Collatz Conjecture Part 1: Alternative Definition for Collatz's Transformations

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Abstract

We consider n to have only odd values, and even values are written in the form; n.2^b. We create a predefined function r_b (n).Define, g(n)=r_b (n)+r_(b-1) (n) and prove g(n)=f(n). g(n) being an identical function to Collatz transformations, we use the properties of said function to probe the conjecture.

The problem

Conjecture: The following operation is applied on an arbitrary positive integer n

$$f(n) = \begin{cases} \dfrac{n}{2}, & if\ n \cong 0\ mod2 \\ 3n+1, & if\ n \cong 1\ mod2 \end{cases}$$

The Collatz's conjecture states: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

Abstract

We consider n to have only odd values, and even values are written in the form; $n.\,2^b$. We create a predefined function $r_b(n)$.Define, $g(n) = r_b(n) + r_{b-1}(n)$ and prove $g(n) = f(n)$. $g(n)$ being an identical function to collatz transformations, we use the properties of said function to probe the conjecture.

Format of the solution: The solution does not adhere to the conventional framework of paragraphed proof writing, every piece of maths that is important(to conjecture) is tabular.

- The solution template is inspired from Leslie Lamport; how to write a 21st century proof
- The Solution is framed in a structured template with every argument followed its proof.
- All the subsections are tabulated to study, IF-THEN clause: for main case and sub cases.
- Tabulation should help the reader understand the larger picture in context to some specific case.

Current understanding: The heuristic and probabilistic arguments that support the conjecture are well known. The conjecture has been proven valid for numbers upto 2^{68} but hasnt been proven yet for all numbers. There has been a lot of interesting work done in this problem by notable mathematicians. Few of the notable efforts have been by; Terras showing almost all values n eventually iterated to a value less than n, Krasikov and Lagarias showed that for any large number x, there were at least x0.84 initial values n between 1 and x whose Collatz iteration reached 1. Terrence tao showed Almost all Collatz orbits attain almost bounded values.

The conjecture has been studied using Benford's law, Markovs chains, binary systems among other approaches. Variants of the Collatz function have been studied, John Conway invented a computer language called fractran in which every program was a variant of the Collatz function, it turned out to be Turing complete.

There has been some interesting commentary by reputed names, regarding the problem; Paul Erdos said about the Collatz conjecture: "Mathematics may not be ready for such problems." Jeffery

Lagarias stated in 2010 that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present day mathematics. Richard K guy stated "Don't try to solve these problems! " Some call it the most dangerous problem in mathematics. All this commentary makes us more interested in looking into the problem.

Definitions:

Definition 0.1

Transformation: Application of 3n+1 on odd number is termed as transformation. We don't consider application of n/2 as a separate kind of transformation. Application of 3n+1 always results the form of $n^{'}.2^{b}$ and we just need to divide $n'.2^{b}$ by 2, b number of times, to get n' which may go through transformation once again.

Notation 1:

{ }: square brackets are used to represent sets. All the sets in the analysis are open ray sets, that is having a certain starting point and can be extended to infinity.

≡ : Equivalence is used for operations under the defined transformations in the problem, that is 3n+1 & n/2. Example; $5 \equiv 1$. One may consider ≡ as applying transformation on odd element and dividing it by max power of 2 with result being an integer.

n is defined to be only odd and we may apply 3n+1 upon it. Any even entity shall be represented as $even = n_{odd}\,.2^{b}$

Definition 0.2

$$\mathrm{n}_{\mathrm{x}(\text{before transformation ; applying } 3n+1)}$$

$$\equiv \mathrm{n}_{\mathrm{s}(\text{after transformation ; applying } 3n+1 \text{ and dividing it by max power of } 2)}$$

$$n_x \;\&\; n_s \; are\; always\; odd$$

The co-application of 3n+1 and n/2 shall be considered as a single step

$$3n_x + 1 = n_s.2^b \mid n_x \;\&\; n_s = 2k+1 \;\&\; k,b \in \mathbb{Z}^+$$

D0.2	$3n_x + 1 = n_s.2^b \; is\; same\; as\; n_x \equiv n_s$

Take the Universal set {U}

$$\{U\} = \{1,2,3,4,5 \ldots .\}$$

On all even elements, apply map (n/2 till we get odd) on {U}, we get:

$$\frac{n}{2} \to \{U\}, we\; get \{U'\} = \{1,3,5,7,9 \ldots .\}$$

We begin our study considering set $\{U'\}$ with only positive odd integers

Rooster Notation: $\{U'\} = \{1,3,5,7,9 \ldots .\}$

Set Builder Notation: $\{U'\} = \{2k+1\} \mid k \in \mathbb{Z}^+$

We define $\{\mathrm{R_y}\}$& $\{\mathrm{R_b}\}$, formulate expansion for $\{\mathrm{r_b}\}$ and establish the relationship between $\mathrm{r_b}$& $\mathrm{n_s}$

Definition 1.0: $\{\mathbf{R_y}\}$ is a set of sets contains elements corresponding to values of {U'} based upon parity of y with the given definition;

D1	Condition	$\mathrm{r_y} = \frac{\mathrm{r_{y-1}} \pm 1}{2} \vert$ $\{\mathrm{R_0}\} = \{\mathrm{U}'\} \Rightarrow \mathrm{r_0} = \mathrm{n_x}\ and\ \mathrm{y} \in \{\mathbb{Z}^+\} \cup \{0\}$
D1.1	$\mathrm{y} \cong 1 \bmod 2$	$\mathrm{r_y} = \frac{\mathrm{r_{y-1}} + 1}{2}$
D1.2	$\mathrm{y} \cong 0 \bmod 2$	$\mathrm{r_y} = \frac{\mathrm{r_{y-1}} - 1}{2}$

$\mathrm{r_{y-1}} \pm 1$ implies, we add or subtract 1 to the value of r for any given subset (y-1)

$\mathrm{r_{y-1}}$ is mapped to $\mathrm{r_y}$ if and only if value of r in $\mathrm{r_{y-1}}$ is odd. The mapping continues till r is even.

For value of r being even, we define said set as $\mathrm{r_b}$.

Example: Say, $\mathrm{n_x} = 13$, $\mathrm{r_0} = 13_0$(by definition)

- For $\mathrm{r_y} = \mathrm{r_1}$: because y is odd, $\mathrm{r_y} = \frac{\mathrm{r_{y-1}}+1}{2}$ implies $\mathrm{r_1} = \frac{\mathrm{r_0}+1}{2} = 7, \mathrm{so}\ \mathrm{r_1} = 7_1$

    Since value of r in $\mathrm{r_1}$ is odd, we extent the set further;
- For $\mathrm{r_y} = \mathrm{r_2}$: because y is even, $\mathrm{r_y} = \frac{\mathrm{r_{y-1}}-1}{2}$ implies $\mathrm{r_2} = \frac{\mathrm{r_1}-1}{2} = 3, \mathrm{so}\ \mathrm{r_2} = 3_2$

    Since value of r in $\mathrm{r_2}$ is odd, we extend the set further.
- For $\mathrm{r_y} = \mathrm{r_3}$: because y is odd, $\mathrm{r_y} = \frac{\mathrm{r_{y-1}}+1}{2}$ implies $\mathrm{r_3} = \frac{\mathrm{r_2}+1}{2} = 2, \mathrm{so}\ \mathrm{r_2} = 2_3$

    Since value of r in $\mathrm{r_3}$ is even, we cannot extend the set further. Thus, b=3 and $\mathrm{r_b} = 2_3$

Definition 2.0: $\{\mathbf{R_b}\}$

$$\mathrm{r_b} = \mathrm{r_y} \vert\ r\ in\ \mathrm{r_y} = 2k, k \in \mathbb{Z}^+$$

Since, $\mathrm{r_b}$ is same as $\mathrm{r_y}$ with the only condition is that value of r in $\mathrm{r_y}$ is even. So, $\mathrm{r_b}$ carries the same defination as $\mathrm{r_y}$

D2	Condition	$\mathrm{r_b} = \frac{\mathrm{r_{b-1}} \pm 1}{2} \vert\ \mathrm{b} \in \mathbb{Z}^+$
D2.1	$\mathrm{b} \cong 1 \bmod 2$	$\mathrm{r_b} = \frac{\mathrm{r_{b-1}} + 1}{2}$
D2.2	$\mathrm{b} \cong 0 \bmod 2$	$\mathrm{r_b} = \frac{\mathrm{r_{b-1}} - 1}{2}$

If one applies relevant map on $\mathrm{r_b}$ where value of r is even, result is a rational solution which is not a positive integer or zero, thus is invalid.

Remark: For condition r=0, we use the classification of zero being even described by Penner 1999, p. 34: Lemma B.2.2

Explanation:

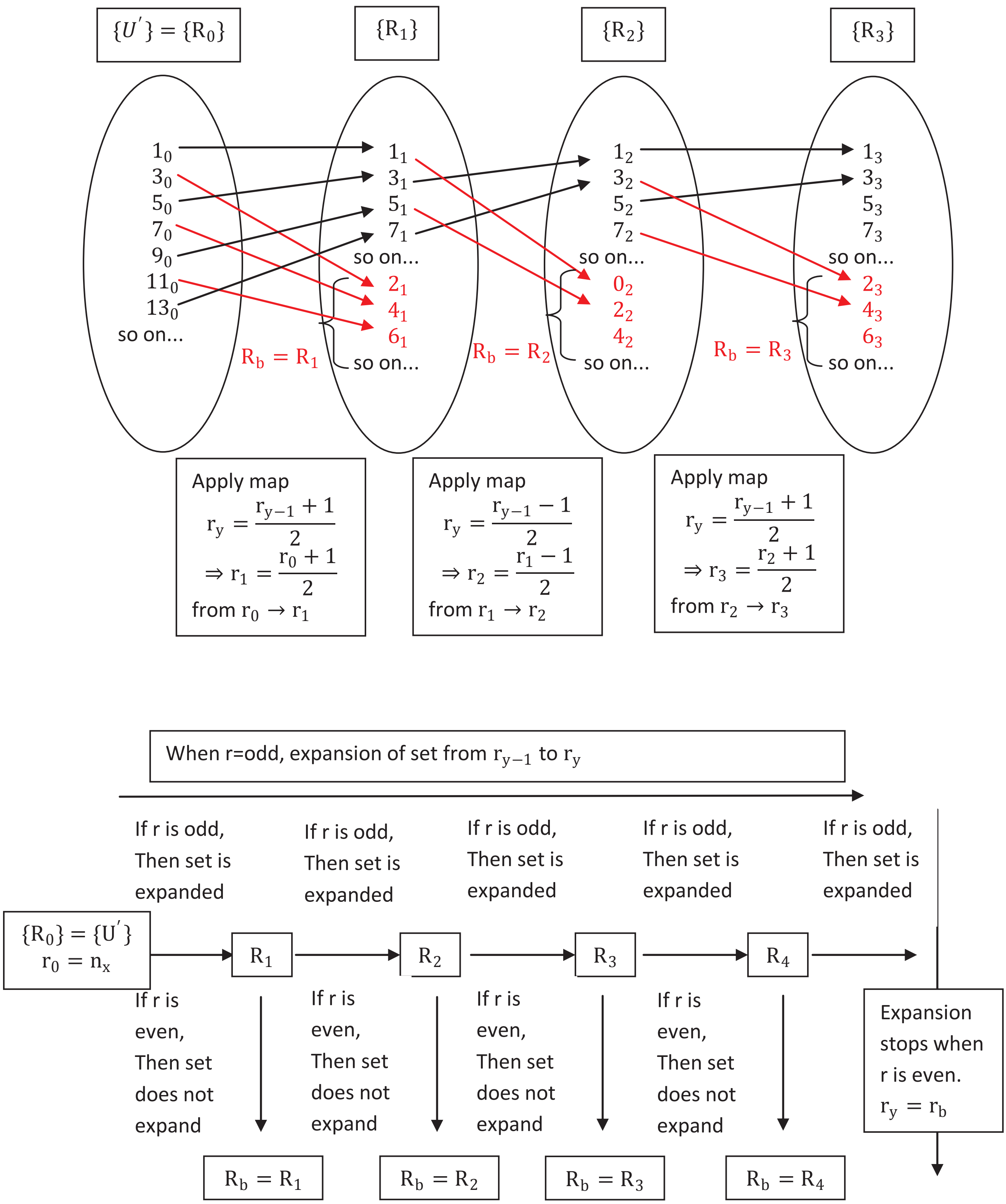

Fig1: extension of $r_0 \to r_y$ & $r_0 \to r_b$

Theorem 1: for all values of $\mathrm{n_x}$, the $\mathrm{r_b}$ has well defined values that depend upon the parity of b

$$\Leftrightarrow \ \mathrm{b} = \mathrm{even}, \mathrm{r_b} = \frac{3\mathrm{n_x} - 2^b + 1}{3.2^b} \wedge \Leftrightarrow \ \mathrm{b} = \mathrm{odd}, \mathrm{r_b} = \frac{3\mathrm{n_x} + 2^b + 1}{3.2^b} \ | 3n_x + 1 = n_s.2^b \ \& \ n_x, n_s = 2k + 1 \ \& \ k, b \in \mathbb{Z}^+$$

Proof:

T1.0	Condition	$\Leftrightarrow \ \mathrm{b} = \mathrm{even}, \mathrm{r_b} = \frac{3\mathrm{n_x} - 2^b + 1}{3.2^b} \wedge \Leftrightarrow \ \mathrm{b} = \mathrm{odd}, \mathrm{r_b} = \frac{3\mathrm{n_x} + 2^b + 1}{3.2^b} \ \vert$ $3n_x + 1 = n_s.2^b \ \& \ n_x, n_s = 2k + 1 \ \& \ k, b \in \mathbb{Z}^+$
T1.1	IF	$\mathrm{r_b} = \frac{\mathrm{r_{b-1}} \pm 1}{2}$
Proof:		By definition D2
T1.2.1	If b=even Base case b=2	$\mathrm{r_2} = \frac{3\mathrm{n_x} - 2^2 + 1}{3.2^2}$
Proof:		$\mathrm{r_2} = \frac{\mathrm{r_{2-1}} - 1}{2^1} = \frac{\frac{\mathrm{n_x} + 1}{2^1} - 1}{2^1} = \frac{\mathrm{n_x} - \frac{3}{3}}{2^2} = \frac{3\mathrm{n_x} - 3}{3.2^2} = \frac{3\mathrm{n_x} - 2^2 + 1}{3.2^2}$
T1.2.2	$b = 2a \vert \in \mathbb{Z}^+$	$\mathrm{r_{2a}} = \frac{3\mathrm{n_x} - 2^{2a} + 1}{3.2^{2a}}$
Proof:		Assumed for induction
T1.2.3	$b = 2a + 2 \vert$ $a \in \mathbb{Z}^+$	$\mathrm{r_{2a+2}} = \frac{3\mathrm{n_x} - 2^{2a+2} + 1}{3.2^{2a+2}}$
Proof:		Using T1.2.2 $\mathrm{r_{2a+2}} = \frac{\mathrm{r_{2a+2-1}} - 1}{2^1} \Rightarrow \mathrm{r_{2a+2}} = \frac{\frac{\frac{3\mathrm{n_x} - 2^{2a} + 1}{3.2^{2a}} + 1}{2^1} - 1}{2^1}$ $\mathrm{r_{2a+2}} = \frac{3\mathrm{n_x} - 2^{2a+2} + 1}{3.2^{2a+2}}$ (by algebra)
T1.2.4	Then,	$\mathrm{r_b} = \frac{3\mathrm{n_x} - 2^b + 1}{3.2^b}$
Proof:		Using mathematical induction in T1.2.2 & T1.2.3 and substituting 2a with b
T1.3.1	If, b=odd Base case b=1	$\mathrm{r_1} = \frac{3\mathrm{n_x} + 2^1 + 1}{3.2^1}$
Proof:		$\mathrm{r_1} = \frac{\mathrm{n_x} + 1}{2^1} = \frac{\mathrm{n_x} + \frac{3}{3}}{2^1} = \frac{\mathrm{n_x} + \frac{2^1 + 1}{3}}{2^1} = \frac{3.\mathrm{n_x} + 2^1 + 1}{3.2^1}$

T1.3.2	$b = 2a + 1\|$ $a \in \mathbb{Z}^{+}$	$$\mathrm{r}_{2\mathrm{a}+1} = \frac{\mathrm{r}_{2\mathrm{a}} + 1}{2^{1}}$$
Proof:		Using definition D2.1
T1.3.3	Then,	$$\mathrm{r}_{2\mathrm{a}+1} = \frac{3\mathrm{n}_{\mathrm{x}} + 2^{2a+1} + 1}{3.\,2^{2a+1}}$$
Proof:		Using T1.2.2 $$\mathrm{r}_{2\mathrm{a}+1} = \frac{\mathrm{r}_{2\mathrm{a}} + 1}{2^{1}} \Longrightarrow \mathrm{r}_{2\mathrm{a}+1} = \frac{\frac{3\mathrm{n}_{\mathrm{x}} - 2^{2a} + 1}{3.\,2^{2a}} + 1}{2^{1}}$$ $$\mathrm{r}_{2\mathrm{a}+1} = \frac{3\mathrm{n}_{\mathrm{x}} + 2^{2a+1} + 1}{3.\,2^{2a+1}} \text{(by algebra)}$$
T1.0	THEN	$$if\ \mathrm{b} = \mathrm{even}, \mathrm{r}_{\mathrm{b}} = \frac{3\mathrm{n}_{\mathrm{x}} - 2^{b} + 1}{3.\,2^{b}} \wedge$$ $$if\ \mathrm{b} = \mathrm{odd}, \mathrm{r}_{\mathrm{b}} = \frac{3\mathrm{n}_{\mathrm{x}} + 2^{b} + 1}{3.\,2^{b}}$$
Proof:		By T1.2.4 & T1.3.3

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Upon calculating based on Theorem 1, for values in r_{b}, we get;

$$r_1 = \frac{\mathrm{n}_{\mathrm{x}} + 1}{2^{1}}, r_2 = \frac{\mathrm{n}_{\mathrm{x}} - 1}{2^{2}}, r_3 = \frac{\mathrm{n}_{\mathrm{x}} + 3}{2^{3}}, r_4 = \frac{\mathrm{n}_{\mathrm{x}} - 5}{2^{4}}, r_5 = \frac{\mathrm{n}_{\mathrm{x}} + 11}{2^{5}}, r_6 = \frac{\mathrm{n}_{\mathrm{x}} - 21}{2^{6}} \ldots$$

Theorem 2:

$$\forall(\mathrm{r}_{\mathrm{b}} + \mathrm{r}_{\mathrm{b}-1}) = \mathrm{n}_{\mathrm{s}} \mid \mathrm{r}_{\mathrm{b}}\ \&\ \mathrm{r}_{\mathrm{b}-1} \in \mathrm{n}_{\mathrm{x}}\ \&\ 3n_x + 1 = n_s.\,2^{b}\ \&\ n_x, n_s = 2k + 1\ \&\ k, b \in \mathbb{Z}^{+}$$

We establish the operation "$\mathrm{r}_{\mathrm{b}} + \mathrm{r}_{\mathrm{b}-1}$" is identical to application of 3n+1 (on odd) followed by n/2 (on even) till we get odd

Proof:

T2.0	Condition	$$\forall(\mathrm{r}_{\mathrm{b}} + \mathrm{r}_{\mathrm{b}-1}) = \mathrm{n}_{\mathrm{s}} \mid$$ $$\mathrm{r}_{\mathrm{b}}\ \&\ \mathrm{r}_{\mathrm{b}-1} \in \mathrm{n}_{\mathrm{x}}\ \&\ 3n_x + 1 = n_s.\,2^{b}\ \&\ n_x, n_s = 2k + 1\ \&\ k, b \in \mathbb{Z}^{+}$$
T2.1	IF	$$\forall(\mathrm{r}_{\mathrm{b}} + \mathrm{r}_{\mathrm{b}-1}) = \mathrm{n}_{\mathrm{s}} \Longrightarrow \forall(\mathrm{r}_{\mathrm{beven}} + \mathrm{r}_{\mathrm{b}-1}) = \mathrm{n}_{\mathrm{s}} \wedge \forall(\mathrm{r}_{\mathrm{bodd}} + \mathrm{r}_{\mathrm{b}-1}) = \mathrm{n}_{\mathrm{s}}$$
Proof:		Since, parity of b seems to play a role, we put in the effort to study each case separately.
T2.2.1	If. Case 1: b=even=2j $\mid j \in \mathbb{Z}^{+}$	$$\mathrm{r}_{2\mathrm{j}} = \frac{3\mathrm{n}_{\mathrm{x}} - 2^{2j} + 1}{3.\,2^{2j}}\ \&\ r_{2k-1} = \frac{3n_x + 2^{2j-1} + 1}{3.\,2^{2j-1}}$$
Proof:		Using Theorem 1
T2.2.2		$$\mathrm{r}_{2\mathrm{j}} + \mathrm{r}_{2\mathrm{j}-1} = \frac{(3n_x + 1)}{2^{2j}}$$

Proof:		Using Algebra
T2.2.3	Then	$\forall(\mathrm{r_{beven}} + \mathrm{r_{b-1}}) = \mathrm{n_s}$
Proof:		Substitute 2j with beven & 2j-1 with b-1 in T2.2.2 and equate with D0.2 $\mathrm{r_{beven}} + \mathrm{r_{b-1}} = \mathrm{n_s} = \frac{(3n_x + 1)}{2^b}$
T2.3.1	If, Case 2: b=odd=2j+1| $j \in \mathbb{Z}^+$	$r_b = \frac{3n_x + 2^{2\mathrm{j}+1} + 1}{3.\,2^{2\mathrm{j}+1}} \,\&\, r_{2\mathrm{j}+1-1} = \frac{3n_x - 2^{2\mathrm{j}+1-1} + 1}{3.\,2^{2\mathrm{j}+1-1}}$
Proof:		Using Theorem 1
T2.3.2		$r_{2\mathrm{j}+1} + r_{2\mathrm{j}+1-1} = \frac{(3n_x + 1)}{2^{2\mathrm{j}+1}}$
Proof:		Using Algebra
T2.3.3	Then,	$\forall(\mathrm{r_{bodd}} + \mathrm{r_{b-1}}) = \mathrm{n_s}$
Proof:		Substitute 2j+1 with bodd & 2j+1-1 with b-1 in T2.3.2 and equate with D0.2 $\mathrm{r_{bodd}} + \mathrm{r_{b-1}} = \mathrm{n_s} = \frac{(3n_x + 1)}{2^b}$
T2.0	THEN,	$\forall(\mathrm{r_b} + \mathrm{r_{b-1}}) = \mathrm{n_s}$
Proof:		Using T2.2.3 & T2.3.3

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Let $\mathrm{g}(n_x) = \mathrm{r_b}(n_x) + \mathrm{r_{b-1}}(n_x)$

Then, $(\mathrm{r_b} + \mathrm{r_{b-1}}) = \mathrm{n_s} \Longrightarrow \mathrm{g}(n_x) = \mathrm{f}(n_x)$

Thus we create an identical function to the collatz transformations

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Now, we explore if there exists some element n_x, which under defined collatz transformations becomes infinity.

$$n_x \equiv n_x \,|\, n_s = \infty$$

Corollary 1: We identify the condition when any given element after undergoing transformation will definitely increase.

$$if b = 1, \forall \mathrm{n_s} > \forall \mathrm{n_x} \wedge if b > 1, \forall \mathrm{n_s} < \forall \mathrm{n_x} | 3n_x + 1 = n_s.\,2^b \,\&\, n_x\,, n_s = 2k + 1 \,\&\, k, b \in \mathbb{Z}^+$$

$$increase/decrease: condition\ for\ any\ transformation = \begin{cases} for\ b = 1, & \forall \mathrm{n_s} > \forall \mathrm{n_s} \\ for\ b > 1, & \forall \mathrm{n_s} < \forall \mathrm{n_s} \end{cases} | \mathrm{n_s} > 1$$

Proof:

C1.0	Condition	$if b = 1, \forall \mathrm{n_s} > \forall \mathrm{n_x} \wedge if b > 1, \forall \mathrm{n_s} < \forall \mathrm{n_x} \mid 3n_x + 1 = n_s.2^b \;\&\; n_x, n_s = 2k + 1 \;\&\; k, b \in \mathbb{Z}^+$
C1.1	IF	$\mathrm{r_b} + \mathrm{r_{b-1}} = \mathrm{n_s}$
Proof:		By Theorem 2
C1.2.1	If Case 1: $b = 1$	$\mathrm{r_1} + \mathrm{r_0} = \mathrm{n_s}$
Proof:		By definition D1: $\mathrm{r_0} = \mathrm{n_x}$
C1.2.2	Then	$\mathrm{n_s} > \mathrm{n_x}$
Proof:		$\mathrm{r_1} + \mathrm{r_0} = \frac{\mathrm{n_x} + 1}{2} + \mathrm{n_x} > \mathrm{n_x} \Longrightarrow \mathrm{n_s} > \mathrm{n_x}$
C1.3.1	If Case 2: $b = 2$	$\mathrm{n_s} = \mathrm{r_2} + \mathrm{r_1}$
Proof:		By Theorem 2
C1.3.2		$\mathrm{n_s} = \frac{3\mathrm{n_x} + 1}{4}$
Proof:		$\mathrm{n_s} = \frac{\mathrm{n_x} - 1}{2^2} + \frac{\mathrm{n_x} + 1}{2} = \frac{3\mathrm{n_x} + 1}{4}$
C1.3.2.1	If $\mathrm{n_x} = 1$ Then	$\mathrm{n_s} = \mathrm{n_x}$
Proof:		$3\mathrm{n_x} + 1 = \mathrm{n_s}.2^2 \;\&\; \mathrm{n_x} = 1 \Longrightarrow \mathrm{n_s} = \frac{3.1 + 1}{4} = 1$
C1.3.2.2	If $\mathrm{n_x} > 1$ Then	$\mathrm{n_x} > \mathrm{n_s}$
Proof:		$3\mathrm{n_x} + 1 = \mathrm{n_s}.2^2 \;\&\; \mathrm{n_x} = 1 + \mathrm{n}' \Longrightarrow \mathrm{n_s} = \frac{3 + 1 + 3n'}{4} = 1 + \frac{3n'}{4}$ $n' = 2k' \;\&\; k' \in \mathbb{Z}^+$
C1.4.1	If Case 3: $b \geq 3$	$3\mathrm{n_x} + 1 = \mathrm{n_s}.2^{\geq 3}$
Proof:		By definition D0.2: because $b \geq 3$
C1.4.2	Then	$\mathrm{n_x} > \mathrm{n_s}$
Proof:		if $\mathrm{n_s} > \mathrm{n_x}$, then $\mathrm{n_s} = \mathrm{n_x} + \mathrm{j} \mid j \in \mathbb{Z}^+$ $3\mathrm{n_x} + 1 = \mathrm{n_s}.2^{\geq 3} \Longrightarrow 3\mathrm{n_x} + 1 = (\mathrm{n_x} + \mathrm{j}).2^{\geq 3}$ $1 - \mathrm{j}.2^{\geq 3} = \mathrm{n_x}.(2^{\geq 3} - 3)$ for $j \geq 1$, left hand side is negative, implying $\mathrm{n_x}$ is negative, implying $\mathrm{n_x} \notin \mathbb{Z}^+$. This is false.

C1.5		$n_s < n_x$ with $b > 2$
Proof:		By C1.3.2.2 & C1.4.2
C1.0	THEN	$if b = 1, \forall n_s > \forall n_x \wedge if b > 1, \forall n_s < \forall n_x$
Proof:		By C1.2.2 & C1.5

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We consider applying transformation of some number multiple times such that it will definitely increase in all the applied transformations. We study the condition n_s is always greater than n_x during these multiple transformations. Corollary 1 states, it is only possible when b is always equal to 1 during all of these multiple transformations.

Corollary 2:

$$r_1(s) = \frac{3}{2} r_1(x) \mid r_1(x)\ is\ r_b\ for\ n_x\ , r_1(s)\ is\ r_b\ for\ n_s\ \&\ b = 1\ , 3n_x + 1 = n_s . 2^b\ \&\ n_x\ , n_s = 2k + 1\ \&\ k \in \mathbb{Z}^+$$

When we repeatedly apply transformation: we always label the element that we apply transformation upon as n_x, the transformed element is always labelled as n_s

Example: Say, $n_x = 9$ then $n_s = 7$, now apply transformation on 7, so 7 becomes n_x
$n_x = 7$ then $n_s = 11$, again continue applying transformation upon 11, so 11 becomes n_x
$n_x = 11$ then $n_s = 17$... and so on.

Proof:

C2.0	Condition	$r_1(s) = \frac{3}{2} r_1(x) \mid r_1(x)\ is\ r_b\ for\ n_x\ , r_1(s)\ is\ r_b\ for\ n_s\ \&\ b = 1\ , 3n_x + 1 = n_s . 2^b\ \&\ n_x\ , n_s = 2k + 1\ \&\ k \in \mathbb{Z}^+$
C2.1	IF	r_b for $n_x = r_b(x)$ & r_b for $n_s = r_b(s) \mid 3n_x + 1 = n_s . 2^b$
Proof:		By definition
C2.2		$n_x = 2r_1(x) - 1\ \&\ n_s = 2r_1(s) - 1$
Proof:		By algebra on definition of r_1 $$r_1(x) = \frac{n_x + 1}{2}\ \&\ r_1(s) = \frac{n_s + 1}{2}$$
C2.3		$r_1(x) = n_s - n_x$
Proof:		$r_1 + r_0 = n_s \Rightarrow r_1(x) + n_x = n_s$
C2.4		$r_{1(x)} = (2r_{1(s)} - 1) - (2r_{1(x)} - 1)$

Proof:		Using substitution of n_s & $\mathrm{n_x}$ from C2.2 in C2.3
C2.0	THEN	$$r_1(\mathrm{s}) = \frac{3}{2} r_1(\mathrm{x})$$
Proof:		Using algebra on C2.4

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Corollary 1 implies for n greater than 1; b greater than 1 is the only condition for increase during transformations.

Corollary 2 implies for n greater than 1, an element can grow finite number of times, as any number ($3r_1(\mathrm{x})$) that is divided by 2 will eventually result; an odd number. Thus after some finite number of transformations, the element n will definitely decrease because b happens to be greater than 1.

Note: we have not concluded that n reaches a value less than itself, we conclude that for all n cannot grow continuously.

Thus, the transformational process, n continuously grows and transforms to infinity; that is described by the following equation

$$n_{u1} \equiv n_{u2} \equiv n_{u3} \equiv n_{u4} \equiv n_{u5} \ldots \equiv n_{u\infty} \mid n_{u1} < n_{u2} < n_{u3} < n_{u4} < n_{u5} < \cdots < n_{u\infty}\, where\, n_{u\infty} = \infty \;\&\; r_b = r_1 \;\forall\; n_{u1}, n_{u2}, n_{u3}, n_{u4}, n_{u5} \ldots$$

is false and invalid. One concludes that *continuous* increase to infinity is not possible.

Notation: 2

$<\neq>$ is used to describe relationship between 2 elements; one element may be greater than or smaller than the other element, but both the elements are not equal.

Note: It would seem improper to use greater than or less than notation describing any series. The problem would arrive out of insufficient information; 'a < b > c ' implies that; a is less than b and b is greater than c, but it is unknown if a is less than or greater than c. However, It is okay to use such notation in the context of our analysis, as we don't know the relationship between a & b and b & c and a & c, all we know is none of the elements in the series can be equal to any other element, all the elements are unique to one and other. We consider every element during the transformational process to be not equal to any other element, as that would imply, the elements loops, thus n cannot transform to infinity.

Consider the transformational process described as:

$$n_{u1} \equiv n_{u2} \equiv n_{u3} \equiv n_{u4} \equiv n_{u5} \ldots \equiv n_{u\infty} \mid n_{u1} <\neq> n_{u2} <\neq> n_{u3} <\neq> n_{u4} <\neq> n_{u5} \ldots$$

The transformation from $n_{u1}\; to\; n_{u\infty}$ with discontinuous growth may be described by the above equation. So, it is still possible for some number to grow to infinity at a relatively slower rate.

Hence, the question of discontinuous growth to infinity remains valid and thus open.

Proposition 1:

$$n_x \not\equiv \infty \mid 3n_x + 1 = n_s . 2^b \;\&\; n_x\, , n_s = 2k + 1 \;\&\; k, b \in \mathbb{Z}^+$$

We prove proposition by contradiction.

Proof:

P1.0	Condition	$n \not\equiv \infty \mid 3n_x + 1 = n_s . 2^b \ \& \ n_x , n_s = 2k + 1 \ \& \ k, b \in \mathbb{Z}^+$
P1.1	IF	$n_x \equiv \infty \mid n_x , n_s = 2k + 1 \ \& \ k, b \in \mathbb{Z}^+$
Proof:		Assumed to establish contradiction
P1.2		$r_b + r_{b-1} = n_s$
Proof:		By Theorem 2
P1.3		$n_x \equiv \infty \Longrightarrow n_s = \infty$
Proof:		By P1.1
P1.4		$r_b + r_{b-1} = \infty$
Proof:		By P1.1 & P1.2
P1.5		$r_b \notin \mathbb{Z}^+$
Proof:		$r_b + r_{b-1} = 3r_b \pm 1 = \infty$ $r_b = \dfrac{\infty \mp 1}{3}$
P1.0	THEN	$n \not\equiv \infty$
Proof:		By contradiction in P1.1 & P1.5

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Thus, no number can transform discontinuously to infinity.

There are three possible conditions in the Collatz system as described by Lagarias;

1. n explodes to infinity (divergent trajectory)
2. numbers looping at some other point other than n=1 (Non-trivial cyclic trajectory)
3. numbers looping at 1 (convergent trajectory)

Our analysis eliminates the first condition. If one can eliminate the possibility of second condition, then the only condition left would be valid for all the numbers establishing the conjecture to be valid.

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References

Also known as $3n + 1$ problem, the $3n + 1$ conjecture, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem. The sequence of numbers involved is sometimes referred to as the hailstone sequence, hailstone numbers or hailstone numerals

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