Euler-Mascheroni Constant

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Abstract

A double integral for Euler-Mascheroni Constant

Introduction

The Euler-Mascheroni constant is defined by

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n\right)=0.577215\ldots \tag{1}$$

In this note we evaluate (via Mathematica 12.3)

$$\varepsilon = \left|\gamma - \int_0^1\int_0^1 \frac{\ln\left(-\ln\left(xy\right)\right)}{\ln\left(xy\right)}dxdy\right| \tag{2}$$

In Mathematica: "EulerGamma" is the symbol representing Euler's constant γ , which is also known as the Euler-Mascheroni constant. Euler's constant has a number of equivalent definitions in mathematics but is most commonly defined as the limiting value

$$\gamma = \lim_{n\to\infty}\left(H_n - \ln n\right) \tag{3}$$

involving Harmonic number and the natural logarithm. γ arises in mathematical computations including sums, products, integrals, and limits.

Numerical Integration in the Wolfram Language

The Wolfram Language function NIntegrate is a general numerical integrator. It can handle a wide range of one-dimensional and multidimensional integrals.

$$\text{NIntegrate}\left[f\left[x_1,x_2,\ldots,x_n\right],\left\{x_1,a_1,b_1\right\},\left\{x_2,a_2,b_2\right\},\ldots,\left\{x_n,a_n,b_n\right\}\right]$$

Find a numerical integral for the function f over the region $\left[a_1,b_1\right]\times\left[a_2,b_2\right]\times\ldots\times\left[a_n,b_n\right]$. For details see Wolfram Mathematica.

Double Integral via Mathematica

Entry 1. Method → GaussKronrodRule

$$\text{Abs}[N[\text{EulerGamma},40]-\text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]},\{x,0,1\},\{y,0,1\},\text{Method}\to \text{"GaussKronrodRule"},\text{WorkingPrecision}\to 40]]\to 0.\times 10^{-40}$$

Entry 2. Method → GaussKronrodRule

$$\text{Abs}[N[\text{EulerGamma},50]-\text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]},\{x,0,1\},\{y,0,1\},\text{Method}\to \text{"GaussKronrodRule"},\text{WorkingPrecision}\to 50]]\to 0.\times 10^{-50}$$

Entry 3. Method → GaussKronrodRule

$$\text{Abs}[N[\text{EulerGamma},60]-\text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]},\{x,0,1\},\{y,0,1\},\text{Method}\to \text{"GaussKronrodRule"},\text{WorkingPrecision}\to 60]]\to 0.\times 10^{-60}$$

Entry 4. Method → GaussKronrodRule

$$\text{Abs}[N[\text{EulerGamma},100]-\text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]},\{x,0,1\},\{y,0,1\},\text{Method}\to \text{"GaussKronrodRule"},\text{WorkingPrecision}\to 100]]\to 0.\times 10^{-100}$$

Entry 5. Method → LobattoKronrodRule

$$\text{Abs}[N[\text{EulerGamma},40]-\text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]},\{x,0,1\},\{y,0,1\},\text{Method}\to \text{"LobattoKronrodRule"},\text{WorkingPrecision}\to 40]]\to 0.\times 10^{-40}$$

Entry 6. Method → LobattoKronrodRule

$$\text{Abs}[N[\text{EulerGamma},50]-\text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]},\{x,0,1\},\{y,0,1\},\text{Method}\to \text{"LobattoKronrodRule"},\text{WorkingPrecision}\to 50]]\to 0.\times 10^{-50}$$

Entry 7. Method → LobattoKronrodRule

$$\text{Abs}[N[\text{EulerGamma},60]-\text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]},\{x,0,1\},\{y,0,1\},\text{Method}\to \text{"LobattoKronrodRule"},\text{WorkingPrecision}\to 60]]\to 0.\times 10^{-60}$$

Entry 8. Method → LobattoKronrodRule

$$\text{Abs}[N[\text{EulerGamma}, 100] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"LobattoKronrodRule"}, \text{WorkingPrecision} \rightarrow 100]] \rightarrow 0.\times 10^{-100}$$

Entry 9. Method → MultidimensionalRule

$$\text{Abs}[N[\text{EulerGamma}, 40] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"MultidimensionalRule"}, \text{WorkingPrecision} \rightarrow 40]] \rightarrow 2.2672752801264518088030039221 44 \times 10^{-9}$$

Entry 10. Method → MultidimensionalRule

$$\text{Abs}[N[\text{EulerGamma}, 40] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \{\text{"MultidimensionalRule"}, \text{"Generators"} \rightarrow 9\}, \text{WorkingPrecision} \rightarrow 40]] \rightarrow 3.56671850080456288 \times 10^{-21}$$

Entry 11. Method → DoubleExponential

$$\text{Abs}[N[\text{EulerGamma}, 40] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"DoubleExponential"}, \text{WorkingPrecision} \rightarrow 40]] \rightarrow 0.0000271956318846658131647347677332106 81$$

Entry 12. Method → DoubleExponential

$$\text{Abs}[N[\text{EulerGamma}, 40] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"DoubleExponential"}, \text{MaxRecursion} \rightarrow 150, \text{WorkingPrecision} \rightarrow 40]] \rightarrow 8.24019798353 \times 10^{-28}$$

Entry 13. Method → DoubleExponential

$$\text{Abs}[N[\text{EulerGamma}, 40] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"DoubleExponential"}, \text{MaxRecursion} \rightarrow 200, \text{WorkingPrecision} \rightarrow 40]] \rightarrow 2.\times 10^{-40}$$

Entry 14. Method → GaussBerntsenEspelidRule

$$\text{Abs}[N[\text{EulerGamma}, 40] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"GaussBerntsenEspelidRule"}, \text{WorkingPrecision} \rightarrow 40]] \rightarrow 0.\times 10^{-40}$$

Entry 15. Method $\rightarrow$ GaussBerntsenEspelidRule

$$\text{Abs}[N[\text{EulerGamma}, 50] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"GaussBerntsenEspelidRule"}, \text{WorkingPrecision} \rightarrow 50]] \rightarrow 0.\times 10^{-50}$$

Entry 16. Method $\rightarrow$ GaussBerntsenEspelidRule

$$\text{Abs}[N[\text{EulerGamma}, 60] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"GaussBerntsenEspelidRule"}, \text{WorkingPrecision} \rightarrow 60]] \rightarrow 0.\times 10^{-60}$$

Entry 17. Method $\rightarrow$ GaussBerntsenEspelidRule

$$\text{Abs}[N[\text{EulerGamma}, 100] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"GaussBerntsenEspelidRule"}, \text{WorkingPrecision} \rightarrow 100]] \rightarrow 0.\times 10^{-100}$$

Entry 18. Method $\rightarrow$ ClenshawCurtisRule

$$\text{Abs}[N[\text{EulerGamma}, 30] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"ClenshawCurtisRule"}, \text{WorkingPrecision} \rightarrow 30]] \rightarrow 9.16 \times 10^{-27}$$

Entry 19. Method $\rightarrow$ ClenshawCurtisRule

$$\text{Abs}[N[\text{EulerGamma}, 40] - \text{NIntegrate}[\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}, \{x, 0,1\}, \{y, 0,1\}, \text{Method} \rightarrow \text{"ClenshawCurtisRule"}, \text{WorkingPrecision} \rightarrow 40]] \rightarrow 1.23 \times 10^{-37}$$

Entry 20. Comparisons of the Rules , Digits=30

Method-Rule	Epsilon- ε	Timing
GaussKronrodRule	$0.\times 10^{-30}$	1.85938
LobattoKronrodRule	2.3×10^{-28}	1.9375
MultidimensionalRule-Generators 9	3.5×10^{-21}	34.5313
DoubleExponential-MaxRecusion 200	$0.\times 10^{-30}$	2.98438
GaussBerntsenEspelidRule	$0.\times 10^{-30}$	1.67188
ClenshawCurtisRule	9.1×10^{-27}	76.8125

Entry 21. Comparisons of the Rules , Digits=40

Method-Rule	Epsilon- ε	Timing
GaussKronrodRule	$0.\times 10^{-40}$	6.60938
LobattoKronrodRule	$0.\times 10^{-40}$	7.42188
MultidimensionalRule-Generators 9	3.5×10^{-21}	35.2031
DoubleExponential-MaxRecusion 200	$0.\times 10^{-40}$	3.375
GaussBerntsenEspelidRule	$0.\times 10^{-40}$	6.42188
ClenshawCurtisRule	1.2×10^{-37}	473.938

Entry 22. Comparisons of the Rules , Digits=50

Method-Rule	Epsilon- ε	Timing
GaussKronrodRule	$0.\times 10^{-50}$	15.9688
LobattoKronrodRule	$0.\times 10^{-50}$	19.0313
DoubleExponential-MaxRecusion 200	$0.\times 10^{-40}$	3.53125
GaussBerntsenEspelidRule	$0.\times 10^{-50}$	15.7813

Entry 23. Comparisons of the Rules , Digits=100

Method-Rule	Epsilon- ε	Timing
GaussKronrodRule	$0.\times 10^{-100}$	169.531
LobattoKronrodRule	$0.\times 10^{-100}$	192.234
GaussBerntsenEspelidRule	$0.\times 10^{-100}$	169.0

Entry 24. Comparisons of the Rules , Digits=150

Method $\to$ GaussBerntsenEspelidRule

Abs[N[EulerGamma, 150] − NIntegrate[$\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}$, {$x$, 0,1}, {$y$, 0,1}, Method → "GaussBerntsenEspelidRule", WorkingPrecision → 150]]//Timing → {613.171875,0.× 10^{-150}}

Method $\to$ GaussKronrodRule

Abs[N[EulerGamma, 150] − NIntegrate[$\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}$, {$x$, 0,1}, {$y$, 0,1}, Method → ”GaussKronrodRule”, WorkingPrecision → 150]]//Timing → {611.59375,0.× 10^{-150}}

Method $\to$ LobattoKronrodRule

Abs[N[EulerGamma, 150] − NIntegrate[$\frac{\text{Log}[-\text{Log}[xy]]}{\text{Log}[xy]}$, {$x$, 0,1}, {$y$, 0,1}, Method → "LobattoKronrodRule", WorkingPrecision → 150]]//Timing → {647.890625,0.× 10^{-150}}

Endnote

$$\gamma = \int_0^1 \int_0^1 \frac{\ln(-\ln(xy))}{\ln(xy)} dxdy \tag{4}$$

$$\gamma = -\int_0^\infty \int_0^\infty \frac{\ln(x+y)}{x+y} e^{-x-y} dxdy \tag{5}$$

$$\gamma = -2\int_0^\infty \int_0^y \frac{\ln(x+y)}{x+y} e^{-x-y} dxdy \tag{6}$$

$$\gamma = -\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\ln(e^x+e^y)}{e^{-x}+e^{-y}} e^{-e^x-e^y} dxdy \tag{7}$$

$$\gamma = -\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\ln(e^{-x}+e^{-y})}{e^x+e^y} e^{-e^{-x}-e^{-y}} dxdy \tag{8}$$

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