +=pb )
                        Segment_i[mb+ib] &= del_bit[i];
                  }
```

```
                }
            }
        }
    }
  }
}
for (kb = nB; kb < std::min (nB+segment_size_b,nB*k_end); kb++)
  count_p+=bit_count[Primes[kb]];
if (kb==nB*k_end && kb<=segment_size_b && kb>(int64_t)0)
  for (ib = 0; ib<nB; ib++)
    for (i = 0; i  < 8; i++)
      if(Primes[kb+ib]& (1 << i) && RW[i+ib*8]<(int64_t)(n%bW-bW))
        count_p++;


if (k_end>segment_size)
{
  int64_t k_low, kb_low;
  std::vector<uint8_t> Segment_t(nB+segment_size_b);
  for (int64_t k_low = segment_size; k_low < k_end; k_low += segment_size)
  {
    kb_low=k_low*nB;
    for (kb = (int64_t)0; kb <(nB+segment_size_b); kb++)
      Segment_t[kb]=Segment_i[kb];
    kmax=(std::min(segment_size,(int64_t)std::sqrt((k_low+segment_size)/bW)+2));
    j=p_mask_i;
    for(k=(int64_t)1; k<=kmax; k++)
    {
      kb=k*nB;
      for (jb = 0; jb<nB; jb++)
      {
        for (; j < 8; j++)
        {
          if (Primes[kb+jb]& (1 << j))
          {
            for (ib = 0; ib<nB; ib++)
            {
              for (i = 0; i < 8; i++)
              {
                pb=bW*k+RW[j+jb*8];
                mmin=-k_low+bW*k*k+ k*C1[(i+ib*8)*nR+j+jb*8] + C2[(i+ib*8)*nR+j+jb*8];
```

```
                        if (mmin<0)
                            mmin=(mmin%pb+pb)%pb;
                        mmin*=nB;
                        pb*=nB;
                        for (mb =mmin; mb <= segment_size_b; mb += pb)
                            Segment_t[mb+ib] &= del_bit[i];
                    }
                }
            }
        }
        j=(int64_t)0;
      }
    }
    for ( kb =nB+kb_low; kb <std::min (kb_low+segment_size_b+nB,nB*k_end); kb++)
        count_p+=bit_count[Segment_t[kb-kb_low]];
  }
  if (kb==nB*k_end && kb-kb_low<=segment_size_b && kb-kb_low>(int64_t)0)
    for (ib = 0; ib<nB; ib++)
        for (i = 0; i  < 8; i++)
            if(Segment_t[kb-kb_low+ib]& (1 << i) && RW[i+ib*8]<(int64_t)(n%bW-bW))
                count_p++;
  }
 }

 return  count_p;
}

int main()
{
  int64_t  count_p;
  // segmented_bit_sieve_wheel(n, max_bW) with max_bW= 30 , 210 , 2310 for set modulus
   count_p = segmented_bit_sieve_wheel(100000000,30);
  std::cout << count_p<< " prime numbers found "<< std::endl;
  return 0;
}
```

References

[1] https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes

[2] https://en.wikipedia.org/wiki/Wheel_factorization

[3] https://en.wikipedia.org/wiki/Modular_arithmetic

[4] https://en.wikipedia.org/wiki/Euler%27s_totient_function