# General Base Decimals With the p-series of Calculus Shows all $\zeta(n)$ Irrational

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**Abstract**

We give a new approach to the question of whether or not all greater than one, integer arguments of Zeta are irrational. Currently only $\zeta(2n)$ and $\zeta(3)$ are known to be irrational. We show that using the denominators of the terms of $\zeta(n) - 1 = z_n$ as decimal bases gives all rational numbers in (0,1) as single decimals, property one. We also show the partial sums of $z_n$ are not given by such single digits so using the denominators of the partial sum's terms as number bases, property two. Next, using integrals contracting upper and lower bounds for partial sum remainders of $z_n$ are generated. Assuming $z_n$ is rational, it is expressible as a single decimal using the denominator of a term of $z_n$ (property one) and eventually these bounds will consist of infinite decimals (property two) with their first decimal equal to this single decimal. But as no single decimal can be between two infinite decimals with the same first digit a contradiction is derived and all $z_n$ are proven irrational.

# Introduction

Apery's $\zeta(3)$ is irrational proof [1] and its simplifications [3, 9] are the only proofs that a specific odd argument for $\zeta(n)$ is irrational.

The irrationality of even arguments of zeta are a consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1}\frac{2^{2n-1}}{(2n!)}B_{2n}\pi^{2n}. \tag{1}$$

Euler's formula is a rational multiple of a power of $\pi$. As $\pi$ is a known transcendental number all its powers are irrational. It follows that $\zeta(2n)$ is irrational.

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs. He replaced Apery's mysterious recursive relationships with multiple integrals. See Poorten [10] for the history of Apery's proof; Havil [5] gives an overview of Apery's ideas and attempts to demystify them. Huylebrouck's [6] gives an historical context for the main technique used by Beukers.

There have been attempts to generalize Apery's one odd success. Rivoal showed that there are an infinite number of odd $n$ such that $\zeta(n)$ is irrational [11] and Zudilin showed at least one of the cases 5,7,9, 11 is irrational [15]. These marginal successes suggest a new approach might be worth exploring.

Let

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

We show that every rational number in $(0,1)$ can be written as a single decimal using the denominators of a term in $z_n$ as a number base: Lemma 1. We also show that partial sums $s_k^n$ can't be expressed with a single decimal using the denominator of one of its terms as a number base: Corollary 1. These two properties combined with lower and upper limits for $z_n$, consisting of partials sums plus a fraction, yield a proof that all $z_n$ are irrational.

## Properties of $z_n$

We define a decimal set.

**Definition 1.** *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n-1)/j^n\} = \{.1, \ldots, .(j^n-1)\} \text{ base } j^n.$$

*That is $d_{j^n}$ consists of all single decimals greater than 0 and less than 1 in* *base $j^n$. The decimal set for $j^n$ is*

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

**Definition 2.**

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

**Example 1.**

$$\Xi_4^2 = \bigcup_{j=2}^{4} D_{j^2} = D_4 \cup D_9 \cup D_{16}$$

$$= \{.1_4, .2_4, .3_4, .1_9, \ldots, .8_9, .1_{16}, .2_{16}, \ldots, .(15)_{16}\},$$

where subscripts give the base of the single decimal.

The union of all decimal sets for a given $z_n$ *covers* all rational numbers in $(0, 1)$.

**Lemma 1.**

$$\bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0, 1)$$

*Proof.* Every rational $a/b \in (0, 1)$ is included in a $d_{b^n}$ and hence in some $D_{r^n}$ with $r \leq b$. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0, 1)$, $ab^{n-1} < b^n$ and so $a/b \in d_{b^n}$. $\square$

Next we show $s_k^n \notin \Xi_k^n$; that is: we show that partial sums of $z_n$ can't be expressed as a single decimal using number bases given by the denominators of the partial's terms; the partials *escape* their terms. The idea of the following Lemmas and Theorem is that every other term has an even denominator and this forces the reduced fraction giving the partial sum to have a power of 2 in the denominator with exponent the index of $z_n$ (Lemma 2). As denominators are powers of all numbers, Bertrand's postulate implies that a power of a prime will occur in the second half of such numbers (Lemma 3 and 4). The power of this prime will also be the index of $z_n$. As twice something greater than half of something is greater than the something, it will be the case that the reduced fraction for partials will exceed the greatest denominator in the partial. This means the reduced fraction giving $s_k^n$ is not in $\Xi_k^n$.

The central technique used in Lemmas 2 and 3 were taken from Hurst's solution [7] of Chapter 1, Problem 30 in Apostol's *Introduction to Analytic Number Theory* [2].

**Lemma 2.** *If $s_k^n = r/s$ with $r/s$ a reduced fraction, then $2^n$ divides $s$.*

*Proof.* The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, $a$; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, $na$. Also $k!$ will have a powers of 2 divisor with exponent $b$; and $(k!)^n$ will have a greatest power of 2 exponent of $nb$. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of $2^{nb-na}$ in their prime factorization, we have the numerator of (2) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and $A$ is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves $2^{na}$ as a factor in the denominator with no powers of 2 in the numerator, as needed. □

**Lemma 3.** *If $s_k^n = r/s$ with $r/s$ a reduced fraction and $p$ is a prime such that $k > p > k/2$, then $p^n$ divides $s$.*

*Proof.* First note that $(k, p) = 1$. If $p|k$ then there would have to exist $r$ such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{3}$$

As $(k,p) = 1$, only the term $(k!)^n/p^n$ will not have $p$ in it. The sum of all such terms will not be divisible by $p$, otherwise $p$ would divide $(k!)^n/p^n$. As $p < k$, $p^n$ divides $(k!)^n$, the denominator of $r/s$, as needed. □

**Lemma 4.** *For any $k \geq 2$, there exists a prime $p$ such that $k < p < 2k$.*

*Proof.* This is Bertrand's postulate [4]. □

**Theorem 1.** *If $s_k^n = \frac{r}{s}$, with $r/s$ reduced, then $s > k^n$.*

*Proof.* Using Lemma 4, for even $k$, we are assured that there exists a prime $p$ such that $k > p > k/2$. If $k$ is odd, $k-1$ is even and we are assured of the existence of prime $p$ such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies $k$ is even, a contradiction.

For both odd and even $k$, using Lemma 4, we have assurance of the existence of a $p$ that satisfies Lemma 3. Using Lemmas 2 and 3, we have $2^n p^n$ divides the denominator of $r/s$ and as $2^n p^n > k^n$, the proof is complete. □

**Corollary 1.**

$$s_k^n \notin \Xi_k^n$$

*Proof.* If $r/s = s_k^n$ is reduced the smallest base that can give $r/s$ as a single fraction is $s$: $.r_s = r/s$. But $s > k^n$ and $k^n$ is the largest base in $\Xi_k^n$. □

**Corollary 2.** *If $j \geq k$ expressing $s_j^n$ using any of the bases $\{2^n, 3^n, \ldots, k^n\}$ will require an infinite decimal.*

*Proof.* The proof of Theorem 1 showed that the denominator of $s_k^n$ had factors 2 and $p$ where $p$ is a prime that occurs only once in $\{2, 3, \ldots, k\}$; it's the prime given by Bertrand's postulate. As any base $b \in \{2^n, 3^n, \ldots, k^n\}$ will not have both prime factors, $s_k^n$ will be a mixed or pure repeating decimal in base $b$.

For $s_j^n$, $j > k$, $s_j^n$ will have factors 2 and the same or greater prime than this $p$ used for $s_k^n$, so it will also be a mixed or pure repeating decimal in this same set of bases.

See Hardy's Chapter 9 for a tutorial on mixed and pure repeating decimals in general bases [4]. □

## Decimals and Series

The partial sums of any series that converges to a rational number $a/b$ will have partials with fixed digits of the form $.(a-1)\overline{(b-1)}_r$, where $r$ is the number of repetitions of the digit $b-1$.

**Example 2.** The telescoping series given by

$$s(tele) = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{j=2}^{\infty} \frac{1}{j} - \frac{1}{j+1} = \sum_{j=2}^{\infty} \frac{1}{j(j+1)} = \frac{1}{2}.$$

This series converges to .5 in base 10. Reasonably close upper and lower bounds must have different first digits. For example, we can make a sequence of contracting intervals with the following pattern

$$.49 < .499 < \cdots < .4\overline{9}_r < s(tele) < .5\overline{0}_r 1 < \cdots < .501 < .51.$$

Any base that gives 1/2 as a single digit will have such a pattern: different first digits in upper and lower bounds.

**Example 3.** The telescopic series of the previous example will have partial sums with fixed digits in base 3 of the form $.\overline{1}_r$. This follows as $1/2 = .\overline{1}_3$. In this case any reasonably close lower and upper bounds for the series will have the same first digit. Consider

$$.1\underline{\epsilon} < \{.1_3, .2_3\} < .1\overline{\epsilon},$$

where the epsilons indicate additional digits that form a strictly lower $\underline{\epsilon}$ and upper bound $\overline{\epsilon}$ for the convergence point.

Both single digits in base 3 (shown in the middle) are impossible. The first, $.1_3$ is smaller than the lower bound on the left and the second is greater than the upper bound on the right.

If a set of bases that covers all pertinent rational numbers led to such constraining intervals, we could conclude that the convergence point is irrational.

The next examples show how the properties given in Lemma 1 and Corollary 1 can be combined with lower and upper bounds for $z_2$ to eliminate all elements of $\Xi_3^2$ as potential convergence points.

**Example 4.** Suppose we have upper and lower bounds for $z_2$ given by

$$s_k^2 + \frac{1}{k+1} < z_2 < s_k^2 + \frac{1}{k}. \tag{4}$$

Then for $s_3^2$ we have $1/4 + 1/9 + 1/4 < z_2 < 1/4 + 1/9 + 1/3$, for example. That is $11/18 < z_2 < 25/36$ or $.2\overline{130}_4 < z_2 < .2\overline{301}_4$. We can infer that $z_2$ is not a single digit in base 4. Each of the elements in $d_4$ will violate the lower or upper bounds.

**Example 5.** To similarly eliminate all single digits in base 9, we need to increase $k$. Although $s_3^2$ and $s_4^2$ are not single decimals in $\Xi_3^2$ per Corollary 2, the lower and upper bounds using (4) have different first digits. With $s_5^2 = 1669/3600$ we have $s_5^2 + 1/6 = 0.5\overline{6042240577}_9$ and $s_5^2 + 1/5 = 0.5\overline{8668513314}_9$; the same first digit. Once again we've eliminated potential convergence points; $z_2$ can't be a single digit in base 9, in $d_9$.

We could have chosen any base $k^2$. We would need to consider partials $s_j^2$ with $j \geq k$ to ensure $s_j^2$ requires an infinite decimal in base $k^2$ per Corollary 2. Then we might have to increase $j$ again, say $j > k' > k$ so as to insure the accuracy of the lower and upper limit forces each to have the same first digit. These ideas can be generalized into a proof that all $z_n$ are irrational.

# Proof

Two additional lemmas are needed. The first uses a formula from a calculus text [13]. Such texts use what is termed the p-series to determine via the comparison test the convergence or divergence of a series. These p-series are just positive rational arguments of the zeta function.

As an application of the integral test upper and lower bounds of a series are found by forming a continuous function with a given series and integrating it in such a way that a pair of integrals under and over estimate the tail of the series. The p-series, for us $z_n$ can be thus approximated using the function $f(x) = x^{-n}$; this function is used in (6) to form (7).

**Lemma 5.**

$$s_k^n + \frac{1}{(n-1)(k+1)^{n-1}} < z_n < s_k^n + \frac{1}{(n-1)k^{n-1}} \tag{5}$$

*Proof.* Applying

$$s_n + \int_{n+1}^{\infty} f(x)\ \mathrm{dx}\ < s < s_n + \int_{n}^{\infty} f(x)\ \mathrm{dx}\ , \tag{6}$$

to $s_k^n$ and $z_n$, we have

$$\int_{k+1}^{\infty} \frac{1}{x^n}\ \mathrm{dx} = \lim_{b\to\infty} \left[\frac{x^{-n+1}}{-n+1}\right]_{k+1}^{b} = \frac{1}{(n-1)(k+1)^{n-1}} \tag{7}$$

and

$$\int_{k}^{\infty} \frac{1}{x^n}\ \mathrm{dx} = \lim_{b\to\infty} \left[\frac{x^{-n+1}}{-n+1}\right]_{k}^{b} = \frac{1}{(n-1)k^{n-1}}, \tag{8}$$

giving (5). □

**Definition 3.** *Let*

$$\overline{\epsilon(n,k)} = \frac{1}{(n-1)k^{n-1}} \ \textit{ and } \underline{\epsilon(n,k)} = \frac{1}{(n-1)(k+1)^{n-1}}.$$

**Definition 4.** *Let $L_k^n = s_k^n + \underline{\epsilon(n,k)}$ be the lower bound and $U_k^n = s_k^n + \overline{\epsilon(n,k)}$ be the upper bound as given in (5).*

**Lemma 6.** *For any base $j^n$, for large enough $k$, $L_k^n$ and $U_k^n$ will have the same first digit.*

*Proof.* As

$$U_k^n - s_k^n = \overline{\epsilon(n,k)}$$

and

$$\lim_{k\to\infty} \overline{\epsilon(n,k)} = 0,$$

the first decimal for $s_k^n$ and $U_k^n$ are the same for sufficiently large $k$. This follows as $s_k^n$ is a mixed or pure repeating decimal and hence its first decimal is non-ambiguous.

As the first decimal of $U_k^n$ is unambiguous and

$$U_k^n - L_k^n = \overline{\epsilon(n,k)} - \underline{\epsilon(n,k)} \tag{9}$$

with

$$\lim_{k\to\infty} \overline{\epsilon(n,k)} - \underline{\epsilon(n,k)} = 0, \tag{10}$$

the first decimal of $L_k^n$ must be the same as that of $U_k^n$, for large enough $k$. □

**Theorem 2.** *$z_n$ is irrational.*

*Proof.* Suppose $z_n$ is rational, then, using Lemma 1, there exists a first $k$ such that $z_n \in \Xi_k^n$.

Per Lemma 6, the first digit in the decimal expansion of $L_j^n$ and $U_j^n$ becomes fixed and the same in base $k^n$. As there is no single decimal in base $k^n$ between $L_j^n$ and $U_j^n$, we have a contradiction. $\square$

# Conclusion

The use of decimals to establish the irrationality of series is not without precedent. Hardy uses decimals to show the juxtaposition of prime numbers in base 10, .2357111317... is irrational. He gives two proofs. One proof uses Bertrand's postulate in a way similar to our use [4].

Finally, this result surviving public scrutiny, there is the possibility of its relevance to the premier number theory open problem: the Riemann hypotheses. I have some hope that the equivalent of number bases (plural) in the complex number system might allow the same exclusions used here (irrational not rational) to carry over to a zero versus not a zero. There are Gaussian integers and Gaussian primes; might there be forms of number bases that inform us of the location of zeros for the zeta function.

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