On the Number of Twin Primes less than a Given Quantity: Legendre's Conjecture for Twin Primes

Junho Choi

ABSTRACT. An equation regarding the density of twin primes had been presented in a preceding research. In this paper, we develop the density equation to explicitly estimate the number of twin primes and propose a conjecture related to the Twin Prime Conjecture based on the method.

1. Introduction

There is a way to get all twin primes by simple repeating process like sieve of Eratosthenes using some properties of twin primes. This method was introduced on the article of Antonie Dinculescu in 2018 as 'covering process'

All prime numbers except 2 and 3 are of form 6k-1 or 6k+1, so all twin primes except (3, 5) are of form $6k \pm 1$. Consider a table consist of numbers of the form 6k-1, 6k, and 6k+1. Since multiples of 2 and 3 are already excluded considering only numbers of form $6k \pm 1$, one would delete the columns with multiples of each prime to leave only twin primes. Two cases can be investigated.

Case 1 : p is a prime number of form 6k-1

...	6(mp-k)-1	...	6mp-1	...	6(mp+k)-1 = (6m+1)p	...
...	6(mp-k)	...	6mp	...	6(mp+k)	...
...	6(mp-k)+1 = (6m-1)p	...	6mp+1	...	6(mp+k)+1	...

Case 2 : p is a prime number of form 6k+1

...	6(mp-k)-1 = (6m-1)p	...	6mp-1	...	6(mp+k)-1	...
...	6(mp-k)	...	6mp	...	6(mp+k)	...
...	6(mp-k)+1	...	6mp+1	...	6(mp+k)+1 = (6m+1)p	...

Here, m is an arbitrary natural number. Both cases give same conclusion.

Theorem 1. ∀ z ∈ N, a pair of two numbers 6z-1 and 6z+1 are not twin primes if and only if $z = mp \pm k$ for some m ∈ N and prime number p. (k depends on p by $p = 6k \pm 1$) [1, p. 3]

This inspires that for all prime number p > 3, the column with no twin primes appears twice every consecutive p columns. Since this property is independent for two arbitrary prime numbers the 'density' of twin primes can be written as

$$\sigma = \frac{1}{6}\prod_{i=3}^{n}\left(1 - \frac{2}{p_i}\right)$$

(p_i is i-th smallest prime number and n is a natural number) [1, p. 4]

However, as the preceding research pointed, if n is fixed, this density would be greater than real on an interval of small numbers and less than real on an interval of large numbers. Some conditions are required to use this expression.

I have found the way to make this 'density of twin primes' completely available by breaking the set of natural numbers into complement intervals and examined the number of twin primes in each of them comparing with the number computed by the density equation.

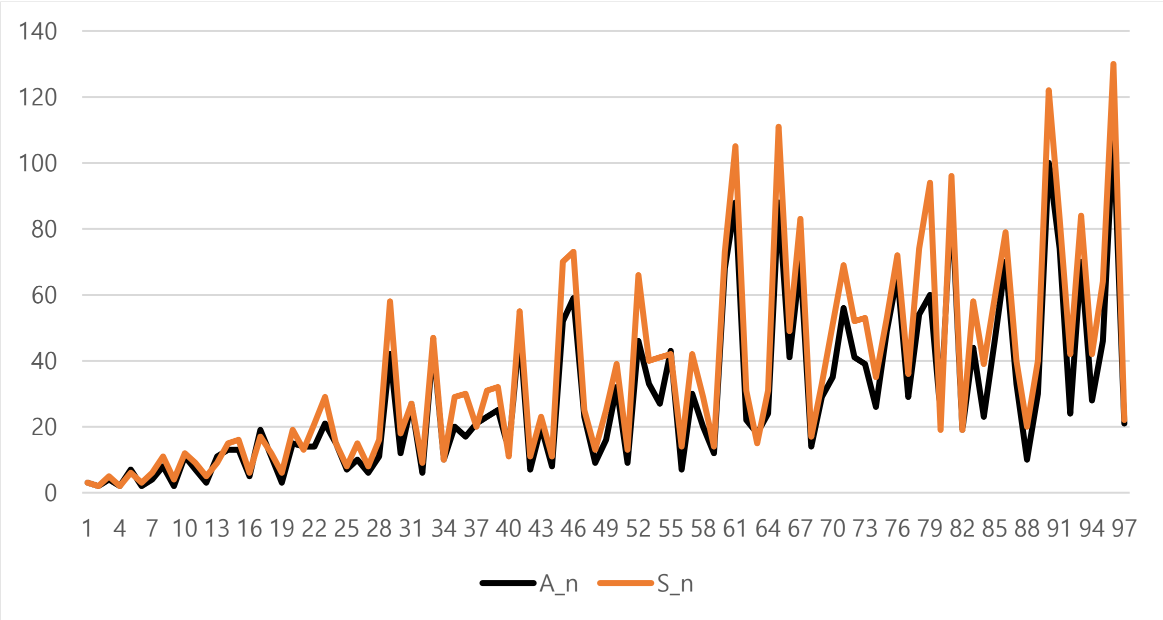
2. Making the Density of Twin Primes Available

One would use only prime numbers less than the square root of a natural number to examine if the number is a prime number or not. Considering this backwards, the range of numbers which should be divided by prime numbers less than a certain prime number p_n is from p_{n-1}^2 to p_n^2. In other words, for every natural number n, the number of twin primes between p_{n-1}^2 and p_n^2 can be estimated by

$$\left(\frac{p_n^2 - p_{n-1}^2}{6} - 1\right)\prod_{i=3}^{n-1}\left(1 - \frac{2}{p_i}\right)$$

Denote this estimation S_n.

Let A_n be the exact number of twin primes in the open interval (p_{n-1}^2, p_n^2), then we can compare A_n and S_n changing n. For the width of the paper, the graph below shows only values for n < 100.



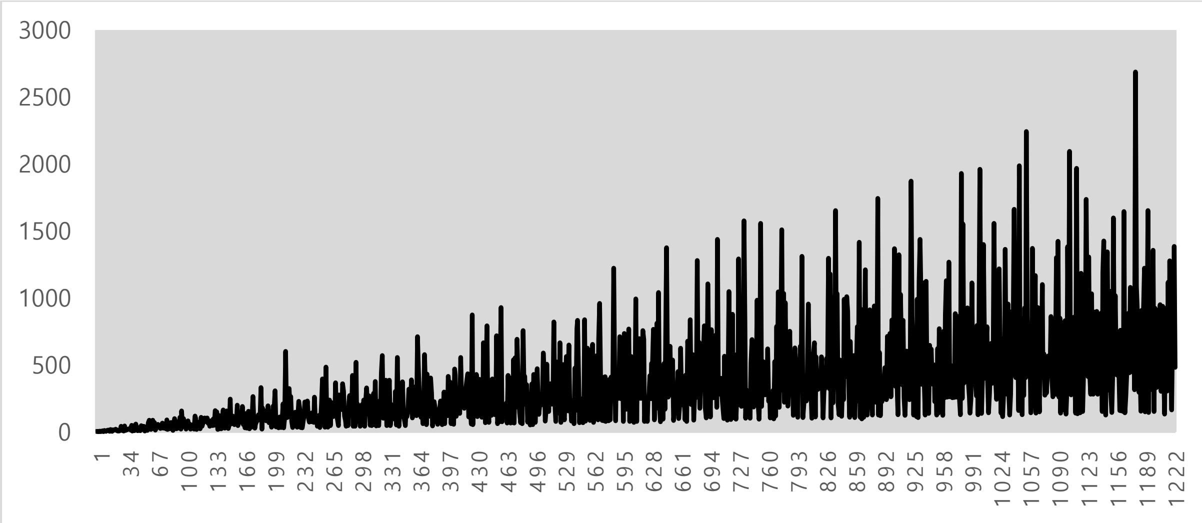
Graph 1 : A_n and S_n values for n < 100

The result is that the trends of A_n and S_n go together but the estimation is not exact(even after 100 and for sufficiently large n) Specifically, the average ratio $\frac{A_n}{S_n}$ is about 0.79

This is not what was expected at the first time considering the fact that the sieving method accurately finds the twin primes, but the cause of this error is simple; the length of each interval is too small compared with the multiple of the prime numbers used in the density equation. And the value of the mean error is reasonable since the ratio of $\pi_2(x)$ to $\frac{x}{6}\prod_{p>3}^{p<\sqrt{x}}\left(1-\frac{2}{p}\right)$ (each are summations of A_n and S_n) is also about 0.79…. One different point is that it is easier to revise the density equation using principle approach when looking at each interval.

3. Legendre's Conjecture for Twin Primes

The result above still has a meaningful point. Let's take a look at the following graph showing A_n values for n < 1230 (prime numbers less than 10^4).



Graph 2 : A_n values for n < 1230

This function oscillates and its local maximums, minimums, and mean grow. Interpreting the results, it seems that as the number gets larger, the effect of the length of the interval $p_n^2 - p_{n-1}^2$ getting longer is greater than that of the density of twin primes getting lower.

In fact, the local minimums appear when p_{n-1} and p_n are twin primes and the values gradually increases. This would be a strong evidence that the Twin Prime Conjecture is more likely to be true. Here, following conjecture can be considered.

Conjecture 1. $\forall$ n $\in$ N, $A_n > 0$
That is, there exists a pair of twin primes between the squares of any two consecutive prime numbers. (the intervals (2^2, 3^2) and (3^2, 5^2) were excluded during process above, but also satisfy this proposition)

This reminds of us Legendre's Conjecture from their similarity and so might be called as Legendre's Conjecture for twin primes. Of course, this conjecture is stronger than the Twin Prime Conjecture. Actually, for this purpose, it is enough to show that $A_n > 0$ for infinitely many n instead of all.

Furthermore, here is a much more excessive but realistic proposition. Though the increase of the local minimums of A_n is not monotone, it is close. And since the difference between the squares of twin primes continuously grows it is not likely to stop increasing. That is, it is expected to satisfy the following condition which leads to the second conjecture.

$\forall$ M > 0 , $\exists$ N > 0 s.t. n > N inspires $A_n > M$

Conjecture 2. $\lim_{n\to\infty} A_n = \infty$

4. Conclusion

Here, I propose a new conjecture which can be a key to prove the Twin Prime Conjecture and somehow related to Legendre's Conjecture; there exists a pair of twin primes between the squares of any two consecutive prime numbers.

It is difficult to prove it, but the estimating formula S_n is expected to help us. For example, it is enough to show $A_n > \frac{1}{2}S_n$ and this looks true. And considering their oscillating graphs, some other conjectures regarding the gap between two consecutive prime numbers like Andrica's Conjecture might help, too.

References

[1] A. Dinculescu. (2018) On the Numbers that Determine the Distribution of Twin Primes. Surveys in Mathematics and its Applications, volume 13, 171 – 181.

[2] Wikipedia: Legendre's Conjecture, Andrica's Conjecture

Department of Mathematics, Korea Advanced Institute of Science and Technology, 291 Daehak-ro, Yuseong-gu, Daejeon, Republic of Korea.