The WKB limit of the Duffin-Kemmer-Petiau equation

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Abstract

The equivalent system of equations corresponding to the Duffin-Kemmer-Petiau (DKP) equation is derived and the WKB approximation of this system is found. It is proved that the Lorentz equation follows from the new DKP-Pardy system.

Key words: Duffin-Kemmer-Petiau equation, Lorentz equation, Duffin-Kemmer-Petiau-Pardy equation.

1 Introduction

The Duffin-Kemmer equation, or, rigorously Duffin-Kemmer-Petiau (DKP) equation is a first-order relativistic wave equation for spin 0 and 1 bosons (Duffin, 1938; Kemmer, 1939; Petiau, 1936). For the vector case, the DKP equation with minimal coupling is equivalent to the Maxwell or Proca equations.

In the DKP formalism, a wave function is multicomponent. That is why the simplest non-minimal interactions with external fields have a more complicated structure than in usual formalism. It has applications in describing of interactions of mesons with nuclei (Clark, et al., 1985), for studies of pionic atoms etc.

The Klein-Gordon equation in the the DKP form for the motion of the spin zero particles was presented in some monographs and articles (Akhiezer et al., 1969, 1981; Pardy 1973a). Now, we follow the five decades old original article by author (Pardy, 1973a) and the monograph by well-known experts (Akhiezer et al., 1969; 1981). The cited equation is of the form:

$$\varphi_1 + \varphi_2 + mc\Psi = 0, \tag{1}$$

where

$$\varphi_1 = \hbar\beta_\mu \frac{\partial\Psi}{\partial x_\mu}; \quad \varphi_2 = -i\frac{e}{c}A_\mu\beta_\mu\Psi. \tag{2}$$

Here Ψ is the five-component spinor and the explicite form of the β-matrices are as follows:

$$\beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

$$\beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{4}$$

$$\beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

$$\beta_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

Rafanelli and Schiller (1964) proved that the Lorentz equation od the equation or spin motion followed from the Dirac equation as a result of the so called WKB approximation, where spin is defined as the axial vector constructed from the γ-matrices and wave functions. The deep understanding what is spin is involved in the well-known texts (Ohanian, 1986; Ternov, 1988; Thomas, 1926; Tomonaga, 1997; Uhlenbeck et al., 1926). In this paper we perform a similar procedure with the the DKP equation. However, it is advantageous to derive the equivalent system of equations for the DKP equation and then to perform WKB approximation in this system. Therefore we first approach the deriving of this equivalent system of equations.

2 The equivalent system of equations to the DKP equation

In deriving the equivalent system to equation (1), we proceed as follows: we put

$$\Sigma_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ldots \tag{7}$$

$$\Sigma_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, .. \tag{8}$$

$$\Sigma_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ldots \tag{9}$$

It means that for the elements of Σ-matrices the following relation holds good:

$$\left(\Sigma_{\alpha\beta}\right)_{\gamma\varrho} = \delta_{\alpha\gamma}\delta_{\beta\varrho} \tag{10}$$

With regard to system of equation (7–9), we can write for the DKP matrices β the relation:

$$\beta_1 = \Sigma_{25} + \Sigma_{52}$$

$$\beta_2 = \Sigma_{35} + \Sigma_{53}$$

$$\beta_3 = \Sigma_{45} + \Sigma_{54}$$

$$\beta_4 = -i\Sigma_{15} + i\Sigma_{51} \tag{11}$$

Further, we can easily see that

$$\Sigma_{55} = \frac{1}{3}\left(\beta_\mu^2 - I\right) \tag{12}$$

If we put

$$\alpha_1 = \beta_4, \quad \alpha_2 = \beta_1, \quad \alpha_3 = \beta_2, \quad \alpha_4 = \beta_3, \quad \Sigma_{55} = P, \tag{13}$$

we can see that the following holds good:

$$\Sigma_{\mu\nu} = \alpha_\mu P \alpha_\nu F(\mu\nu), \tag{14}$$

where

$$F(\mu\nu = \begin{cases} i; \quad \mu = 1, \quad \nu \neq 1 \\ -i; \quad \mu \neq 1, \quad \nu = 1, \quad \mu \neq 5 \\ 1; \quad \mu \neq 1, \quad \nu \neq 1, \quad \nu \neq 5 \\ 1; \quad \mu = 1, \quad \nu = 1 \end{cases} \tag{15}$$

$$\Sigma_{\mu 5} = \alpha_\mu P; \quad \Sigma_{5\mu} = P\alpha_\mu, \tag{16}$$

Using the relation (10) we find that

$$\left(\Sigma_{\mu\nu}\right)_{\sigma\varrho}\left(\Sigma_{\nu\mu}\right)_{\epsilon\lambda} = \delta_{\sigma\lambda}\delta_{\varrho\epsilon} \tag{17}$$

With regard to relation (15) we can see that the following holds good:

$$F(\mu\nu)F(\nu\mu) = 1; \quad \text{(no sum up over} \quad \mu, \nu). \tag{18}$$

Now, if we substitute eq. (15) and eq. (16) into eq. (17) and if we return to the matrix β, we get:

$$\sum_{\mu,\nu=1}^{4}\left(\beta_\mu P \beta_\nu\right)_{\sigma\varrho}\left(\beta_\nu P \beta_\mu\right)_{\epsilon\lambda} + \sum_{\mu=1}^{4}\left(P\beta_\mu\right)_{\sigma\varrho}\left(\beta_\nu P\right)_{\epsilon\lambda} +$$
$$\sum_{\mu=1}^{4}\left(\beta_\mu P\right)_{\sigma\varrho}\left(P\beta_\mu\right)_{\epsilon\lambda} + (P)_{\sigma\varrho}(P)_{\epsilon\lambda} = \delta_{\sigma\lambda}\delta_{\varrho\epsilon} \tag{19}$$

Now, let us still remark that equation (1) involving the electromagnetic interaction has the explicit form:

$$\hbar\beta_\lambda\frac{\partial\Psi}{\partial x_\lambda} - i\frac{e}{c}A_\lambda\beta_\lambda\Psi + mc\Psi = 0, \tag{20}$$

Let us put further according to Akhiezer (1969) the following denotaions $\bar{\Psi} = \Psi^+\omega, \Psi^+ = (\Psi^*)^T, \omega = 2\beta_4^2 - I$, where we denote by means of the symbol $*$ the complex conjugate quantity and by means of the symbol T the operation of transposition and let us accept the designation:

$$T_{\mu\nu} = \beta_\mu P \beta_\nu, \quad V_\mu = \beta_\mu P, \quad W_\mu = P\beta_\mu \tag{21}$$

Now, let us multiply eq. (19) by spinors $\bar{\Psi}, \varphi, \Psi$. With regard to the designation (21) we get:

$$T_{\nu\mu}\Psi.\bar{\Psi}T_{\mu\nu}\varphi + V_\mu\Psi.\bar{\Psi}V_\mu\varphi + W_\mu\Psi.\bar{\Psi}W_\mu\varphi + P\Psi.\bar{\Psi}P\varphi = \varphi(\bar{\Psi}\Psi) \tag{22}$$

and simultaneously for $\varphi = \Psi$:

$$T_{\nu\mu}\Psi.\bar{\Psi}T_{\mu\nu}\Psi + V_\mu\Psi.\bar{\Psi}V_\mu\Psi + W_\mu\Psi.\bar{\Psi}W_\mu\Psi + P\Psi.\bar{\Psi}P\Psi = \Psi(\bar{\Psi}\Psi) \tag{23}$$

Finally let us multiply eq. (1) gradually by spinors $\bar{\Psi}T_{\mu\nu}, \bar{\Psi}V_\mu, \bar{\Psi}W_\mu, \bar{\Psi}P$. We get following system of equations:

$$\bar{\Psi}T_{\mu\nu}\varphi_1 + \bar{\Psi}T_{\mu\nu}\varphi_2 + mc\bar{\Psi}T_{\mu\nu}\Psi = 0 \tag{24}$$

$$\bar{\Psi}V_\mu\varphi_1 + \bar{\Psi}V_\mu\varphi_2 + mc\bar{\Psi}V_\mu\Psi = 0 \tag{25}$$

$$\bar{\Psi}W_\mu\varphi_1 + \bar{\Psi}W_\mu\varphi_2 + mc\bar{\Psi}W_\mu\Psi = 0 \tag{26}$$

$$\bar{\Psi}P\varphi_1 + \bar{\Psi}P\varphi_2 + mc\bar{\Psi}P\Psi = 0 \tag{27}$$

After applying eq. (24) by the spinor $T_{\mu\nu}\Psi$, eq. (25) by the spinor $V_\mu\Psi$, eq. (26) by the spinor $W_\mu\Psi$, eq. (27) by the spinor $P\Psi$ and after by the summing uo the resulting equations, we get with regard to relations (22) and (23):

$$(\bar{\Psi}\Psi)(\varphi_1 + \varphi_2 + mc\Psi) = 0, \tag{28}$$

If we assume now that $(\bar{\Psi}\Psi) \neq 0$, then

$$\varphi_1 + \varphi_2 + mc\Psi = 0, \tag{29}$$

which is the original equation (1).

We have seen therefore, that the system of equations (24–27) follows from eq. (1) and at the same time eq. (1) follows from equation system (24–27). Therefore the system of equation (24–27) and equation (1) are equivalent to each other. In the theoretical applications it is possible to proceed both from eq. (1) and from the equivalent system (24–27). For our purpose it is more advantageous to use the majestic system (24–27).

3 The WKB approximation of the majestic form of the Duffin-Kemmer-Petiau equation

Before we start to find the WKB solution of the system (24–27), let us determine the transformation properties of the mathematical objects $\bar{\Psi}T_{\mu\nu}\Psi, \bar{\Psi}V_\mu\Psi, \bar{\Psi}W_\mu\Psi, \bar{\Psi}P\Psi$.

It is known that the Lorentz transformation (Muirhead, 1965)

$$x'_\lambda = a_{\lambda\alpha}x_\alpha; \quad a_{\alpha\lambda}a_{\alpha\varrho}\delta_{\lambda\varrho} \tag{30}$$

implies the transformation of the wave function $\bar{\Psi}$, or Ψ respectively in the following way:

$$\Psi'(x') = \Lambda\Psi(x), \quad \bar{\Psi}'(x') = \bar{\Psi}(x)\Lambda^{-1}, \tag{31}$$

where Λ is the unitary matrix. As a result of a Lorentz invariance, the relations for the matrices follows (Muirhead, 1965):

$$\beta_\alpha = a_{\alpha\lambda}\Lambda\beta_\lambda\Lambda^{-1} \tag{32}$$

Using eq. (30) and (32) we easily find that

$$P = \Lambda P \Lambda^{-1} \tag{33}$$

By means of equations (31) (32) and (33) we can easily see that we get:

$$\bar{\Psi}'(x')T_{\mu\nu}\Psi'(x') = a_{\mu\alpha}a_{\nu\beta}\bar{\Psi}(x)T_{\alpha\beta}\Psi(x) \tag{34}$$

$$\bar{\Psi}(x')V_\mu\Psi'(x') = a_{\mu\nu}\bar{\Psi}V_\nu\Psi(x) \tag{35}$$

$$\bar{\Psi}(x')W_\mu\Psi'(x') = a_{\mu\nu}\bar{\Psi}W_\nu\Psi(x) \tag{36}$$

$$\bar{\Psi}(x')P\Psi'(x') = \bar{\Psi}P\Psi(x) \tag{37}$$

Let us write now equations (24– 27) in the explicit form

$$\bar{\Psi}T_{\mu\nu}\hbar\beta_\lambda\frac{\partial\Psi}{\partial x_\lambda} - i\bar{\Psi}T_{\mu\nu}\frac{e}{c}A_\lambda\beta_\lambda\Psi. + mc\bar{\Psi}T_{\mu\nu}\Psi = 0 \tag{38}$$

$$\bar{\Psi}V_\mu\hbar\beta_\lambda\frac{\partial\Psi}{\partial x_\lambda} - i\bar{\Psi}V_\mu\frac{e}{c}A_\lambda\beta_\lambda\Psi + mc\bar{\Psi}V_\mu\Psi = 0 \tag{39}$$

$$\bar{\Psi}W_\mu\hbar\beta_\lambda\frac{\partial\Psi}{\partial x_\lambda} - i\bar{\Psi}W_\mu\frac{e}{c}A_\lambda\beta_\lambda\Psi + mc\bar{\Psi}W_\mu\Psi = 0 \tag{40}$$

$$\bar{\Psi}P\hbar\beta_\lambda\frac{\partial\Psi}{\partial x_\lambda} - i\bar{\Psi}P\frac{e}{c}A_\lambda\beta_\lambda\Psi + mc\bar{\Psi}P\Psi = 0 \tag{41}$$

Now, let us try to find the WKB solutions of eqs. (38–41). This method is named after physicists Wentzel, Kramers, and Brillouin, who all developed it in 1926. In 1923, mathematician Harold Jeffreys had developed a general method of approximating solutions to linear, second-order differential equations, a class that includes the Schrödinger equation. Early quantum mechanics contained any number of combinations of their initials, including WBK, BWK, WKBJ, JWKB and BWKJ. The critical survey has been given by Dingle (1973).

This asymptotic solution is the series in the small parameter $\hbar$ of the following form (Akhiezer et al., 1969; 1981):

$$\Psi = e^{\frac{i}{\hbar}S}(a_0 + \hbar a_1 + \hbar^2 a_2 + ...), \quad \bar{\Psi} = e^{-\frac{i}{\hbar}S}(\bar{a}_0 + \hbar\bar{a}_1 + \hbar^2\bar{a}_2 + ...), \tag{42}$$

where S is the scalar real function and the coefficients $a_0, a_1, a_2, ..$ are the four-component complex spinors.

It will suffice for our purposes, if we restrict series to the first term. For this reason we replace the function by the following one

$$\Psi_{WKB} = a_0 e^{\frac{i}{\hbar}S} \tag{43}$$

in the system (38–41). If we annul the coefficients with $\hbar^0$ we get:

$$\left(\partial_\lambda S - \frac{e}{c}A_\lambda\right)\overline{a}_0 T_{\mu\nu}\beta_\lambda a_0 = imc\overline{a}_0 T_{\mu\nu} a_0 \tag{44}$$

$$\left(\partial_\lambda S - \frac{e}{c}A_\lambda\right)\overline{a}_0 V_{\mu}\beta_\lambda a_0 = imc\overline{a}_0 V_{\mu} a_0 \tag{45}$$

$$\left(\partial_\lambda S - \frac{e}{c}A_\lambda\right)\overline{a}_0 W_{\mu}\beta_\lambda a_0 = imc\overline{a}_0 W_{\mu} a_0 \tag{46}$$

$$\left(\partial_\lambda S - \frac{e}{c}A_\lambda\right)\overline{a}_0 P\beta_\lambda a_0 = imc\overline{a}_0 P a_0 \tag{47}$$

Let us put further

$$a_0 = R\Phi, \tag{48}$$

where R is the scalar function and Φ is the unit spinor, which we normalize in the following way

$$\bar{\Phi}P\Phi = 1. \tag{49}$$

Now, let us consider that the particle with the spin zero does not perform the spin motion. Therefore, in order to obtain some information on the motion of this particle, it is sufficient to keep the vector equation (45), or, (46). Putting

$$v_\mu = ic\bar{\Phi}P\beta_\mu\Phi \tag{50}$$

$$\left(\partial_\mu S - \frac{e}{c}A_\mu\right) = mv_\mu \tag{51}$$

we get with regard to the relation $v_\mu\beta_\lambda = P\delta_{\mu\lambda}$ from equation (45) and (46)

$$v_\mu v_\mu = c^2 \tag{52}$$

and we can therefore interpret the quantity v_μ as a four-vector of velocity (Landau, 1988).

If we put $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and if we take account of the identity which holds for the proper time (Rafanelli et al., 1964)

$$v_\mu\partial_\mu = \frac{d}{d\tau}, \tag{53}$$

we get from the equations (51) and (52) the following equation:

$$-\frac{e}{c}F_{\mu\nu}v_\nu = m\frac{dv_\mu}{d\tau} \tag{54}$$

The last equation is the Lorentz equation for motion of the charged particle in the electromagnetic field expressed by the tensor $F_{\mu\nu}$.

4 Discussion

We have derived the equivalent system of equations corresponding to the DKP equation and the WKB approximation of this system has been found. The new tensor equation was derived and it is not excluded that the verification of this author original equation will be performed by such adequate laboratories as CERN. Equation (42) is the original tensor equation following from DKP-Pardy majestic system. This author equation was not considered in physics for five decades (Pardy, 1973a, 1973b). We have analogy in the history of science, where the Mendel, Heaviside, Planck, Moessbauer, Sommerfeld, and so on, ideas were ignored for some decades. Nevertheless, the experimental verification of the author ideas will be, no doubt, interesting and the grand DKP-Pardy system will be considered as relevant.

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