LIBOR Market Model Pricing

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ABSTRACT

This article presents a lattice approach for LIBOR Market Model by using several fast drift approximation methods. The fast convergence behavior requires fewer discretization nodes that gives better performance without losing much accuracy. Moreover, the calibration is almost automatic and it is simple and easy to implement.

Key Words: LIBOR Market Model, lattice model, drift approximation, risk management, calibration

This paper presents a lattice approach within the LMM. The model has similar accuracy to the current pricing models in the market, but is much faster. Some other merits of the model are that calibration is almost automatic and the approach is less complex and easier to implement than other current approaches.

We introduce a shifted forward measure that uses a variable substitution to shift the center of a forward rate distribution to zero. This ensures that the distribution is symmetric and can be represented by a relatively small number of discrete points. The shift transformation is the key to achieve high accuracy in relatively few discrete finite nodes. In addition, we present several fast and novel drift approximation approaches. Other concepts used in the model are probability distribution structure exploitation, numerical integration and the long jump technique (we only position nodes at times when decisions need to be made).

This model is actually quite useful for risk management because normally full-revaluations of an entire portfolio under hundreds of thousands of different future scenarios are required for a short time window. Without an efficient algorithm, one cannot properly capture and manage the risk exposed by the portfolio.

I. LIBOR MARKET MODEL

Let (Ω , $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{P}$) be a filtered probability space satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration. Consider an increasing maturity structure $0=T_0<T_1<\ldots<T_N$ from which expiry-maturity pairs of dates (T_{k-1} , T_k) for a family of spanning forward rates are taken. For any time $t\leq T_{k-1}$, we define a right-continuous mapping function $n(t)$ by $T_{n(t)-1}\leq t<T_{n(t)}$. The simply compounded forward rate reset at *t* for forward period (T_{k-1} , T_k) is defined by

$$F_k(t) := F(t;T_{k-1},T_k) = \frac{1}{\delta_k}\left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right) \qquad (1)$$

where $P(t,T)$ denotes the time *t* price of a zero-coupon bond maturing at time *T* and $\delta_k := \delta(T_{k-1},T_k)$ is the accrual factor or day count fraction for period (T_{k-1} , T_k).

Inverting this relationship (1), we can express a zero-coupon bond price in terms of forward rates as:

$$P(t,T_k) = P(t,T_{n(t)})\prod_{j=n(t)}^{k} \frac{1}{1+\delta_j F_j(t)} \qquad (2)$$

LIBOR Market Model Dynamics

For brevity, we discuss the one-factor LMM only. The one-factor LMM (Brace et al. [1997]) under forward measure Q^i can be expressed as

If $i < k, t \le T_i$, $$dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=i+1}^{k} \frac{\delta_j \sigma_j(t) F_j(t)}{1+\delta_j F_j(t)} dt + \sigma_k(t)F_k(t)dX_t \qquad (3a)$$

If $i = k, t \le T_{k-1}$, $$dF_k(t) = \sigma_k(t)F_k(t)dX_t \qquad (3b)$$

If $i > k, t \le T_{k-1}$, $$dF_k(t) = -\sigma_k(t)F_k(t)\sum_{j=k+1}^{i} \frac{\delta_j \sigma_j(t) F_j(t)}{1+\delta_j F_j(t)} dt + \sigma_k(t)F_k(t)dX_t \qquad (3c)$$

where X_t is a Brownian motion.

There is no requirement for what kind of instantaneous volatility structure should be chosen during the life of the caplet. All that is required is (see Hull-White [2000]):

$$(\widehat{\sigma}_k)^2 := \left(\widehat{\sigma}_k(T_{k-1},\mathrm{K})\right)^2 = \frac{1}{T_{k-1}}\int_0^{T_{k-1}} \sigma_k^2(u)du \qquad (4)$$

where $\widehat{\sigma}_k$ denotes the market Black caplet volatility and K denotes the strike. Given this equation, it is obviously not possible to uniquely pin down the instantaneous volatility function. In fact, this specification allows an infinite number of choices. People often assume that a forward rate has a piecewise constant instantaneous volatility. Here we choose the forward rate $F_k(t)$ has constant instantaneous volatility regardless of *t* (see Brigo-Mercurio [2006]).

Shifted Forward Measure

The $F_k(t)$ is a Martingale or driftless under its own measure Q^k. The solution to equation (3b) can be expressed as

$$F_k(t) = F_k(0)\exp\left(-\frac{1}{2}\int_0^t \sigma_k(s)^2 ds + \int_0^t \sigma_k(s) dX_s\right) \tag{5}$$

where $F_k(0) = F(0; T_{k-1}, T_k)$ is the current (spot) forward rate. Under the volatility assumption described above, equation (5) can be further expressed as

$$F_k(t) = F_k(0)\exp\left(-\frac{\sigma_k^2}{2}t + \sigma_k X_t\right) \tag{6}$$

Alternatively, we can reach the same Martingale conclusion by directly deriving the expectation of the forward rate (6); that is

$$\begin{aligned} E_0\left(F_k(t)\right) &= F_k(0)\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma_k^2}{2}t + \sigma_k X_t\right)\exp\left(-\frac{X_t^2}{2t}\right)dX_t \\ &= F_k(0)\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(X_t - t\sigma_k)^2}{2t}\right)dX_t = F_k(0)\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\exp\left(-\frac{Y_t^2}{2t}\right)dY_t = F_k(0) \end{aligned} \tag{7}$$

where X_t, Y_t are both Brownian motions with a normal distribution (0, *t*) at time *t*, $E_t(\bullet) := E(\bullet \mid \mathcal{F}_t)$ is the expectation conditional on the $\mathcal{F}_t$, and the variable substitution used for derivation is

$$Y_t = X_t - t\sigma_k \tag{8}$$

After applying this variable substitution (8), equation (6) can be expressed as

$$F_k(t) = F_k(0)\exp\left(-\frac{\sigma_k^2}{2}t + \sigma_k X_t\right) = F_k(0)\exp\left(\frac{\sigma_k^2}{2}t + \sigma_k Y_t\right) \tag{9}$$

Since the LMM models the complete forward curve directly, it is essential to bring everything under a common measure. The terminal measure is a good choice for this purpose, although this is by no means the only choice. The forward rate dynamic under terminal measure Q^N is given by

$$dF_k(t) = -\sigma_k F_k(t) \sum_{j=k+1}^{N} \frac{\delta_j \sigma_j F_j(t)}{1+\delta_j F_j(t)} dt + \sigma_k F_k(t) dX_t \tag{10}$$

The solution to equation (10) can be expressed as

$$F_k(t) = F_k(0) \exp\left(\mu_k(t) - \int_0^t \frac{\sigma_k^{\ 2}}{2} ds + \int_0^t \sigma_k dX_s \right) = F_k(0) \exp\left(\mu_k(t) - \frac{\sigma_k^{\ 2}}{2} t + \sigma_k X_t \right) \tag{11a}$$

where the drift is given by

$$\mu_k(t) = -\int_0^t \sum_{j=k+1}^{N} \psi_j(s) \sigma_k \sigma_j ds = -\int_0^t \sum_{j=k+1}^{N} \frac{\delta_j F_j(s)}{1+\delta_j F_j(s)} \sigma_k \sigma_j ds \tag{11b}$$

where $\psi_j(s) = \delta_j F_j(s) / \lfloor 1 + \delta_j F_j(s) \rfloor$ is the drift term.

Applying (8) to (11a), we have the forward rate dynamic under the shifted terminal measure as

$$F_k(t) = F_k(0) \exp\left(\mu_k(t) + \frac{\sigma_k^{\ 2}}{2} t + \sigma_k Y_t \right) \tag{12}$$

Drift Approximation

Under terminal measure, the drifts of forward rate dynamics are state-dependent, which gives rise to sufficiently complicated non-lognormal distributions. This means that an explicit analytic solution to the forward rate stochastic differential equations cannot be obtained. Therefore, most work on the topic has focused on ways to approximate the drift, which is the fundamental trickiness in implementing the Market Model.

Frozen Drift (FD). Replace the random forward rates in the drift by their deterministic initial values, i.e.,

$$\mu_k(t) = -\int_0^t \sum_{j=k+1}^{N} \frac{\delta_j F_j(s)}{1+\delta_j F_j(s)} \sigma_k \sigma_j ds \approx -\sum_{j=k+1}^{N} \frac{\delta_j F_j(0)}{1+\delta_j F_j(0)} \sigma_k \sigma_j t \qquad (13)$$

Arithmetic Average of the Forward Rates (AAFR)*.* Apply the midpoint rule (rectangle rule) to the random forward rates in the drift, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^{N} \frac{\delta_j \frac{1}{2}\left(F_j(0)+F_j(t)\right)}{1+\delta_j \frac{1}{2}\left(F_j(0)+F_j(t)\right)} \sigma_k \sigma_j t \qquad (14)$$

Arithmetic Average of the Drift Terms (AADT)*.* Apply the midpoint rule to the random drift terms, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^{N} \frac{1}{2}\left(\frac{\delta_j F_j(0)}{1+\delta_j F_j(0)} + \frac{\delta_j F_j(t)}{1+\delta_j F_j(t)} \right) \sigma_j \sigma_k t \qquad (15)$$

Geometric Average of the Forward Rates (GAFR). Replace the random forward rates in the drift by their geometric averages, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^{N} \frac{\delta_j \sqrt{F_j(0) \times F_j(t)}}{1+\delta_j \sqrt{F_j(0) \times F_j(t)}} \sigma_j \sigma_k t \qquad (16)$$

Geometric Average of the Drift Terms (GADT). Replace the random drift terms by their geometric averages, i.e.,

$$\mu_k(t) \approx -\sum_{j=k+1}^{N} \sqrt{\frac{\delta_j F_j(0)}{1+\delta_j F_j(0)} \times \frac{\delta_j F_j(t)}{1+\delta_j F_j(t)}} \sigma_j \sigma_k t \qquad (17)$$

Conditional Expectation of the Forward Rate (CEFR). In addition to the two endpoints, we can further enhance our estimate based on the dynamics of the forward rates. The forward rate $F_j(s)$ follows the dynamic (9) (The drift term is ignored). We can derive the expectation of the forward rate conditional on the two endpoints and replace the random forward rate in the drift by the conditional expectation of the forward rate.

Proposition 1. *Assume the forward rate* $F_j(s)$ *follows the dynamic (9), with the two known endpoints given by* $F_j(0)$ *and* $F_j(t)$. *Based on the conditional expectation of the forward rate* $F_j(s)$, *the drift of* $F_k(t)$ *can be expressed as*

$$\mu_k(t) \approx -\sum_{j=k+1}^{N}\int_0^t \frac{\delta_j E_0[F_j(s)|_{F_j(0),F_j(t)}]}{1+\delta_j E_0[F_j(s)|_{F_j(0),F_j(t)}]}\sigma_j\sigma_k ds \qquad (18a)$$

where the conditional expectation of the forward rate is given by

$$E_0[F_j(s)|_{F_j(0),F_j(t)}] = F_j(0)\left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{s}{t}}\exp\left(\frac{\sigma_j^2 s(t-s)}{2t}\right) \qquad (18b)$$

Proof. See Appendix A.

Conditional Expectation of the Drift Term (CEDT). Similarly, we can calculate the conditional expectation of the drift term and replace the random drift term by the conditional expectation.

Proposition 2. *Assume the forward rate* $F_j(s)$ *follows the dynamic (9), with the two known endpoints given by* $F_j(0)$ *and* $F_j(t)$. *Based on the conditional expectation of the drift term* ψ_j, *the drift of* $F_k(t)$ *can be expressed as*

$$\mu_k(t) \approx -\sum_{j=k+1}^{N}\int_0^t E_0\left(\left.\frac{\delta_j F_j(s)}{1+\delta_j F_j(s)}\right|_{F_j(0),F_j(t)}\right)\sigma_j\sigma_k ds \qquad (19a)$$

where the conditional expectation of the drift term is given by

$$E_0\left(\psi_j(s)|_{F_j(0),F_j(t)}\right) = E_0\left(\left.\frac{\delta_j F_j(s)}{1+\delta_j F_j(s)}\right|_{F_j(0),F_j(t)}\right) = 1-\frac{1+\nu_{Cj}(s)/\mu_{Cj}^2(s)}{\mu_{Cj}(s)} \qquad (19b)$$

$$\mu_{Cj}(s) = 1+\delta_j F_j(0)\left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{s}{t}}\exp\left(\frac{\sigma_j^2 s(t-s)}{2t}\right) \qquad (19c)$$

$$\nu_{Cj}(s) = \delta_j^2 F_j^2(0)\left(\frac{F_j(t)}{F_j(0)}\right)^{\frac{2s}{t}}\left(\exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right)-1\right)\exp\left(\frac{\sigma_j^2 s(t-s)}{t}\right) \qquad (19d)$$

Proof. See Appendix A.

The accuracy and performance of these drift approximation methods are discussed in section IV.

II. THE LATTICE PROCEDURE IN THE LMM

There are two primary types of lattices for pricing financial products: tree lattices and grid lattices (or rectangular lattices or Markov chain lattices). The tree lattices, e.g., traditional binomial tree, assume that the underlying process has two possible outcomes at each stage. In contrast with the binomial tree lattice, the grid lattices (see Amin [1993], Gandhi-Hunt [1997], Martzoukos-Trigeorgis [2002], Hagan [2005], and Das [2011]) shown in Exhibit 1, which permit the underlying process to change by multiple states, are built in a rectangular finite difference grid (not to be confused with finite difference numerical methods for solving partial differential equations). The grid lattices are more realistic and convenient for the implementation of a Markov chain solution.

This article presents a grid lattice model for the LMM. To illustrate the lattice algorithm, we use a callable exotic as an example. Callable exotics are a class of interest rate derivatives that have Bermudan style provisions that allow for early exercise into various underlying interest rate products. In general, a callable exotic can be decomposed into an underlying instrument and an embedded Bermudan option.

We will simplify some of the definitions of the universe of instruments we will be dealing with for brevity. Assume the payoff of a generic underlying instrument is a stream of payments $Z_i = \delta_i \left[F_i(T_{i-1}) - C_i \right]$ for *i=1,…,N*, where C_i is the structured coupon. The callable exotic is a Bermudan style option to enter the underlying instrument on any of a sequence of notification dates $t_1^{ex}, t_2^{ex}, \ldots, t_M^{ex}$. For any notification date $t = t_j^{ex}$, we define a right-continuous mapping function $n(t)$ by $T_{n(t)-1} \le t < T_{n(t)}$. If the option is exercised at t, the reduced price of the underlying instrument, from the structured coupon payer's perspective, is given by

$$\tilde{I}(t) := \frac{I(t)}{P(t,T_N)} = \sum_{i=n(t)}^{N} E_t\left(\frac{Z_i}{P(T_i,T_N)}\right) = \sum_{i=n(t)}^{N} E_t\left(\frac{\delta_i\left(F_i(T_{i-1}) - C_i\right)}{P(T_i,T_N)}\right) \tag{20}$$

where the ratio $\tilde{I}(t)$ is usually called the reduced value of the underlying instrument or the reduced exercise value or the reduced intrinsic value.

Lattice approaches are ideal for pricing early exercise products, given their "backward-in-time" nature. Bermudan pricing is usually done by building a lattice to carry out a dynamic programming calculation via backward induction and is standard. The lattice model described below also uses backward induction but exploits the Gaussian structure to gain extra efficiencies.

First, we need to create the lattice. The random process we are going to model in the lattice is the LMM (12). Unlike traditional trees, we only position nodes at the determination dates (the payment and exercise dates). At each determination date, the continuous-time stochastic equation (12) shall be discretized into a discrete-time scheme. We have the discrete form of the forward rate as

$$F_k(t; y_{i,t}) = F_k(0)\exp\left(\mu_k(t, y_{i,t}) + \frac{\sigma_k^2}{2}t + \sigma_k y_{i,t}\right) \tag{21}$$

The zero-coupon bond (2) can be expressed in discrete form as

$$P(t,T_k; y_{i,t}) = P(t,T_{n(t)}; y_{i,t}) \prod_{j=n(t)}^{k} \frac{1}{1+\delta_j F_j(t; y_{i,t})} \tag{22}$$

We now have expressions for the forward rate (21) and discount bond (22), conditional on being in the state $y_{i,t}$ at time t, and from these we can perform valuation for the underlying instrument.

At the maturity date, the value of the underlying instrument is equal to the payoff, i.e.,

$$I(T_N, y_{i,T_N}) = Z_N(y_{i,T_N}) \qquad (23)$$

The underlying state process X_t in the LMM (11) is a Brownian motion. The transition probability density from state ($x_{i,t}$, t) to state ($x_{j,T}$, T) is given by

$$p(x_{i,t}, t; x_{j,T}, T) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(x_{j,T} - x_{i,t})^2}{2(T-t)} \right] \qquad (24)$$

Applying the variable substitution (8), equation (24) can be expressed as

$$p(y_{i,t}, t; y_{j,T}, T) = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left[-\frac{(y_{j,T} - y_{i,t} + \sigma_T T - \sigma_t t)^2}{2(T-t)} \right] \qquad (25)$$

Equation (20) can be further expressed as a conditional value on any state ($y_{i,t}$, t) as:

$$\frac{I(t; y_{i,t})}{P(t, T_N; y_{i,t})} = \sum_{j=n(t)}^{N} \frac{1}{\sqrt{2\pi(T_j - t)}} \int \frac{Z_j(y_{T_j})}{P(T_j, T_N; y_{T_j})} \exp\left[-\frac{(y_{T_j} - y_{i,t} + \sigma_{T_j} T_j - \sigma_t t)^2}{2(T_j - t)} \right] dy_{T_j} \qquad (26)$$

This is a convolution integral. Some fast integration algorithms, e.g., Cubic Spline Integration, Fast Fourier Transform (FFT), etc., can be used for optimization. We use the Trapezoidal Rule Integration in this paper for ease of illustration.

Next, we determine the option values in each final notification node. On the last exercise date, if we have not already exercised, the reduced option value in any state $y_{i,M}$ is given by

$$\frac{V(t_M^{ex}, y_{i,M})}{P(t_M^{ex}, T_N; y_{i,M})} = \max\left(\frac{I(t_M^{ex}; y_{i,M})}{P(t_M^{ex}, T_N; y_{i,M})}, 0 \right) \qquad (27)$$

Then, we conduct the backward induction process that is performed by iteratively rolling back a series of long jumps from the final exercise date t_M^{ex} across notification dates and exercise opportunities until we reach the valuation date. Assume that in the previous rollback step t_j^{ex}, we calculated the reduced option value: $V(t_j^{ex}, y_{i,j}) / P(t_j^{ex}, T_N; y_{i,j})$. Now, we go to t_{j-1}^{ex}. The reduced option value at t_{j-1}^{ex} is

$$\frac{V(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})} = \max\left\{\frac{I(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})}, \frac{V^c(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})}\right\} \tag{28a}$$

where the reduced continuation value is given by

$$\frac{V^c(t_{j-1}^{ex}, y_{i,j-1})}{P(t_{j-1}^{ex}, T_N; y_{i,j-1})} = \frac{1}{\sqrt{2\pi(t_j^{ex} - t_{j-1}^{ex})}} \int \frac{V(t_j^{ex}, y_j)}{P(t_j^{ex}, T_N; y_j)} \exp\left[-\frac{(y_j - y_{i,j-1} + \sigma_j t_j^{ex} - \sigma_{j-1} t_{j-1}^{ex})^2}{2(t_j^{ex} - t_{j-1}^{ex})}\right] dy_j \tag{28b}$$

We repeat the rollback procedure and eventually work our way through the first exercise date. Then the present value of the Bermudan option is found by a final integration given by

$$pv_{Bermudan}(0) = P(0, T_N) \frac{1}{\sqrt{2\pi t_1^{ex}}} \int \frac{V(t_1^{ex}, y_1)}{P(t_1^{ex}, T_N; y_1)} \exp\left[-\frac{\left(y_1 + \sigma_1 t_1^{ex}\right)^2}{2t_1^{ex}}\right] dy_1 \tag{29}$$

The present value or the price of the callable exotic from the coupon payer's perspective is:

$$pv_{payer}(0) = pv_{Bermudan}(0) - pv_{underly\_instrument}(0) \tag{30}$$

This framework can be used to price any interest rate products in the LMM setting and can be easily extended to the Swap Market Model (SMM).

III. CONCLUSION

We use the following techniques in our model: shifted forward measure, drift approximation, probability distribution structure exploitation, long jump, numerical integration, incomplete information handling, and calibration. Combining these techniques, the model achieves sufficient accuracy in relatively few time steps and discrete nodes, which makes it a very efficient method.

For ease of illustration, we present the lattice model based on the Trapezoidal Rule integration. A better but slightly more complicated solution is to spline the payoff functions. The cubic spline of the option payoffs can achieve higher accuracy, especially for Greeks calculations, and higher speed. Although cubic spline takes some time, the lattice will require much fewer nodes (23 ~ 28 nodes are good enough) and can perform a much faster integration. In general, the spline method can provide a speedup factor around 3 ~ 5 times.

We have implemented the lattice model to price a variety of interest rate exotics. The algorithm can always achieve a fast convergence rate. The accuracy, however, is a bit trickier, depending on many factors: drift approximation approaches, numerical integration schemes, volatility selections, and calibration, etc. Some work, such as calibration, is more of an art than a science.

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