On The Collatz Conjecture.

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Abstract.

This article proves that the *Collatz Conjecture* is valid for all positive integers. The main formula (and rules) for the *Collatz Conjecture* is as follows:

$$f(n) = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ 3n+1 & if\ n\ is\ odd \end{cases}$$

Keywords: Logic; Series/Sequences; Dynamical Systems; Probability; Functional Analysis; Stochastic Processes; Number Theory.

1. Existing Literature.

The *Collatz Conjecture* is described in Carletti & Fanelli (2018), Barina (2020), Lagarias (2003, 2009) and Thomas (2017). On Dynamical Systems in the Context of *Collatz Conjecture*, see: Bourgain (1994) and Wirsching (1998). On related topics, see: Chamberland (2010), Crandall (1978), Kontorovich & Miller (2005), Kontorovich & Sinai (2002), Krasikov & Lagarias (2003), Lagarias (1985), Lagarias & Soundararajan (2006), and Oliveira e Silva (2010). On the "3x+1" Problem and stochastic models, see: Lagarias & Weiss (1992), Kontorovich & Lagarias (2010) and Sinai (2003). On Analytic Number Theory, see: Niven (1951), Steuding (2002), Tenenbaum (1995) and Everett (1977).

Barina (2020) and Oliveira e Silva (2010) attempted to empirically verify the Collatz Conjecture. Barina (2020) noted that as of 2020, the *Collatz Conjecture* had been verified by computer for all positive integers up to 10^{20}.

Many proffered solutions of the *Collatz Conjecture* are heavily or partly based on Modulo Arithmetic, but Nwogugu (revised 2020) illustrated why *Modulo Arithmetic* can be very inaccurate in Number Theory.

Several researchers have noted that in any *Collatz Orbit*, once the (4, 2, 1) sequence is reached, it (4, 2, 1) repeats itself perpetually. That is because:

i) Where n=4, then n/2=2, and 2/2 =1.
ii) Where n=2, then 2/2 = 1.
iii) Where n=1, the next number in the sequence is defined by (3n+1) which is 4, and 4/2 = 2, and 2/2=1.

On logarithms and the use of Logarithmic-Density and Natural-Density within the context of *Collatz Conjecture*, see: Baker (1966), Terras (1979), Tao (2022, 2016) and Korec (1994). Some proffered solutions to the *Collatz Conjecture* are also partly based on finding the Natural-Density and or Logarithmic-Density of the counter-factual to the Collatz Conjecture, which is an inappropriate method. If *X* is a set of positive integers, and X ϵ N, X has a Natural-Density (β) if the proportion of elements of *X* in (1, *N*) converges to β as *N* tends to infinity. A counting function *a*(N) is the number of elements of *X* that are less than or equal to *N*, and β implies that $a(N)/N \rightarrow \beta$ as $N \rightarrow \infty$; and if β exists, then $0 \leq \beta \leq 1$. The *Davenport–Erdős theorem* states that for the set of multiples of an integer sequence, if the Natural-Density exists, then its equal to the Logarithmic-Density. Theoretically and practically and in the context of Tao (2022) and the *Collatz Conjecture*, the Natural-Density and the Logarithmic Density are akin to probabilities (that measure whether the counterfact/counter-example of the *Collatz Conjecture* can occur), but they cannot be applied to correctly prove the *Collatz Conjecture* (partly because of reasons stated herein and below).

2. The Proofs.

Theorem-1: All Valid *Collatz Orbits* For All Integers Greater Than 2 (two) Except 3 (Three) Include The Declining Sequence (16, 8, 4, 2, 1).
Proof:
There isn't any integer that is divided by two to result in 1 except 2. There isn't any integer that is divided by two to result in 2 except 4. There isn't any integer that is divided by two to result in 4 except 8.

Let:
n/2 = "Rule1".
(3n+1) = "Rule2".
Collatz Process = the process and results of repeatedly applying Rule1 and or Rule2 in any order/sequence, in an attempt to derive or reach the number 1 (one).
"n#" = any odd-number that is obtained by applying *Rule1* at any stage of the *Collatz Process*.
"*n#-Orbit*" = the sub-set of all the n# that are in a *Collatz Orbit*.
"Lower-n#" = these are smaller n# (odd-numbers in a Collatz Orbit) that are typically less than 500.

The smallest *n* for which (3n+1) is equal to an even number (4) is one. After that, the next smallest n for which (3n+1) is equal to an even number (10) is 3; and the next smallest n for which (3n+1) is equal to an even number (16) is 5. Since there cannot be a n# that is less than 5 (five) for which the sequential application of Rule2 and then Rule1 will result in 8, 4 or 2, then all valid *Collatz Orbits* for all positive integers greater than four must include the declining sequence (16,8,4,2,1).

If the subject (first) integer is:
i) 1, then the Collatz Orbit is: 1, 4, 2,1,4, 2, 1,4,2,1……..
ii) 2, then the Collatz Orbit is: 2, 1, 4, 2, 1, 4………..
iii) 3, then the Collatz Orbit is: 3, 10, 5, 16, 8, 4, 2, 1, 4,2,1,…..
iv) 4, then the Collatz Orbit is: 4, 2, 1, 4, 2, 1, ……… ■

Theorem-2: All Valid "n#-Orbits" For 3 (Three) And Positive Integers Greater Than Four Contain The Number 5 (Five); And All Collatz Orbits For 3 (Three) And Positive Integers Greater Than Four Contain The Number 5 (Five).
Proof:
As proved above, all valid *Collatz Orbits* for all positive integers that are greater than 4 (four) contain the declining sequence (16,8,4,2,1). In all valid *Collatz Orbits* of 3 (three) and all integers that are greater than four (4), the resulting Lower-n# includes at least one odd-number that complies with all the following conditions, and if there is only one such odd-number in the *Collatz Orbit*, then it's a "*Critical Data-Point*":
i) (3n+1) * 0.5= 8; (or n=5)
ii) (3n+1) * 0.5 * 0.5 = 4; (or n=5)
iii) (3n+1) * 0.5 * 0.5 * 0.5 = 2; (or n=5)
iv) (3n+1) * 0.5 * 0.5 * 0.5 * 0.5 = 1; (or n=5)

That confirms that all Collatz Orbits of 3 (three) and all positive integers that are greater than four (4) include the integer 5 (five), and all their valid "n#-Orbits" include the number 5 (five); and 5 (five) is a *Critical Data-point*. ■

Theorem-3: In The *Collatz Orbit* For Any Positive Integer, The Number Of Even-Numbers Exceeds The Number of Odd-Numbers; And The Number Of *Rule-1 Procedures* Exceeds The Number Of *Rule2 Procedures*; and For Any Set Of Contiguous/Sequential Numbers In Any Collatz Orbit, The Average Ratio

Of *Rule2 Procedures* To *Rule1 Procedures* Is A Minimum Of Between 1:1 And 1:5; And The Longest-Chain Of A *Rule1 Procedure* Contains At Least Five (5) Numbers.

***Proof*:**

In each *Collatz Process*, there cannot be any two sequential *Rule2 Procedures*, and thus the maximum number of sequential *Rule2 Procedures* (the "Longest-Chain") is one. However, the minimum of the maximum number of sequential *Rule1 Procedures* (the "Longest-Chain") is five (5) numbers (the Longest-Chain is at least five numbers). For even-numbers in a *Collatz* Orbit that end in:

i) 2 (such as 12, 22, 32, 42, 82, etc.), the resulting numbers (in the "Longest-Chain") when divided by 2 will contain the following last digits: 2, 6, 3, 1.

ii) 4 (such as 24, 14, 44, 64, etc.), the resulting numbers in the "*Longest-Chain*" when divided by 2 will contain the following last digits: 4, 2, 1.

iii) 6 (such as 16, 36, 56, 96, etc.), the resulting numbers in the "*Longest-Chain*" when divided by 2 will have the following last digits: 6, 8, 4, 2, 1.

iv) 8 (such as 18, 28, 48, etc.), the resulting "*Longest*-Chain" when divided by 2 will have the following last digits: 8, 4, 2, 1.

v) 0 (such as 360, 160, 80, 40, etc.), the resulting "*Longest-Chain*" when divided by 2 will have at least six numbers.

Thus, the Longest-Chain of *Rule1 Procedures* contains at least five integers. For any set of contiguous/sequential numbers in any *Collatz Orbit*, the average ratio of *Rule2 Procedures* to *Rule1 Procedures* is at least between 1:1 and 1:5, and the average probability of occurence of *Rule2 Procedure* is a maximum of 50%, while the average probability of occurence of *Rule1 Procedure* is at least 50%. The range-of-probabilities (instead of a conditional probability) is relevant here because the *Collatz Conjecture* and its rule imposes unusual conditions (that create non-uniform instances of sequences/series in each *Collatz Orbit* that cannot be readily or verifiably quantified by a single probability), such as the following:

i) There can never be two sequential *Rule2 Procedures*.

ii) There must be a *Rule1 Procedure* immediately after each *Rule2 Procedure*.

iii) A *Rule1 Procedure* that results in an odd-number must be immediately followed by a *Rule2 Procedure*.

iv) *Rule1 Procedure* produces both even and odd numbers, while *Rule2 Procedure* produces only even numbers; and the process continues until the series converges to 1 (one).

Thus, in any applicable *Collatz Orbit*, the absolute number of even-numbers exceeds odd-numbers. Since *Rule1 Procedure* even-numbers are smaller than their "inputs", the existence of a majority of Rule1 even-numbers increases the probability and speed of convergence of the *Collatz Orbit* series to 1 (one). ■

Theorem-4: For All Positive Integers, The *Collatz Conjecture* Is Correct.

***Proof*:**

See *Theorem-3* herein and above. As proved above, all valid *Collatz Orbits* for all number that are greater than 4 (four) contain the declining sequence (16, 8, 4, 2, 1). A "*Rule1-Rule1 Procedure*" refers to sequential application of Rule1 twice as part of a *Collatz Process*. A "*Rule1-Rule2 Procedure*" refers to sequential application of Rule1 and then Rule2 as part of a *Collatz Process*. A "*Rule2-Rule1 Procedure*" refers to sequential application of Rule2 and then Rule1 as part of a *Collatz Process*. *Rule2Rule2 Procedures* are impossible.

Any positive integer greater than four and of any size that is subjected to a Collatz Process will eventually result in Lower-n# that eventually declines to the number 5 (as proved above, the number 5 is included in the *n#-Orbits* of all integers that are greater than four). In a *Collatz Process*, the number 5 (five) is the "*Critical Data-Point*" that automatically triggers the number 16 (sixteen) which in turn triggers the (16,8,4,2,1) sequence (which proves that the *Collatz Conjecture* is correct). The *Collatz Process* is a "*Reduction Procedure*" that produces "*Declining-Numbers Series*" that converge to 1 (one). That is, for any integer that is greater than five, the Collatz Orbit numbers will eventually begin to decline in magnitude until they reach the number five (5), upon which they enter the (16,8,4,2,1) sequence. The term *Declining Number Series* means that:

i) Each *Rule1-Rule1 Procedure* and each *Rule1 Procedure* results in a smaller integer that is part of the series.

ii) Each *Rule2 Procedure* always produces an even number, which when divided by two (Rule1), produces a smaller integer.

iii) In the Collatz Process for 3 (three) and for any integer that is greater than four (4), the absolute number of *Rule1-Rule1 Procedures* is always greater than the number of *Rule2-Rule1 Procedures* ("Inequality-1"). This phenomenon and inequality is attributable to the following:

1) *Rule1 Procedure* produces both even and odd numbers, while *Rule2 Procedure* produces only even numbers; and if the *Rule1 Procedure* produces an odd-number number, the Collatz Process must switch to *Rule2 Procedure*, and the process continues until the series converges to 1 (one). Thus, in any applicable Collatz Process, the absolute number of *Rule1-Rule1 Procedures* is always greater than the number of *Rule2-Rule1 Procedures* ("Inequality-1"). That increases the probability and speed of convergence of the series to 1 (one).

2) *Rule1 Procedure* reduces the magnitude of numbers in the Collatz Orbit series, whereas *Rule2 Procedure* increases the magnitude of numbers in the Series. The series can never contain zero or a negative number or a fraction, and given the foregoing, the Collatz Orbit always converges to 1 (one).

3) *Rule1 Procedure* is applicable only to integers that end with 0, 2, 4, 6, and 8 (the "Rule1-evens"), whereas *Rule2 Procedure* is applicable only to integers that end with 1,3,5,7, and 9 (the "Rule2-odds"); but the combined application of both Rule1 and Rule2 always ensures that: a) the number of *Rule-1 Procedures* exceeds *Rule2 Procedures*; b) there can never be *Rule2Rule2 Procedures*; c) *Rule1Rule2 Procedures* are less than *Rule1Rule1 Procedures* and *Rule1Rule2 Procedures*; d) a majority of the resulting numbers in the Series are *Rule1-evens*.

4) Given the foregoing and the formulas for Rule1 and Rule2, *Rule1-evens* are more likely to occur in any Collatz Orbit than *Rule2-odds*. That increases the probability and speed of convergence of the series to 1 (one).

iv) In the *Collatz Process* for 3 (three) and for any integer that is greater than four, the absolute number of *Rule1 Procedures* is always greater than the number of *Rule2 Procedures* ("Inequality-2"), and that results in "*Declining-Numbers Series*" in the *Collatz Process* series until the (8,4,2,1) or (5,16,8,4,2,1) or (4,2,1) sequence is reached. This phenomenon and inequality are attributable to the following factors:

1) *Rule1 Procedure* produces both even and odd numbers, while *Rule2 Procedure* produces only even numbers; and if the *Rule1 Procedure* produces an odd-number number, the Collatz Process must switch to *Rule2 Procedure*, and the process continues until the series converges to 1 (one). Thus, in any applicable Collatz Process, the absolute number of *Rule1 Procedures* is always greater than the number of *Rule2 Procedures* ("Inequality-2"). That increases the probability and speed of convergence of the series to 1 (one).

2) *Rule1 Procedure* reduces the magnitude of numbers in the Collatz Orbit series, whereas *Rule2 Procedure* increases the magnitude of numbers in the Series. The series can never contain zero or a negative number or a fraction, and given the foregoing, the Collatz Orbit always converges to 1 (one).

3) *Rule1 Procedure* is applicable only to integers that end with 0, 2, 4, 6, and 8 (the "Rule1-evens"), whereas *Rule2 Procedure* is applicable only to integers that end with 1,3,5,7, and 9 (the "Rule2-odds"). As mentioned above, the combined application of both Rule1 and Rule2 but the combined application of both Rule1 and Rule2 always ensures that: a) the number of *Rule-1 Procedures* exceeds *Rule2 Procedures*; b) there can never be *Rule2Rule2 Procedures*; c) *Rule1Rule2 Procedures* are less than *Rule1Rule1 Procedures* and *Rule1Rule2 Procedures*; d) a majority of the resulting numbers in the Series are *Rule1-evens*.

4) Given the foregoing and the formulas for Rule1 and Rule2, Rule1-evens are more likely to occur in any Collatz Orbit than Rule2-odds. That increases the probability and speed of convergence of the series to 1 (one).

Thus, every Collatz Orbit will eventually converge to 1 (one) regardless of the number of digits in the base-number (first number). ■

Theorem-5*: Natural Density Or Logarithmic Density* Cannot Be Used To Solve The *Collatz Conjecture*.

Proof: The following are the conditions for existence of *Natural Density* and *Logarithmic Density* (and all such conditions must exist simultaneously), and why such conditions don't and or can't exist:

i) *All elements/members of the set X (the Collatz Orbit) must be known at the outset* – in the Collatz Conjecture context, that is impossible, and X can be infinite. Also, all elements/members of the set (1, N) must be known at the outset.

ii) The implied (used) probability distribution has little relationship with the facts/circumstances of the Collatz Conjecture.

iii) The main distorting factors are that:

1) In this circumstance, the Natural Density or Logarithmic Density measures apply only to the counterfactual of the Collatz Conjecture – and it's not clear that such conjecture can be completely proved by using the probability of its counterfactual;

2) The Natural Density or Logarithmic Density measures are substantially dependent on the natural-log scale which may not fit the circumstances;

3) The Natural Density or Logarithmic Density measures are partly based on Modulo Arithmetic which has been shown to be inaccurate - see Nwogugu (revised 2020);

iv) Given the theorems herein and above, any set in a Collatz Orbit will have a Lower density and an Upper Density that will be different; and thus the Collatz Orbit doesn't have any density.

v) The Natural Density or Logarithmic Density strictly refers to how sparse the numbers in the Collatz Orbit are (Low Density means the numbers are farther apart, while High Density means the numbers are closer together). Given the theorems herein and above, the Natural Density or Logarithmic Density of the counterfactual of the Collatz Conjecture doesn't have any relationship to whether or not the Collatz Conjecture is valid. That is, in each Collatz Orbit, there is typically more even-integers than odd-integers, and the probability of a *Rule1-Procedure* always equals or exceeds the Probability of a *Rule2 Procedure*. For these same reasons, the Collatz Orbit doesn't have any limits in all instances.

vi) Its not clear that all Collatz Orbits have limits, where there is no limit, there can't be a Natural- Density or Logarithmic-Density.

vii) For the Natural-Density or Logarithmic Density to be correctly used to prove the *Collatz Conjecture*, each such density needs to have a value of one (1) for each Collatz Orbit of each Positive Integer, and that's highly improbable.

viii) As mentioned herein and above, the *Collatz Conjecture* and its rule imposes unusual conditions (that create non-uniform instances of sequences/series in each *Collatz Orbit* that cannot be readily or verifiably quantified by a single probability), such as the following:

1) There can never be two sequential *Rule2 Procedures*.

2) There must be a *Rule1 Procedure* immediately after each *Rule2 Procedure*.

3) A *Rule1 Procedure* that results in an odd-number must be immediately followed by a *Rule2 Procedure*.

4) *Rule1 Procedure* produces both even and odd numbers, while *Rule2 Procedure* produces only even numbers; and the process continues until the series converges to 1 (one).

■

3. Computer Codes For Verifying The *Collatz Conjecture*.

A simple Python Code for verifying the Collatz Conjecture for Positive Integers up to at least 10^{224} is as follows:

```
Int n
Int a
CollatzGroup = [n]

For (n in CollatzGroup; n=1; n <=
(1000000000000000000000000000000000000000000000000000000000000000000000000000000000000 **
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
```

```
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000)):
    a = n
    While a <=
    (1000000000000000000000000000000000000000000000000000000000000000000000000000000000000 **
    10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000):
        if(a % 2 == 0):
            a == (a/2)
        else(a == ((3*a)+1))
        if(a==1):
            break
    if(n ==
    (1000000000000000000000000000000000000000000000000000000000000000000000000000000000000 **
    10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000)):
            print("CollatzGroup Phase-3 verification completed")
        elif(n == (100000000000000000000000000000000000000000000000000000000000000000000 **
        10000000000000000000000000000000000000000000000000000000000000000000000000000000000**
        100000000000000000000000000000000000000000000000000000000000000000000000000000000000**
        100000000000000000000000000000000000000000000000000000000000000000000000000000000000)):
            print("CollatzGroup Phase-2 verification completed")
        elif(n == (100000000000000000000000000000000000000000000000000000000000000000000 **
        1000000000000000000000000000000000000000000000000000000000)):
            print("CollatzGroup Phase-1 verification completed")
    n += 1
```

As mentioned herein and above, once the Collatz Orbit sequence reaches five, the (16,8, 4,2,1) sequence automatically starts and converges the Collatz Orbit to one (1). A simple Java Code for verifying both the *Collatz Conjecture* (for Positive Integers up to at least 10^{224}) and the fact the number 5 (five) must be in every Collatz Orbit for all Positive Integers that are greater than four, is as follows (and the count begins at six because the Collatz Orbit for five contains five).

```
Int n;
Int a;

For (n=i; i=6; n <=
(1000000000000000000000000000000000000000000000000000000000000000000000000000000000000 **
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000)):
    For (a=i; i<=n; n <=
    (1000000000000000000000000000000000000000000000000000000000000000000000000000000000000 **
    10000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
```

```
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
100000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
100000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)):
    if(a % 2 == 0):
        a == (a/2);
    else(a == ((3*a)+1));
    if(a==5):
        break;
if(n ==
(100000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000 **
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
100000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
1000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
100000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
100000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)):
        system.out.print("CollatzGroup Phase-3 verification completed");
    else if(n == (100000000000000000000000000000000000000000000000000000000000000000000000000 **
    1000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000**
    10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000)):
        system.out.print("CollatzGroup Phase-2 verification completed");
    else if(n == (100000000000000000000000000000000000000000000000000000000000000000000000000 **
    100000000000000000000000000000000000000000000000000000000000000)):
        system.out.print("CollatzGroup Phase-1 verification completed");
n += 1;
```

4. Conclusion.

The *Collatz Conjecture* is valid for all positive integers.

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5. Bibliography.

Baker, A. (1966). Linear forms in the logarithms of algebraic numbers. *Mathematika: A Journal of Pure and Applied Mathematics*, 13, 204–216.

Barina, D. (2020). Convergence verification of the Collatz problem. *The Journal of Supercomputing*, ____.

Bourgain, J. (1994). Periodic nonlinear Schrodinger equation and invariant measures. *Comm. Math. Phys*., 166, 1–26.

Carletti, T. & Fanelli, D. (2018). Quantifying the degree of average contraction of Collatz orbits. *Boll. Unione Mat. Ital*., 11, 445–468.

Chamberland, M. (2010). A 3x+ 1 survey: number theory and dynamical systems. Chapter in: "The ultimate challenge: the 3x+ 1 problem" (pp. 57–78, Amer. Math. Soc., Providence, RI, 2010).

Crandall, R. (1978). On the '3x+ 1' problem. *Mathematics Of Computation*, 32, 1281–1292.

Everett, C. J. (1977). Iteration of the number-theoretic function f(2n)=n , f\left(2n+1\right)=3n+2 , Adv. Math., 25(1), 42–45.

Korec, I. (1994). A density estimate for the 3x+ 1problem. *Math. Slovaca*, 44(1), 85–89.

Kontorovich, A. & Lagarias, J. (2010). *Stochastic models for the 3x+ 1 and 5x+1 problems and related problems*. Chapter in: "*The ultimate challenge: the 3x+ 1 problem*" (pp. 131–188, Amer. Math. Soc., Providence, RI, USA; 2010).

Kontorovich, A. & Miller, S. (2005). Benford's law, values of L-functions and the 3x+ 1 problem. *Acta Arithmetica*, 120(3), 269–297.

Kontorovich, A. & Sinai, Y. (2002). Structure theorem for (d,g,h)-maps. *Bull. Braz. Math. Soc*., 33(2), 213–224.

Krasikov, I. & Lagarias, J. (2003). Bounds for the 3x+1 problem using difference inequalities. *Acta Arith*., 109, 237–258.

Lagarias, J. (1985). The 3x+1 problem and its generalizations. *Amer. Math. Monthly*, 92(1), 3–23.

Lagarias, J. & Soundararajan, K. (2006). Benford's law for the 3x+1 function. *Journal Of The London Math. Society*, 74(2), 289–303.

Lagarias, J. & Weiss, A. (1992). The 3x+ 1 problem: two stochastic models. *Annals Of Applied Probability*, 2(1), 229–261.

Niven, I. (1951). "The asymptotic density of sequences". *Bulletin of the American Mathematical Society*, 57(6), 420–434.

Nwogugu, M. (revised 2020). *On Conjectures About The Simultaneous Pell Equations* $x^2-(a^2-1)y^2=1$ *and* $y^2-pz^2=1$. Available in www.ssrn.com.

Oliveira e Silva, T. (2010). *Empirical verification of the 3x+1 and related conjectures, The ultimate challenge: the 3x+ 1 problem*. 189207, Amer. Math. Soc., Providence, RI, 2010.

Sinai, Y. (2003). Statistical (3x+ 1) problem, Dedicated to the memory of Jurgen K. Moser. *Communications In Pure & Applied Math*., 56(7), 1016–1028.

Steuding, J. (2002). "*Probabilistic number theory*". Working paper.

Tenenbaum, G. (1995). Introduction to analytic and probabilistic number theory. *Cambridge Studies in Advanced Mathematics. Vol. 46*. (Cambridge University Press).

Tao, T. (2016*)*. The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. *Forum Of Mathematics, Pi*, 4(2016), e8.

Tao, T. (2022). Almost all orbits of the Collatz map attain almost bounded values. *Forum Of Mathematics, Pi*, Volume (10).

Terras, R. (1979). On the existence of a density. *Acta Arithmetica*, 35, 101–102.

Thomas, A. (2017). A non-uniform distribution property of most orbits, in case the 3x+ 1 conjecture is true. *Acta Arithmetica*, 178(2), 125–134.

Wirsching, G. (1998). The Dynamical System Generated by the 3n+ 1 Function. *Lecture Notes in Math*. #1681, (Springer-Verlag: Berlin 1998).