Schrödinger's equation in non-Euclidean metric using Geometric Algebra

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Abstract

In this paper, it is used geometric algebra to derive the Schrödinger equation in non-Euclidean metric. The properties of the basis vectors transporting information of the metric, will be used for this goal.

The result for the time dependent Schrödinger's equation is:

$$\frac{\hbar\hat{t}}{g_{tt}}\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{g_{tt}}{{g_{xx}}^2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V\psi \qquad (1)$$

And for the time independent equation:

$$g_{tt}\left(\frac{1}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{{g_{yy}}^2}\frac{\partial^2\psi}{\partial y^2}+\frac{1}{{g_{zz}}^2}\frac{\partial^2\psi}{\partial z^2}\right)+\frac{2m}{\hbar^2}(E-V)\psi=0 \qquad (2)$$

Being g_{ii}, the corresponding metric elements of each coordinate.

Also, other possibilities that offer geometric algebra to work in Quantum Mechanics will be commented.

Keywords

Geometric Algebra, non-Euclidean metric, Quantum Mechanics, basis vectors, Schrödinger equation

1. Introduction

In this paper, we will use the basis vectors used in geometric algebra (that transport information regarding the metric) to derive the Schrödinger equation (both time dependent and time independent) in non-Euclidean metric.

Also, we will give hints of what other things can be done in Quantum Mechanics using this process.

2. Basis vectors in geometric algebra

If you do not anything regarding geometric algebra, I recommend you these papers [1][2].

In paper [2] a first hint of how, in geometric algebra, you can use the basis vectors to transmit information regarding the metric was given. Anyhow, I will comment here the minimum necessary that you will need for this paper.

If we consider a basis of four vectors (not necessarily orthonormal or orthogonal):

$$\hat{t} \quad \hat{x} \quad \hat{y} \quad \hat{z}$$

In the paper [2] the nomenclature was e_0, e_1, e_2 and e_3. But in this case, we will use the above nomenclature as it is more natural and is more typical in Quantum Mechanics literature.

The only things you need to know about GA for this paper are the following.

The square of a vector in GA is its norm to the square. The norm of a vector is a scalar (not a vector anymore).

This applies also to the basis vectors:

$$\hat{x}^2 = ||\hat{x}||^2$$
$$\hat{y}^2 = ||\hat{y}||^2$$
$$\hat{z}^2 = ||\hat{z}||^2$$

For the basis vector of time, it is the same, but with a negative signature. This means, its square is negative:

$$\hat{t}^2 = -||\hat{t}||^2$$

About why the signature of time is negative will enter in more detail later.

Another point, commented in the paper [2] is that the square of the norm of the basis vectors correspond to the metric element of that coordinate:

$$\hat{x}^2 = ||\hat{x}||^2 = g_{xx}$$
$$\hat{y}^2 = ||\hat{y}||^2 = g_{yy}$$
$$\hat{z}^2 = ||\hat{z}||^2 = g_{zz}$$
$$\hat{t}^2 = -||\hat{t}||^2 = g_{tt}$$

You can see above that the basis is not necessarily orthonormal. In fact, the norm of the basis vectors is not necessarily one, but the value of the element of the metric.

Another interesting property of the vectors in geometric algebra is that you can divide by them:

$$\frac{1}{\hat{x}} = \hat{x}^{-1} = \frac{\hat{x}}{||\hat{x}||^2} = \frac{\hat{x}}{g_{xx}}$$
$$\frac{1}{\hat{y}} = \hat{y}^{-1} = \frac{\hat{y}}{||\hat{y}||^2} = \frac{\hat{y}}{g_{yy}}$$
$$\frac{1}{\hat{z}} = \hat{z}^{-1} = \frac{\hat{z}}{||\hat{z}||^2} = \frac{\hat{z}}{g_{zz}}$$

Again, for time, the minus sign appears.

$$\frac{1}{\hat{t}} = \hat{t}^{-1} = -\frac{\hat{t}}{||\hat{t}||^2} = \frac{\hat{t}}{g_{tt}}$$

You can check that the division of a vector by itself is 1.

$$\frac{\hat{x}}{\hat{x}} = \hat{x}\hat{x}^{-1} = \frac{\hat{x}\hat{x}}{||\hat{x}||^2} = \frac{||\hat{x}||^2}{||\hat{x}||^2} = 1$$

As a convention, we will consider a division always a post-multiplication by the inverse of the vector. (In GA, in general, products are not commutative).

In this paper, we will not enter in products crossing different coordinates but just for info, this is type of equation to perform those products [2]:

$$\hat{x}\hat{y} = 2g_{xy} - \hat{y}\hat{x} \quad (3)$$

Again, in an orthonormal (or just orthogonal) basis, the component g_{xy} would be zero, simplifying the calculations. But with the above equation, we do not need an orthonormal or orthogonal basis anymore. We can use above equation and the alike for different coordinates to perform the calculations in non-orthogonal and non-orthonormal basis.

Also, you can check above an example that the geometric product is not in general commutative (or even anticommutative) [1][2][3].

For your information, the product we are using between vectors is neither the scalar product nor the cross product but a kind of mix of both, that is called geometric product.

I have commented the very minimum about GA that you will need for this paper. If you need more info regarding geometric algebra, you can check these papers and book [1][2][3]. If you want to have a glimpse of what GA is, at least you need to get the understanding about the geometric product.

3. Vectorial magnitudes in geometric algebra

In standard physics, the vectorial magnitudes have always the value and the units of measure multiplying by a vector (normally a basis vector). You will never see the magnitude *divided* by a vector (it does not have any meaning).

Nevertheless, one of the characteristics of the Geometric Algebra is that you can divide by vectors (using the inversion of vectors that exists in Geometric Algebra, see previous chapter).

But is this useful at all in physics? Does it have any meaning?

As we will see later, in Euclidean metric (in an orthonormal basis) dividing by a basis vector, in general, is exactly the same as multiplying by it (except a possible change of sign in certain cases, as we will see later).

But, in non-Euclidean metric (using a non-orthonormal basis) dividing by a basis vector gives a completely different result than multiplying by it.

Is this important? And how do we know when to multiply and when to divide by a vector?

In the paper [2] Annex A1.4, a first hint was given. The units of measure in a magnitude, tell us in which position (multiplying or dividing) the vector should be located.

Let's go for a typical example of a vectorial magnitude, the velocity. If we consider a car going to the right at speed v in the x axis, normally we would represent this as:

$$\vec{v} = v \cdot \hat{x} \quad (4)$$

Where the x-hat, represents the basis vector in the x axis.

But is this correct? Or being more precise, is this complete?
Let's go the units of measure, the speed has the following units:

$$\frac{m}{s} \quad (5)$$

So, the unit of length is in the numerator and the time in the denominator. Being more precise, the velocity vector should read:

$$\vec{v} = v \cdot \frac{\hat{x}}{\hat{t}} \qquad (6)$$

The issue is that the t-hat (the unit vector of time) can be kept hidden there, with no effect at all, for different reasons.

First one, the time always flows in the same direction. In fact, except in special/general relativity, time is not really considered as a vector (as the direction of time is always the same). Only in special/general relativity the time as a vector (or as one of the inputs for the tensors if you prefer) starts having a role.

Second one, as commented before, in Euclidean metric dividing by a basis unit vector does not have any effect in the value (it is just like multiplying by one). In fact, the time is one of the strange vectors where dividing or multiplying could have an effect in sign, but let's leave this discussion for a later point.

So, the summary is, dividing by the vector t-hat in equation (6) does not have any effect in 99.9% of the situations you could think about. That is the reason, it can be kept hidden in most of the occasions.

But what happens in non-Euclidean metric? Following the rule that the division by a vector is a post-multiplication by its inverse:

$$\vec{v} = v \cdot \frac{\hat{x}}{\hat{t}} = v\hat{x}\,(\hat{t})^{-1} = \frac{v}{\|\hat{t}\|^2}\hat{x}\hat{t} = \frac{v}{g_{tt}}\hat{x}\hat{t} \quad (7)$$

We can see that if we leave the t-hat basis vector in the denominator (as it is normally done implicitly) it does not have any effect. But when we raise the vector to the numerator, the non-Euclidean effects start appearing. The square norm of the basis vector affects the value of the velocity. And do not forget that in general g_{tt} is not a constant value (it depends on the position and time coordinate of the particle).

If we were in Euclidean metric g_{tt} would be 1, so no effect at all would happen.

In general, for velocity, the norm of the time is always kept hidden in the denominator, so it does not have any influence. Or better said, normally when calculating or directly measuring the speed, this hidden factor is already there so it is implicitly already included and it is not necessary to make it explicit to get the correct results (it is there, for the "good and for the bad" let's say).

Let's continue and let's raise the bar and go to the force.

If we have a force action in the x axis, its vector will take the form:

$$\vec{F} = F \cdot \hat{x}$$

But if we take into account the units of measure, we know that the force is:

$$kg\frac{m}{s^2}$$

Considering mass a scalar (although it could be considered vectorial through time, but this will be another conversation), we would have the vectorial force as:

$$\vec{F} = F \cdot \frac{\hat{x}}{(\hat{t})^2}$$

And this is directly:

$$\vec{F} = F \cdot \frac{\hat{x}}{(\hat{t})^2} = F \cdot \frac{\hat{x}}{||\hat{t}||^2} = \frac{F}{||\hat{t}||^2}\hat{x} = \frac{F}{g_{tt}}\hat{x}$$

Same comments as with velocity, apply. And here, as the square of the time basis vector is already a scalar, in the force the vector units were correct from the beginning, just the basis vector x-hat. Although the influence of time vector is shown as a scalar (g_{tt}) affecting the value. (But hidden as commented before). And take into account that g_{tt} is not constant, depends on the coordinates of the particle including time.

In [2] Annex A1.5 you can see that we can apply this philosophy even to operators like the gradient where the vectors normally appear multiplying, but they should be dividing instead. Again, this is not noticed in Euclidean metric, so normally it does not have any consequence.

Ok, knowing this, let's start with quantum mechanics.

4. Wavefunction using Geometric Algebra

For the next points, we will follow the steps in [4] to obtain the Schrödinger equation. But we will apply everything that we have considered in the previous chapters regarding geometric algebra, vectorial magnitudes, and units of measure in non-Euclidean metric.

Just a note, in the reference [4] they call η the quantity $h/2\pi$ (being h the Planck constant [5]). Here, we will call it $\hbar = h/2\pi$ (h-bar or reduced Planck constant) as it is a more common nomenclature.

We consider a free particle of mass m moving in the positive x direction. Its potential energy is V, its momentum is p, and its total energy is E.

For that particle, the free particle wave equation is [4]:

$$\psi = Ae^{-\frac{i}{\hbar}(Et-px)} \qquad (8)$$

We see that it appears i (the imaginary unit) and the vectorial magnitudes of momentum, space and time (regarding why time is a vectorial unit, you can see the discussion in Annex A2).

As commented, we will consider E as a scalar magnitude (see Annex A1 for a discussion in this issue).

A is just a scalar coefficient (dimensionless, with no units of measure).

So, let's start with the time. As we have seen it is a vectorial unit, so we will write in the equation as $t\hat{t}$. Where the t-hat ($\hat{t}$) represents the basis vector in t. And t without hat is a scalar (the value of the time coordinate).

It is important to note that although normally a hat means a unit vector, it is not necessarily the case. The $\hat{t}$ in this case, just means the basis vector that could have a non-unitary norm. We will work in general with non-orthonormal basis (so not necessary unit basis vectors).

But what it is very important about $\hat{t}$ is that it has a negative signature. This means:

$$\hat{t}^2 = \hat{t}\hat{t} = -||\hat{t}||^2 \qquad (9)$$

We see that a minus sign appears when trying to get the norm of $\hat{t}$ (when squaring it). And in general, the norm $||\hat{t}||$ does not have to be 1, as commented. In non-Euclidean metric, we work in a more general case with non-orthonormal bases.

So, summing up the Energy and time element in the exponential will be written as:

$$Et\hat{t} \qquad (10)$$

Now, let's go to the momentum p. The units of measure of the linear momentum p are:

$$kg \cdot \frac{m}{s}$$

Here, we have a problem. Clearly the mass is a scalar, but if we have considered the Energy a scalar, this could imply incoherences. See Annex A1 for an explanation. Anyhow, for the sake of simplicity, we will consider the mass a scalar and follow straightforward, as commented in Annex A1. There, you can check that it is the most coherent step and it does not really matter, to explain the process we are following.

So, considering the mass a scalar, the vectors we have for momentum (remember that we are considering the mass moving in the x axis direction):

$$\frac{\hat{x}}{\hat{t}}$$

You can see the difference with the vector normally associated with momentum, that is only a direction in space (the x-hat in this case) but not with t-hat (as in general, you cannot divide by vectors).

Following the criteria for division of vectors established in chapter 2, we have:

$$\hat{x}(\hat{t})^{-1} = -\frac{\hat{x}\hat{t}}{||t||^2} \quad (11)$$

Recall that as t-hat has negative signature the minus sign appears when getting the inverse (it works as the imaginary unit i). Anyhow, in this case, the sign is just indicating the direction of the momentum, the criteria used in general is that a minus sign means moving to the positive x. So, we will keep the minus sign as in the original equation to meet this criterion. If you change the sign to positive, the outcome will be perfectly equivalent but with an opposite criterion regarding the signs.

So, the momentum, including its vectors, will be:

$$-\frac{p}{||t||^2}\hat{x}\hat{t} \quad (12)$$

Again, the hats do not mean unitary vectors, just basis vectors (with not necessarily unit norm).

The last magnitude to check is the x in the second element in the exponential. The magnitude including the basis vector is (nothing much to discuss here):

$$x\hat{x} \quad (13)$$

So, if we transfer everything to the equation (8) we have:

$$\psi = Ae^{-\frac{i}{\hbar}(Et\hat{t}-\frac{p}{\|t\|^2}\hat{x}\hat{t}x\hat{x})} \qquad (14)$$

For the sake of simplicity, we will consider an orthogonal basis (but not necessarily orthonormal) so we can interchange the position of t-hat and the x-hat. In a non-orthogonal basis, this can be done also using a similar equation as (3) but complicating the outcome. As this is just to show a process of obtaining Schrödinger equation in non-Euclidean metric, we will make it simple not choosing the most complicated examples.

In fact, even in an orthogonal basis, exchanging the t-hat and x-hat could imply a change of sign. As commented before, we will not take this into account as we consider this sign just a convention regarding the direction of the wave. No change in the explanation would affect having the wave in one direction or another.

$$\psi = Ae^{-\frac{i}{\hbar}(Et\hat{t}-\frac{p}{\|t\|^2}x\hat{t}\hat{x}\hat{x})} \qquad (15)$$

Now, we just operate the multiplication of the two x-hat vectors:

$$\psi = Ae^{-\frac{i}{\hbar}(Et\hat{t}-\frac{p}{\|t\|^2}x\hat{t}\|x\|^2)} \qquad (16)$$

Reordering all the scalars:

$$\psi = Ae^{-\frac{i}{\hbar}(Et\hat{t}-px\frac{\|x\|^2}{\|t\|^2}\hat{t})} \qquad (17)$$

As commented in chapter 2, the square of the norms of the basis vectors correspond to the corresponding metric element.

$$\psi = Ae^{-\frac{i}{\hbar}(Et\hat{t}-p\frac{g_{xx}}{g_{tt}}x\hat{t})} \qquad (18)$$

Also, we can take the t-hat time vector basis outside the parenthesis.

$$\psi = Ae^{-\frac{i\hat{t}}{\hbar}(Et-p\frac{g_{xx}}{g_{tt}}x)} \qquad (19)$$

As we have seen before, the signature of the t-hat is negative. So, this vector is making the work of the imaginary unit already. The imaginary unit is not necessary anymore (it is redundant). Or better said, normally the imaginary unit is an ad-hoc element added to get the equation of the wave. In this case, the vector t-hat has a physical and geometrical meaning and includes implicitly the negative signature. It is not an add-on, but a natural appearance.

So, we remove the imaginary unit, as it is not necessary anymore. The vector t is making its function.

$$\psi = Ae^{-\frac{\hat{t}}{\hbar}(Et-p\frac{g_{xx}}{g_{tt}}x)} \qquad (20)$$

Once we have the wavefunction in this form, let's continue in our goal of getting the Schrödinger equation in non-Euclidean metric using Geometric Algebra.

5. Partial derivatives of the wavefunction

Following the steps in [4], to obtain the Schrödinger equation, now we will differentiate partially the wavefunction (20) with respect to t:

$$\psi = Ae^{-\frac{\hat{t}}{\hbar}(Et-p\frac{g_{xx}}{g_{tt}}x)} \qquad (20)$$

$$\frac{\partial\psi}{\partial t} = -\frac{\hat{t}}{\hbar}EAe^{-\frac{\hat{t}}{\hbar}\left(Et-p\frac{g_{xx}}{g_{tt}}x\right)} \qquad (21)$$

$$\frac{\partial\psi}{\partial t} = -\frac{\hat{t}}{\hbar}E\psi \qquad (22)$$

$$E\psi = -\frac{\hbar}{\hat{t}}\ \frac{\partial\psi}{\partial t} \qquad (23)$$

$$E\psi = \frac{\hbar\hat{t}}{\|t\|^2}\ \frac{\partial\psi}{\partial t} \qquad (24)$$

$$E\psi = \frac{\hbar\hat{t}}{g_{tt}}\ \frac{\partial\psi}{\partial t} \qquad (25)$$

Again, following the steps in [4], now, we differentiate partially the wavefunction (20) with respect to x:

$$\psi = Ae^{-\frac{\hat{t}}{\hbar}(Et-p\frac{g_{xx}}{g_{tt}}x)} \qquad (20)$$

$$\frac{\partial\psi}{\partial x} = \frac{\hat{t}}{\hbar}p\frac{g_{xx}}{g_{tt}}Ae^{-\frac{\hat{t}}{\hbar}\left(Et-p\frac{g_{xx}}{g_{tt}}x\right)} \qquad (26)$$

$$\frac{\partial\psi}{\partial x} = \frac{\hat{t}}{\hbar}p\frac{g_{xx}}{g_{tt}}\psi \qquad (27)$$

$$p\psi = \frac{g_{tt}}{g_{xx}}\frac{\hbar}{\hat{t}}\frac{\partial\psi}{\partial x} \qquad (28)$$

$$p\psi = -\frac{g_{tt}}{g_{xx}}\frac{\hbar\hat{t}}{\|t\|^2}\frac{\partial\psi}{\partial x} \qquad (29)$$

$$p\psi = -\frac{g_{tt}}{g_{xx}}\frac{\hbar\hat{t}}{g_{tt}}\frac{\partial\psi}{\partial x} \qquad (30)$$

$$p\psi = -\frac{\hbar\hat{t}}{g_{xx}}\frac{\partial\psi}{\partial x} \qquad (31)$$

If we consider p as an operator, we have [6]:

$$p = -\frac{\hbar\hat{t}}{g_{xx}}\frac{\partial}{\partial x} \qquad (32)$$

And we can obtain the square of p as:

$$p^2 = \left(-\frac{\hbar\hat{t}}{g_{xx}}\frac{\partial}{\partial x}\right)^2 = \frac{\hbar^2\hat{t}^2}{{g_{xx}}^2}\frac{\partial^2}{\partial x^2} = \frac{\hbar^2(-\|t\|^2)}{{g_{xx}}^2}\frac{\partial^2}{\partial x^2} = -\frac{\hbar^2 g_{tt}}{{g_{xx}}^2}\frac{\partial^2}{\partial x^2} \qquad (33)$$

Remind that the square of t-hat is negative (negative signature).
It could seem weird that the square of a differential is the second differential instead of being the square of the first differential. But this is exactly as it is done in [6] and [4] for the operators getting the correct result, so we deduct that the process is ok.

So, if we apply the operator square of p (33) to the wavefunction, we have:

$$p^2\psi = -\frac{\hbar^2 g_{tt}}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2} \qquad (34)$$

6. Schrödinger time-dependent wave equation

The total energy of the particle is the sum of the kinetic energy and the potential energy [4]:

$$E = K.E. + P.E. \qquad (35)$$
$$E = \frac{1}{2}mv^2 + V \qquad (36)$$
$$E = \frac{1}{2}\frac{m^2}{m}v^2 + V \qquad (37)$$
$$E = \frac{p^2}{2m} + V \qquad (38)$$

We multiply both sides by ψ:

$$E\psi = \left(\frac{p^2}{2m} + V\right)\psi \qquad (39)$$
$$E\psi = \frac{p^2\psi}{2m} + V\psi \qquad (40)$$

We substitute the values of $E\psi$ and $p^2\psi$ obtained in equations (25) and (33):

$$\frac{\hbar\hat{t}}{g_{tt}}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{g_{tt}}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2} + V\psi \qquad (41)$$

If we compare with the Schrödinger equation [4]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \qquad (42)$$

We see the following differences:

Again, the imaginary unit does not appear. Its function is done by the t-hat vector, which square has a negative signature.

And, in the non-Euclidean geometric algebra Schrödinger equation (41) appear some elements of the metric tensor (g_{tt} and g_{xx}). Do not forget that these elements depend in general also in the position and in the time, so complicates extremely solving the equation.

For the three-dimensional expression we can just generalize the equation as done in [4]:

$$\frac{\hbar\hat{t}}{g_{tt}}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{g_{tt}}{{g_{xx}}^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi \qquad (1)$$

And again, comparing with the original Schrödinger equation, we see the same differences as commented above (the imaginary unit substituted by the t-hat vector and the appearance of the metric tensor elements g_{tt} and g_{xx}).

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi \qquad (43)$$

7. Time-independent Schrödinger's equation

Following [4], we start from equation (40):

$$E\psi = \frac{p^2\psi}{2m} + V\psi \qquad (40)$$

We substitute the value of the square of p operator () in above equation:

$$E\psi = -\frac{\hbar^2}{2m}\frac{g_{tt}}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2} + V\psi \qquad (44)$$

$$\frac{\hbar^2}{2m}\frac{g_{tt}}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2} + (E-V)\psi = 0 \qquad (45)$$

$$\frac{g_{tt}}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0 \qquad (46)$$

If we compare with the original Schrödinger's equation [4]:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0 \qquad (47)$$

We see that the difference again is in the elements of the metric tensor (g_{tt} and g_{xx}) appearing in the non-Euclidean geometric algebra equation. In general, they are not constant (they depend in position and time) complicating the solution of the equation.

If we want to get the three-dimensional equation, we just add the other two dimensions:

$$g_{tt}\left(\frac{1}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{{g_{yy}}^2}\frac{\partial^2\psi}{\partial y^2} + \frac{1}{{g_{zz}}^2}\frac{\partial^2\psi}{\partial z^2}\right) + \frac{2m}{\hbar^2}(E-V)\psi = 0 \qquad (2)$$

Again, comparing with the original equation [4] (see below), we see that the difference is the metric tensor elements not appearing. In this case there is no imaginary unit or t-hat vector so, these elements do not appear in any of the equations:

$$\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \frac{2m}{\hbar^2}(E-V)\psi = 0 \qquad (48)$$

8. One last comment regarding possibilities of geometric algebra

In this paper, we have tried to keep the calculations and the usage of geometric algebra (as it is a not known tool for most people) to the bare minimum.

The idea was not to present a perfect and unique solution but to show a process that can be followed when joining geometric algebra and quantum mechanics in non-Euclidean metric.

Let's go with an example. In (14) we arrived to the following equation:

$$\psi = Ae^{-\frac{i}{\hbar}\left(Et\hat{t} - \frac{p}{\|t\|^2}\hat{x}\hat{t}x\hat{x}\right)} \qquad (14)$$

This equation was calculated supposing that the direction of p and x is the same.
If we go to a more general situation where p goes in the direction of a vector a and the displacement (due to original or boundary conditions) is starting in another direction of a vector b, we would have:

$$\psi = Ae^{-\frac{i}{\hbar}\left(Et\hat{t}-\frac{p}{\|t\|^2}\hat{a}\hat{t}x\hat{b}\right)} \qquad (49)$$

Not entering in detail, but the geometric product of two different vectors give as a result a scalar and a different entity called bivector. In the original equations (14) (16) (19) the result is just a scalar (the square norm of x as both vectors x are the same (they are colinear)).

But in equation (49) the result of the product of a and b would include this new entity known as a bivector. The bivectors are related to rotations (it could affect the position the axis of the spin somehow) or to undulatory/vibratory movements (like zitterbewegung for example). You can check [3] and [7]. So very probably the solving the Schrödinger equation for certain cases where momentum and the displacement are not colinear could give as a result strange spin behaviors or zitterbewegung movements without needing to add any ad-hoc explanation.

As I commented in the annexes of the paper [2], the possibilities of geometric algebra are infinite and still to be explored.

9. Conclusions

In this paper, it has been explained a process to obtain the Schrödinger equation for non-Euclidean metric using geometric algebra.

The solutions obtained are (time dependent solution):

$$\frac{\hbar\hat{t}}{g_{tt}}\ \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{g_{tt}}{{g_{xx}}^2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V\psi \qquad (1)$$

And (time independent solution):

$$g_{tt}\left(\frac{1}{{g_{xx}}^2}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{{g_{yy}}^2}\frac{\partial^2\psi}{\partial y^2}+\frac{1}{{g_{zz}}^2}\frac{\partial^2\psi}{\partial z^2}\right)+\frac{2m}{\hbar^2}(E-V)\psi = 0 \qquad (2)$$

Where g_{tt}, g_{xx}, g_{yy} and g_{zz} are the corresponding metric elements (and not necessary constant, but depending in general in all the coordinates).

The idea of the paper was to show how to work in Quantum mechanics in non-Euclidean coordinates using geometric algebra. In the chapter 8, it is given an example of how just making the momentum and the displacement not colinear would change above equations and lead to changes in spin axis or in zitterbewegung movements in the corresponding particle.

Same thing can be commented regarding crossing different coordinates in the equations (something that has not been done in this paper but could be done using equation (3) and alike).

Bilbao, 18th August 2022 (viXra-v1).

16. Acknowledgements

AAAAÁBCCCDEEIIILLLLLMMMOOOPSTU

To my family and friends.

If you consider this helpful, do not hesitate to drop your BTC here:

bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f

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[4]https://imbooz.com/engineering-physics/schrodinger-wave-equation-time-dependent-and-time-independent-wave-equation/

[5] https://en.wikipedia.org/wiki/Planck_constant

[6]http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/schr.html

[7] https://arxiv.org/abs/0802.2728v1

A1. Annex A1. Energy and mass as scalars

In the present paper, we have considered mass as a scalar, as all the literature and logic states. The problem comes when we want to consider energy also a scalar because both things seem incompatible following literally the units of measure of energy.

If we consider mass as a scalar and now, we write the units of measure of the energy we have:

$$kg \cdot \frac{m^2}{s^2} = \frac{\hat{x}^2}{\hat{t}^2} = \frac{\|\hat{x}\|^2}{-\|\hat{t}\|^2} \quad (a1)$$

This leaves us with a lot of questions. Which norm of space should we use? x-hat, y-hat, z-hat? The trace of the metric tensor? The Ricci scalar? Because there is not any preferred direction that we should use.

If we go to the transformation of mass to energy, we have:

$$mc^2$$

The speed of light is always the same independently of the direction and independently of the curvature of space-time that affects the mass particles, but not the speed of the photons (its value) that it is always the same. So, it is logic to think that applying the factors in (a1) to energy, does not have any sense.

So, the most logic is to think that both mass and energy could be considered scalars at the same time.

The problem comes when we want to calculate the vectors of momentum coming from the energy as a scalar (instead of coming from the mass as a scalar).

If we consider the energy as a scalar and we want to get momentum from it, we should multiply by the opposite of velocity:

$$E \cdot \frac{s}{m} = kg \cdot \frac{m^2}{s^2} \cdot \frac{s}{m} = kg \cdot \frac{m}{s} = p$$

This means, if we start the process from energy instead of from mass (as made in the main body of the paper). The vectors in momentum would be the opposite:

$$\frac{\hat{t}}{\hat{s}}$$

This would mean that all equations in the main body (chapter 4) of the paper would be wrong. As an example, the equation (11) would read:

$$\hat{t}(\hat{x})^{-1} = -\frac{\hat{t}\hat{x}}{\|x\|^2}$$

$$-\frac{p}{\|x\|^2}\hat{t}\hat{x}$$

$$\psi = Ae^{-\frac{i}{\hbar}\left(Et\hat{t}-\frac{p}{\|x\|^2}\hat{t}\hat{x}x\hat{x}\right)}$$

$$\psi = Ae^{-\frac{i}{\hbar}\left(Et\hat{t}-\frac{p}{\|\hat{x}\|^2}\hat{t}x\|\hat{x}\|^2\right)}$$

Eliminating the i (because the t-hat makes its function as done in (20)), we have:

$$\psi = Ae^{-\frac{\hat{t}}{\hbar}(Et-px)} \qquad (a2)$$

You can see that the equation (a2) changes completely from (20), disappearing the square of the norm of x in the denominator. All the subsequent calculations, of course would be completely different.

So, what is correct? Is it incompatible to have mass as a scalar and the Energy as a scalar also?

As commented in the beginning of this annex, the m^2/s^2 in the energy come from c^2 that should not be affected by the norms (the metric) of the space and time regarding its value.

So, for the momentum, it is clear that the most straightforward way is to start from mass being a scalar and then to multiply it by space (and we know the direction) to divide it by time. Instead of trying to force an anti-intuitive move converting energy in momentum dividing by speed (and forcing the unit vectors and norms being in the inverse position where they should have an effect in the momentum value). This is, it is logic that the metric of space affect in the numerator (where the space is in the momentum) and the time in the denominator (where it is in the momentum).

So, in principle, the steps followed in the body in the paper regarding the vector for momentum seem the correct ones.

A2. Annex A2. Time as a vector

In general relativity, the time is just considered as another dimension. There are three space dimensions with its corresponding basis vectors and another dimension (time) that has another basis vector. Time is a vectorial dimension (same as the space ones).

The same as the space dimensions have a direction x-hat, y-hat, z-hat or a combination of all, time has its own direction defined by t-hat.

The only difference between time and space is its signature. The signature of time is negative while the space dimensions are positive. Depending in the convention, it could be the opposite (time positive, space negative) but the time will be always different from the space ones. This different signature of time has to be included ad-hoc somehow, in whatever theory we use. This is not the case for geometric algebra, where you can find a speculative explanation for the negative signature of time (See Annex A3) without needing any ad-hoc addition.

A3. Annex A3. Negative signature of time

In the paper [2] I already proposed a solution to the negative signature of time. But I guess we are not prepared for this conversation.

The issue is that the geometric product of the three space components:

$$\hat{x}\hat{y}\hat{z}$$

Is called pseudoscalar in geometric algebra in three dimensions. And its signature, in fact, it is negative. The following equation just applies in an orthogonal basis, but it is sufficient as an example (to see that the signature of the pseudoscalar is negative). In a non-orthogonal basis, the idea would be similar (although the calculations would be more complicated).

$$\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = -\|\hat{x}\|^2\|\hat{y}\|^2\|\hat{z}\|^2$$

Also, normally, as commented in [2], the time norm (or metric element g_{tt}) is in general the inverse of the product of the space elements (check Schwarzschild metric for example), so the proposal I made in [2] was:

$$\hat{t} = \frac{1}{\hat{x}\hat{y}\hat{z}} = \hat{x}^{-1}\hat{y}^{-1}\hat{z}^{-1} = \frac{\hat{x}\hat{y}\hat{z}}{\|\hat{x}\|^2\|\hat{y}\|^2\|\hat{z}\|^2}$$

Another possibility is that t-hat instead of being the exact inverse of the space basis vectors, it is the inverse multiplicated by a constant k.

$$\hat{t} = \frac{k}{\hat{x}\hat{y}\hat{z}} = k \cdot \hat{x}^{-1}\hat{y}^{-1}\hat{z}^{-1} = k\,\frac{\hat{x}\hat{y}\hat{z}}{\|\hat{x}\|^2\|\hat{y}\|^2\|\hat{z}\|^2}$$

Where k could be the trace of the space metric tensor, the product of the space metric elements, the Ricci scalar… Let's say a constant that it is necessary to normalize the value of t-hat compared with the space elements.

It is important to remark, as I did in [2], that if the basis vector t-hat is composed by the space basis vectors, it does not mean that the dimension time is not independent from the space ones. The parameter t (without hat) that multiplies the basis vector t (with hat) is completely free and independent. The dimension of time exists although its basis vector is somehow related to the space ones. In fact, in geometric algebra, having three space vectors imply the existence of 8 dimensions (scalars, 3 basis vectors, 3 bi-vectors and one pseudo-scalar (the time in this approach)). You can check this in [3] for example. So, time would be just one of these 8 dimensions (the pseudoscalar) appearing from the three space dimensions.