Exploration of Expression for Pi

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Abstract:
In this paper, the author explores the expression for Pi.

1. Introduction

Recall that

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx = 3.1415926535... \tag{1}$$

In this note we give an integral for π.

2. Integral for Pi

$$\pi = 6\int_0^\infty \frac{\operatorname{sech} x}{\sqrt[4]{1+4\cosh x\cos\left(x\sqrt{3}\right)+4\left(\cosh x\right)^2}}\cos\left(\frac{x\sqrt{3}}{2}-\frac{1}{2}\tan^{-1}\left(\frac{\sin\left(x\sqrt{3}\right)}{2\cosh x+\cos\left(x\sqrt{3}\right)}\right)\right)dx \tag{2}$$

3. Alternative formula

$$\pi = 3\sqrt{2}\int_0^\infty \frac{\left(\operatorname{sech} x\right)^{3/2}}{\sqrt{4+4\operatorname{sech} x\cos\left(x\sqrt{3}\right)+\left(\operatorname{sech} x\right)^2}}f\left(x\right)dx \tag{3}$$

where

$$f\left(x\right)=\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{\sqrt{4+4\operatorname{sech} x\cos\left(x\sqrt{3}\right)+\left(\operatorname{sech} x\right)^2}+2+\operatorname{sech} x\cos\left(x\sqrt{3}\right)}$$
$$+sign\left(\sin\left(x\sqrt{3}\right)\right)\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{\sqrt{4+4\operatorname{sech} x\cos\left(x\sqrt{3}\right)+\left(\operatorname{sech} x\right)^2}-2-\operatorname{sech} x\cos\left(x\sqrt{3}\right)} \tag{4}$$

The function $sign\left(x\right)$ is defined by

$$sign\left(x\right)=\begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x>0 \end{cases} \tag{5}$$

4. Inequality

$$\left|\pi - 6\int_0^n \frac{\operatorname{sech} x}{\sqrt[4]{1+4\cosh x\cos\left(x\sqrt{3}\right)+4\left(\cosh x\right)^2}}g\left(x\right)dx\right| \le \frac{12e^{-n}}{\sqrt[4]{1+4\cosh n\left(\cosh n-1\right)}} \quad ,n=1,2,3,... \tag{6}$$

where

$$g\left(x\right)=\cos\left(\frac{x\sqrt{3}}{2}-\frac{1}{2}\tan^{-1}\left(\frac{\sin\left(x\sqrt{3}\right)}{2\cosh x+\cos\left(x\sqrt{3}\right)}\right)\right) \tag{7}$$

5. Series

$$\pi = 6\int_0^a \frac{\operatorname{sech} x}{\sqrt[4]{1+4\cosh x\cos\left(x\sqrt{3}\right)+4\left(\cosh x\right)^2}}\cos\left(\frac{x\sqrt{3}}{2}-\frac{1}{2}\tan^{-1}\left(\frac{\sin\left(x\sqrt{3}\right)}{2\cosh x+\cos\left(x\sqrt{3}\right)}\right)\right)dx$$
$$+12\sum_{k=0}^{\infty}\left(-1\right)^k e^{-2ka}\sum_{n=0}^{k}\binom{2n}{n}2^{-2n}\sum_{m=0}^{n}\binom{n}{m}e^{a\left(m-\frac{3}{2}\right)}\Re\left(\frac{e^{a\sqrt{3}\left(m+\frac{1}{2}\right)i}}{2k-m+\frac{3}{2}-\sqrt{3}\left(m+\frac{1}{2}\right)i}\right) \tag{8}$$

where

$$a > \sigma = 0.43205... \tag{9}$$
$$e^{-2\sigma}\left(1+2e^{-\sigma}\cos\left(\sigma\sqrt{3}\right)+e^{-2\sigma}\right)=1 \tag{10}$$

and $i=\sqrt{-1}$, $\Re(z)$ is the real part of complex z.

6. Future Research

$$\int_0^{\infty}\frac{1}{1+4\cosh x\cos\left(x\sqrt{3}\right)+4\left(\cosh x\right)^2}\cos\left(2\tan^{-1}\left(\frac{\sin\left(x\sqrt{3}\right)}{2\cosh x+\cos\left(x\sqrt{3}\right)}\right)\right)dx=\frac{1}{6} \tag{11}$$

$$\int_0^{\infty}\frac{1}{1+4\cosh x\cos\left(x\sqrt{3}\right)+4\left(\cosh x\right)^2}\sin\left(2\tan^{-1}\left(\frac{\sqrt{2}\cosh x-\sin\left(x\sqrt{3}-\frac{\pi}{4}\right)}{\sqrt{2}\cosh x+\sin\left(x\sqrt{3}+\frac{\pi}{4}\right)}\right)\right)dx=\frac{1}{6} \tag{12}$$

References

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