Via Geometric Algebra: Rotating a Vector to Locate its Endpoint at a Specific Distance d from a Given Point P

July 27, 2022

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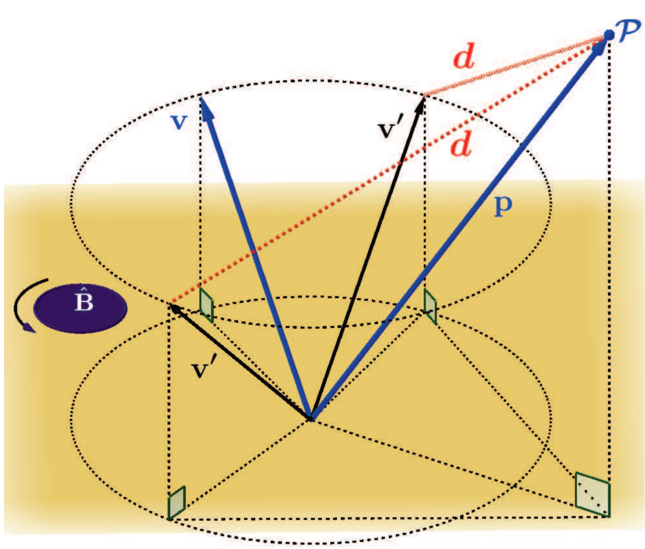
Abstract

To help fill the need for examples of introductory-level problems that have been solved via Geometric Algebra (GA), we show how to calculate the angle through which a given vector must be rotated in order that its endpoint be at a given distance d from a specified point P. The three solution methods that are employed here start from a trigonometric equation is derived from GA's formula for rotating vectors. The first two solutions use methods that are "automatic", but produce formulas that are not readily interpreted. In contrast, the third method —which does produce a readily interpreted formula —is based upon an examination of the geometric significance of terms in the initial trigonometric equation.



1 Introduction

The ability to express rotations conveniently is one of GA's strong points, and offers many opportunities to show newcomers how to use GA identities. For example, GA can easily express the vector ($\mathbf{v}'$, (Fig. 1)) that results from the rotation of vector $\mathbf{v}$ through the angle θ about an axis that is perpendicular to the bivector $\hat{\mathbf{B}}$:

$$\mathbf{v}' = \left[e^{-\hat{\mathbf{B}}\theta/2}\right] \mathbf{v} \left[e^{\hat{\mathbf{B}}\theta/2}\right]. \tag{1}$$

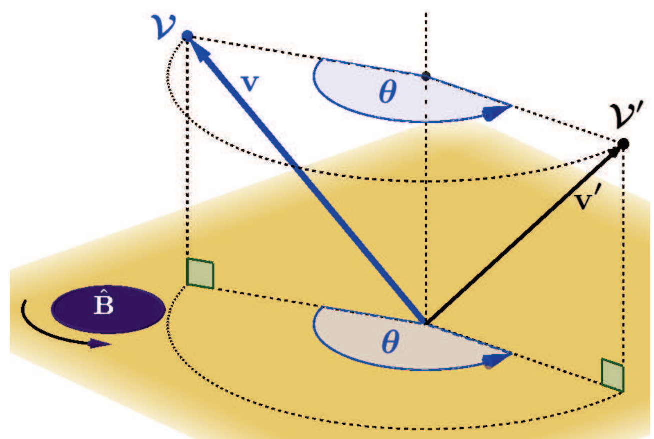


Figure 1: The vector $\mathbf{v}'$ that results when vector $\mathbf{v}$ is rotated through the bivector angle $\hat{\mathbf{B}}\theta$: $\mathbf{v}' = \left[e^{-\hat{\mathbf{B}}\theta/2}\right] \mathbf{v} \left[e^{\hat{\mathbf{B}}\theta/2}\right]$.

As explained in Refs. [1] and [2], Eq. (1) can be transformed to obtain

$$\mathbf{v}' = \mathbf{v}_{\perp} + \mathbf{v}_{\parallel} \cos\theta + \mathbf{v} \cdot \hat{\mathbf{B}} \sin\theta, \tag{2}$$

in which $\mathbf{v}_{\perp}$ and $\mathbf{v}_{\parallel}$ are as shown in Fig. 2.

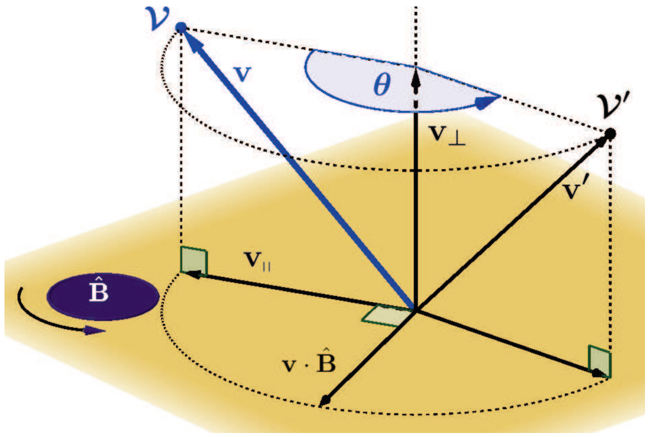


Figure 2: Relations between vector $\mathbf{v}$; its components perpendicular and parallel to $\hat{\mathbf{B}}$; and the rotated vector $\mathbf{v}'$.

In this document, we show how to determine the specific angle θ at which the endpoint of $\mathbf{v}'$ is at a specified distance d from a given point $\mathcal{P}$ (Fig. 3). For that purpose, we will first use GA identities to further transform Eq. (1), then use additional identities to simplify the equation that we find for d. In this way, we will arrive at a trigonometric equation, which we will solve in three ways. We will see that by investing a little time to understand the geometric significance of terms in the trigonometric equation, we can arrive quickly at a solution that is readily interpretable.

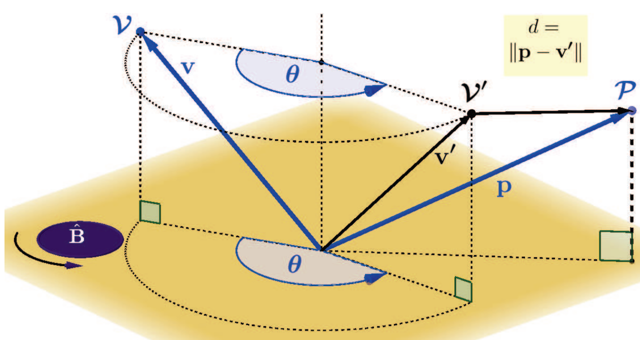


Figure 3: At what angle θ will the endpoint of $\mathbf{v}'$ be at a specified distance d from a given point $\mathcal{P}$? The distance d is equal to $\|\mathbf{p}-\mathbf{v}'\|$, which is $\sqrt{(\mathbf{p}-\mathbf{v}')^2}$.

2 Preliminary Examination

One observation that stands out is that for any given distance d (except for the maximum and minimum possible distances), there will be **two** vectors $\mathbf{v}'$ whose endpoints are at distance d from the endpoint of $\mathbf{p}$. The traces of those two vectors upon $\hat{\mathbf{B}}$ will be symmetric with respect to the trace of $\mathbf{p}$ (Fig. 4).

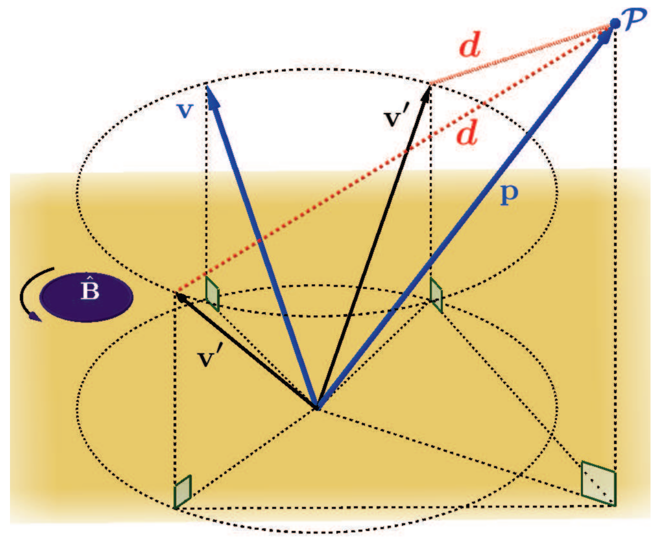


Figure 4: For any given distance d (except for the maximum and minimum possible distances), there will be two vectors $\mathbf{v}'$ whose endpoints are at distance d from the endpoint of $\mathbf{p}$. The traces of those two vectors upon $\hat{\mathbf{B}}$ will be symmetric with respect to the trace of $\mathbf{p}$.

3 Some of the Ideas that We Will Use

1. An expression of the form $A\sin\theta + B\cos\theta$ can be rewritten as

   $$A\sin\omega + B\cos\omega = \sqrt{A^2+B^2}\left\{\left[\frac{A}{\sqrt{A^2+B^2}}\right]\sin\omega + \left[\frac{B}{\sqrt{A^2+B^2}}\right]\cos\omega\right\},$$

   We can then define $A/\sqrt{A^2+B^2} = \cos\alpha$, $B/\sqrt{A^2+B^2} = \sin\alpha$, so that

   $$A\sin\omega + B\cos\omega = \left(\sqrt{A^2+B^2}\right)\sin(\omega+\alpha).$$

2. Various half-angle formulas: $\sin\psi = 2\sin\frac{\psi}{2}\cos\frac{\psi}{2}$;
   $\cos\psi = \cos^2\frac{\psi}{2} - \sin^2\frac{\psi}{2} = 1 - 2\sin^2\frac{\psi}{2}$.

3. If $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a set of perpendicular unit vectors, then any vector $\mathbf{u}$ that is parallel to the bivector $\mathbf{b}_1\mathbf{b}_2$ can be written as

   $$\mathbf{u} = (\mathbf{u}\cdot\mathbf{b}_1)\,\mathbf{b}_1 + (\mathbf{u}\cdot\mathbf{b}_2)\,\mathbf{b}_2.$$

   Then,

   $$\|\mathbf{u}\|^2 = (\mathbf{u}\cdot\mathbf{b}_1)^2 + (\mathbf{u}\cdot\mathbf{b}_2)^2.$$

4. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\|\mathbf{a}\wedge\mathbf{b}\|^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$. A simple proof: $\mathbf{ab} = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}$, and $\mathbf{ba} = \mathbf{b}\cdot\mathbf{a}+\mathbf{b}\wedge\mathbf{a} = \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\wedge\mathbf{b}$. Therefore, $(\mathbf{ab})(\mathbf{ba}) = [\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}][\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\wedge\mathbf{b}]$; $a^2b^2 = (\mathbf{a}\cdot\mathbf{b})^2 - (\mathbf{a}\wedge\mathbf{b})^2 = (\mathbf{a}\cdot\mathbf{b})^2 +{}^{-}\|\mathbf{a}\wedge\mathbf{b}\|^2$, etc.

5. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}\wedge\mathbf{b} = \mathbf{a}\cdot(\mathbf{bi})\mathbf{i}$, where $\mathbf{i}$ is the unit bivector that is parallel to both $\mathbf{a}$ and $\mathbf{b}$.

6. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}\wedge\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\sin\psi\mathbf{i}$, where ψ is the angle of rotation from $\mathbf{a}$ to $\mathbf{b}$, and $\mathbf{i}$ is the unit bivector that is parallel to both $\mathbf{a}$ and $\mathbf{b}$, and whose sense of rotation is from $\mathbf{a}$ to $\mathbf{b}$.

4 Deriving the Trigonometric Equation

Why does $(v')^2 = v^2$? First, because both of those vectors have the same length. An algebraic justification begins by writing $(v')^2$ as

$$\underbrace{\left[e^{-\hat{\mathbf{B}}\theta/2}\right]\mathbf{v}\left[e^{\hat{\mathbf{B}}\theta/2}\right]}_{\mathbf{v}'}\underbrace{\left[e^{-\hat{\mathbf{B}}\theta/2}\right]\mathbf{v}\left[e^{\hat{\mathbf{B}}\theta/2}\right]}_{\mathbf{v}'}$$

Then, we can proceed as follows:

$$\left[e^{-\hat{\mathbf{B}}\theta/2}\right]\mathbf{v}\underbrace{\left[e^{\hat{\mathbf{B}}\theta/2}\right]\left[e^{-\hat{\mathbf{B}}\theta/2}\right]}_{=1}\mathbf{v}\left[e^{\hat{\mathbf{B}}\theta/2}\right]$$

$$\left[e^{-\hat{\mathbf{B}}\theta/2}\right]\underbrace{\mathbf{vv}}_{=v^2}\left[e^{\hat{\mathbf{B}}\theta/2}\right]$$

$$v^2\underbrace{\left[e^{\hat{\mathbf{B}}\theta/2}\right]\left[e^{-\hat{\mathbf{B}}\theta/2}\right]}_{=1}$$

The required distance d is equal to $\|\mathbf{p}-\mathbf{v}'\|$, which is $\sqrt{(\mathbf{p}-\mathbf{v}')^2}$. Because our goal is to identify the necessary value of θ, we will write that requirement more conveniently as

$$\begin{aligned}
d^2 &= (\mathbf{p}-\mathbf{v}')^2 \\
&= p^2 - 2\mathbf{p}\cdot\mathbf{v}' + (\mathbf{v}')^2 \\
&= p^2 - \underbrace{\left[2\mathbf{p}\cdot\mathbf{v}_\perp + 2\mathbf{p}\cdot\mathbf{v}_\parallel\cos\theta + 2\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\sin\theta\right]}_{=\mathbf{p}\cdot\mathbf{v}'} + v^2.
\end{aligned}$$

Therefore,

$$\mathbf{p}\cdot\mathbf{v}_\parallel\cos\theta + \mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\sin\theta = \frac{p^2+v^2-d^2-2\mathbf{p}\cdot\mathbf{v}_\perp}{2}.$$

We can simplify the numerator by noting that $(\mathbf{p}-\mathbf{v})^2 = p^2+v^2-2\mathbf{p}\cdot\mathbf{v}$. Thus, $p^2+v^2 = (\mathbf{p}-\mathbf{v})^2 + 2\mathbf{p}\cdot\mathbf{v}$. In addition,

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{v} - \mathbf{p}\cdot\mathbf{v}_\perp &= \mathbf{p}\cdot(\mathbf{v}-\mathbf{v}_\perp) \\
&= \mathbf{p}\cdot\mathbf{v}_\parallel.
\end{aligned}$$

Putting all of these ideas together,

$$\begin{aligned}
\mathbf{p}\cdot\mathbf{v}_\parallel\cos\theta + \mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\sin\theta &= \frac{p^2+v^2-d^2-2\mathbf{p}\cdot\mathbf{v}_\perp}{2} \\
&= \frac{(\mathbf{p}-\mathbf{v})^2 + 2\mathbf{p}\cdot\mathbf{v} - 2\mathbf{p}\cdot\mathbf{v}_\perp - d^2}{2} \\
&= \frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}\cdot(\mathbf{v}-\mathbf{v}_\perp)}{2} \\
&= \frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}\cdot\mathbf{v}_\parallel}{2} \qquad (3)
\end{aligned}$$

We will transform that equation further in each of our three solutions.

5 Transforming and Solving the Trigonometric Equation

5.1 The First Solution

If we examine Eq. (3) while keeping the half-angle formulas in mind, we will see that the occurrence of a $\mathbf{p}\cdot\mathbf{v}_{\|}$ term on both side give us the opportunity to rewrite Eq. (3) as follows:

$$\mathbf{p}\cdot\mathbf{v}_{\|}\left(\cos\theta-1\right)+\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\sin\theta=\frac{\left[\left(\mathbf{p}-\mathbf{v}\right)^2-d^2\right]}{2},$$

$$\mathbf{p}\cdot\mathbf{v}_{\|}\left(-2\sin^2\frac{\theta}{2}\right)+\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]=\frac{\left[\left(\mathbf{p}-\mathbf{v}\right)^2-d^2\right]}{2},$$

$$\left\{2\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^2=\left\{\frac{\left[\left(\mathbf{p}-\mathbf{v}\right)^2-d^2\right]}{2}+2\mathbf{p}\cdot\mathbf{v}_{\|}\sin^2\frac{\theta}{2}\right\}^2,$$

and

$$4\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2\sin^2\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right)=\left\{\frac{\left[\left(\mathbf{p}-\mathbf{v}\right)^2-d^2\right]}{2}+2\mathbf{p}\cdot\mathbf{v}_{\|}\sin^2\frac{\theta}{2}\right\}^2. \tag{4}$$

After solving that quadratic for the two possible values of $\sin^2\dfrac{\theta}{2}$, we then find the corresponding values of $\cos\theta$ from the formula $\cos\theta=1-2\sin^2\dfrac{\theta}{2}$. We will not delve into the resulting simplifications, except to note that the identity $\|\mathbf{a}\wedge\mathbf{b}\|^2=a^2b^2-\left(\mathbf{a}\cdot\mathbf{b}\right)^2$ is useful.

5.2 The Second Solution

The left-hand side of Eq. (3) is of the form $A\sin$
$omega+B\cos\omega$. In such a case, we almost automatically use item 1 in Section 3. First, we define

By defining both $\sin\alpha$ and $\sin\alpha$, we have identified the angle α uniquely.

$$\sin\alpha=\left(\mathbf{p}\cdot\mathbf{v}_{\|}\right)/\sqrt{\left(\mathbf{p}\cdot\mathbf{v}_{\|}\right)^2+\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2},\ \text{ and}$$

$$\cos\alpha=\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]/\sqrt{\left(\mathbf{p}\cdot\mathbf{v}_{\|}\right)^2+\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2}$$

That "automatic" choice turns out to be unfortunate, as will become clear when we see the third solution. However, we will follow through on that choice because of the opportunities that it offers for practicing the use of GA identities.

With $\sin\alpha$ and $\cos\alpha$ defined as above, we may rewrite Eq. (3) as

$$\left[\sqrt{\left(\mathbf{p}\cdot\mathbf{v}_{\|}\right)^2+\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2}\right]\sin\left(\theta+\alpha\right)=\frac{\left[\left(\mathbf{p}-\mathbf{v}\right)^2-d^2\right]+2\mathbf{p}\cdot\mathbf{v}_{\|}}{2},$$

and

$$\sin\left(\theta+\alpha\right)=\frac{\left[\left(\mathbf{p}-\mathbf{v}\right)^2-d^2\right]+2\mathbf{p}\cdot\mathbf{v}_{\|}}{2\sqrt{\left(\mathbf{p}\cdot\mathbf{v}_{\|}\right)^2+\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2}},$$

from which

$$\sin\theta=\sin^{-1}\left[\frac{\left[\left(\mathbf{p}-\mathbf{v}\right)^2-d^2\right]+2\mathbf{p}\cdot\mathbf{v}_{\|}}{2\sqrt{\left(\mathbf{p}\cdot\mathbf{v}_{\|}\right)^2+\left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2}}\right]-\alpha. \quad (5)$$

We will simplify that result shortly. The ideas that we use will lead directly to the third solution (Section 5.3), which is not only more efficient, but easier to interpret.

The algebraic sign of $\sin\left(\theta+\alpha\right)$ is a key aspect of the present solution. By analyzing Eq. (8), we can see that $\sin\left(\theta+\alpha\right)\geq 0$ when $d^2<\left(\mathbf{p}-\mathbf{v}\right)^2+2\mathbf{p}\mathbf{\cdot}\mathbf{v}_{\|}$, and negative when $d^2>\left(\mathbf{p}-\mathbf{v}\right)^2+2\mathbf{p}\mathbf{\cdot}\mathbf{v}_{\|}$. For each of these cases, there are two angles "$\theta+\alpha$" that satisfy Eq. (8). Hence, the angles and their respective quadrants in the unit circle are

The range of $\arcsin\left(x\right)$ is $\left[-\frac{\pi}{2},\,\frac{\pi}{2}\right]$.

$$\left.\begin{array}{r}\arcsin\left(\theta+\alpha\right),\text{ in Quadrant I}\\ \text{and}\\ \pi-\arcsin\left(\theta+\alpha\right),\text{ in Quadrant II}\end{array}\right\},\quad d^2<\left(\mathbf{p}-\mathbf{v}\right)^2+2\mathbf{p}\mathbf{\cdot}\mathbf{v}_{\|}\ ;$$

$$(6)$$

$$\left.\begin{array}{r}\arcsin\left(\theta+\alpha\right),\text{ in Quadrant IV}\\ \text{and}\\ -\pi+\left|\arcsin\left(\theta+\alpha\right)\right|,\text{ in Quadrant III}\end{array}\right\},\quad d^2>\left(\mathbf{p}-\mathbf{v}\right)^2+2\mathbf{p}\mathbf{\cdot}\mathbf{v}_{\|}.$$

Therefore, the values of θ are

$$\left.\begin{array}{r}\arcsin\left(\theta+\alpha\right)-\alpha\\ \text{and}\\ \pi-\arcsin\left(\theta+\alpha\right)-\alpha\end{array}\right\},\quad d^2<\left(\mathbf{p}-\mathbf{v}\right)^2+2\mathbf{p}\mathbf{\cdot}\mathbf{v}_{\|}\ ;$$

$$(7)$$

$$\left.\begin{array}{r}\arcsin\left(\theta+\alpha\right)-\alpha\\ \text{and}\\ -\pi+\left|\arcsin\left(\theta+\alpha\right)\right|-\alpha\end{array}\right\},\quad d^2>\left(\mathbf{p}-\mathbf{v}\right)^2+2\mathbf{p}\mathbf{\cdot}\mathbf{v}_{\|}.$$

To simplify the denominator of the right-hand side of Eq. (5), we begin by writing $\mathbf{p}$ as the sum of its components perpendicular and parallel to $\hat{\mathbf{B}}$:

Simplifying the second solution.

$\mathbf{p} = \mathbf{p}_\perp + \mathbf{p}_\|$. Thus,

$$\begin{aligned}\mathbf{p}\cdot\mathbf{v}_\| &= \left(\mathbf{p}_\perp + \mathbf{p}_\|\right)\cdot\mathbf{v}_\| \\ &= \underbrace{\mathbf{p}_\perp\cdot\mathbf{v}_\|}_{=0} + \mathbf{p}_\|\cdot\mathbf{v}_\| \\ &= \mathbf{p}_\|\cdot\mathbf{v}_\|.\end{aligned}$$

Similarly, because $\mathbf{p}_\perp$ is perpendicular to the vector $\mathbf{v}\cdot\hat{\mathbf{B}}$,

$$\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right) = \mathbf{p}_\|\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right).$$

Now, we recall that $\mathbf{v}_\|$ and $\mathbf{v}\cdot\hat{\mathbf{B}}$ are perpendicular to each other, and that the length of each of these vectors is $\|\mathbf{v}_\|\|$. Thus, $\left\{\frac{\mathbf{v}_\|}{\|\mathbf{v}_\|\|}, \frac{\mathbf{v}\cdot\hat{\mathbf{B}}}{\|\mathbf{v}_\|\|}\right\}$ is a set of perpendicular unit vectors. Furthermore, $\mathbf{p}_\|$ is parallel to the bivector $\frac{\mathbf{v}_\|}{\|\mathbf{v}_\|\|}\wedge\frac{\mathbf{v}\cdot\hat{\mathbf{B}}}{\|\mathbf{v}_\|\|}$. Therefore, according to Item 3 of Section 3, we can write $\mathbf{p}_\|$ as

$$\mathbf{p}_\| = \left[\mathbf{p}_\|\cdot\frac{\mathbf{v}_\|}{\|\mathbf{v}_\|\|}\right]\left[\frac{\mathbf{v}_\|}{\|\mathbf{v}_\|\|}\right] + \left[\mathbf{p}_\|\cdot\left(\frac{\mathbf{v}\cdot\hat{\mathbf{B}}}{\|\mathbf{v}_\|\|}\right)\right]\left[\frac{\mathbf{v}\cdot\hat{\mathbf{B}}}{\|\mathbf{v}_\|\|}\right].$$

Consequently,

$$\begin{aligned}\|\mathbf{p}_\|\|^2 &= \left[\mathbf{p}_\|\cdot\frac{\mathbf{v}_\|}{\|\mathbf{v}_\|\|}\right]^2 + \left[\mathbf{p}_\|\cdot\left(\frac{\mathbf{v}\cdot\hat{\mathbf{B}}}{\|\mathbf{v}_\|\|}\right)\right]^2; \\ \|\mathbf{p}_\|\|^2 &= \left[\frac{\mathbf{p}_\|\cdot\mathbf{v}_\|}{\|\mathbf{v}_\|\|}\right]^2 + \left[\frac{\mathbf{p}_\|\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)}{\|\mathbf{v}_\|\|}\right]^2; \\ \|\mathbf{p}_\|\|^2\|\mathbf{v}_\|\|^2 &= \left[\mathbf{p}_\|\cdot\mathbf{v}_\|\right]^2 + \left[\mathbf{p}_\|\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2, \text{ and} \\ \sqrt{\left[\mathbf{p}_\|\cdot\mathbf{v}_\|\right]^2 + \left[\mathbf{p}_\|\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2} &= \|\mathbf{p}_\|\|\|\mathbf{v}_\|\|.\end{aligned}$$

Because $\mathbf{p}_\|\cdot\mathbf{v}_\| = \mathbf{p}\cdot\mathbf{v}_\|$ and $\mathbf{p}_\|\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right) = \mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)$,

$$\begin{aligned}\sqrt{\left[\mathbf{p}_\|\cdot\mathbf{v}_\|\right]^2 + \left[\mathbf{p}_\|\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2} &= \sqrt{\left(\mathbf{p}\cdot\mathbf{v}_\|\right)^2 + \left[\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\right]^2} \\ &= \|\mathbf{p}_\|\|\|\mathbf{v}_\|\|.\end{aligned}$$

Thus, Eq. (5) becomes

$$\sin(\theta+\alpha) = \frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}\cdot\mathbf{v}_\|}{2\|\mathbf{p}_\|\|\|\mathbf{v}_\|\|}. \tag{8}$$

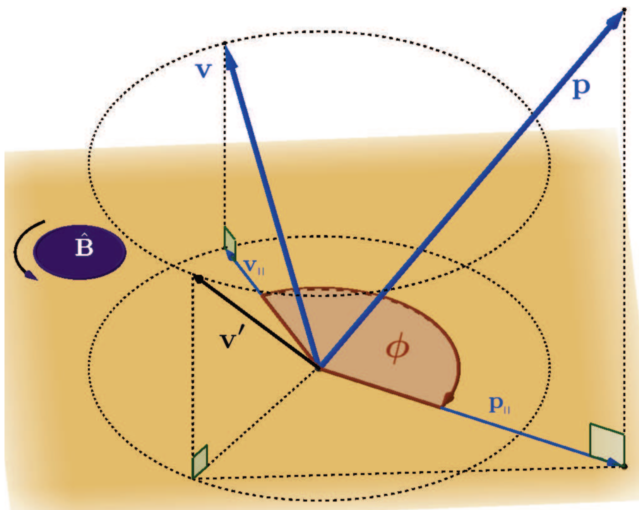


Figure 5: The angle ϕ from $\mathbf{v}_{\|}$ to $\mathbf{p}_{\|}$.

5.3 The Third Solution

The ideas that we used to simplify the second solution might lead us to re-examine Eq. (3), which we reproduce here for convenience:

$$\mathbf{p}\cdot\mathbf{v}_{\|}\cos\theta+\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\sin\theta=\frac{\left[(\mathbf{p}-\mathbf{v})^2-d^2\right]+2\mathbf{p}\cdot\mathbf{v}_{\|}}{2}\ .$$

In the process of simplifying the second solution, we saw that $\mathbf{p}\cdot\mathbf{v}_{\|}=\mathbf{p}_{\|}\cdot\mathbf{v}_{\|}$, and $\mathbf{p}\cdot\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)=\mathbf{p}_{\|}\cdot\left(\mathbf{v}_{\|}\cdot\hat{\mathbf{B}}\right)$. Therefore, Eq. (3) can be rewritten as

$$\mathbf{p}_{\|}\cdot\mathbf{v}_{\|}\cos\theta+\mathbf{p}_{\|}\cdot\left(\mathbf{v}_{\|}\cdot\hat{\mathbf{B}}\right)\sin\theta=\frac{\left[(\mathbf{p}-\mathbf{v})^2-d^2\right]+2\mathbf{p}_{\|}\cdot\mathbf{v}_{\|}}{2}\ .$$

We know that $\hat{\mathbf{p}}_{\|}\cdot\hat{\mathbf{v}}_{\|}$ is the cosine of the angle (ϕ, in Fig. 5) between $\mathbf{p}_{\|}$ and $\mathbf{v}_{\|}$. But what is $\mathbf{p}_{\|}\cdot\left(\mathbf{v}_{\|}\cdot\hat{\mathbf{B}}\right)$? Might it be $\|\mathbf{p}_{\|}\|\|\mathbf{v}_{\|}\|$ times the sine of that same angle? When we start to think in this way, we soon see that because $\mathbf{v}_{\|}\wedge\hat{\mathbf{B}}=0$, $\mathbf{v}_{\|}\hat{\mathbf{B}}=\mathbf{v}_{\|}\cdot\hat{\mathbf{B}}$. Thus, $\mathbf{p}_{\|}\cdot\left(\mathbf{v}_{\|}\hat{\mathbf{B}}\right)$. From Item 5 of Section 3, we can see that $\mathbf{p}_{\|}\cdot\left(\mathbf{v}_{\|}\hat{\mathbf{B}}\right)\hat{\mathbf{B}}$ would be $\mathbf{p}_{\|}\wedge\mathbf{v}_{\|}$. Hence, from Item 6 of Section 3, $\mathbf{p}_{\|}\cdot\left(\mathbf{v}_{\|}\hat{\mathbf{B}}\right)$ is indeed $\|\mathbf{p}_{\|}\|\mathbf{v}_{\|}\|$ times the sine of the angle of rotation from $\mathbf{v}_{\|}$ to

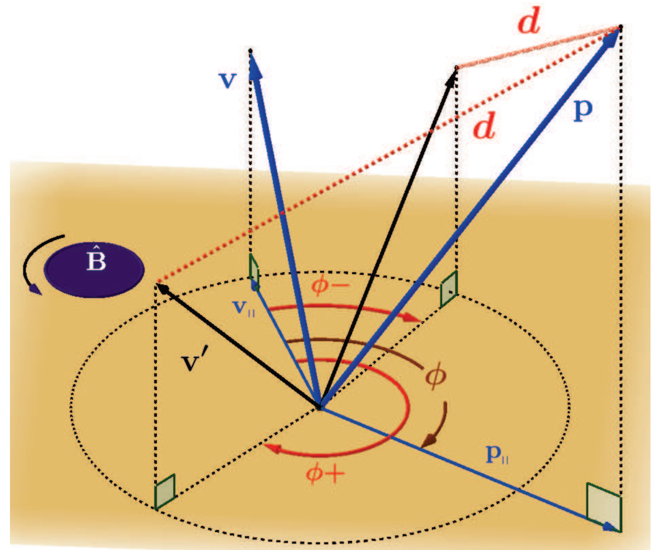


Figure 6: The two angles of θ for which the distance between the endpoint of $\mathbf{p}$ and that of $\mathbf{v}'$ is d. "$\phi+$" is the angle $\phi + \cos^{-1}\left\{\frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}\cdot\mathbf{v}_{\parallel}}{2\|\mathbf{p}_{\parallel}\|\|\mathbf{v}_{\parallel}\|}\right\}$, and "$\phi-$" is $\phi - \cos^{-1}\left\{\frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}\cdot\mathbf{v}_{\parallel}}{2\|\mathbf{p}_{\parallel}\|\|\mathbf{v}_{\parallel}\|}\right\}$.

$\mathbf{p}_{\parallel}$. Putting all of these ideas together, Eq. 3 becomes

$$\|\mathbf{p}_{\parallel}\|\|\mathbf{v}_{\parallel}\|\cos\theta\cos\phi + \|\mathbf{p}_{\parallel}\|\|\mathbf{v}_{\parallel}\|\sin\theta\sin\phi = \frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}_{\parallel}\cdot\mathbf{v}_{\parallel}}{2} ;$$

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}_{\parallel}\cdot\mathbf{v}_{\parallel}}{2\|\mathbf{p}_{\parallel}\|\|\mathbf{v}_{\parallel}\|} ;$$

and

$$\cos(\theta - \phi) = \frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}_{\parallel}\cdot\mathbf{v}_{\parallel}}{2\|\mathbf{p}_{\parallel}\|\|\mathbf{v}_{\parallel}\|} .$$

To proceed further, we note that by definition, the arc-cosine of a given number x is an angle between $0°$ and $180°$. In addition, for any angle β, $\cos\beta = \cos(-\beta)$. Thus, there are two angles whose cosine is x. Namely, $\cos^{-1}x$ and $-\cos^{-1}x$. Hence, there are two values of θ for which the distance between the endpoint of $\mathbf{p}$ and that of $\mathbf{v}'$ is d (Fig. 6):

$$\theta = \phi \pm \cos^{-1}\left\{\frac{\left[(\mathbf{p}-\mathbf{v})^2 - d^2\right] + 2\mathbf{p}_{\parallel}\cdot\mathbf{v}_{\parallel}}{2\|\mathbf{p}_{\parallel}\|\|\mathbf{v}_{\parallel}\|}\right\} .$$

In this solution, it is clear that the two angles are symmetric with respect to the trace of $\mathbf{p}_{\parallel}$ upon $\hat{\mathbf{B}}$ —a condition that was noted in Section 2 .

6 Discussion

Perhaps the most important lesson to be learned from this exercise (and more specifically, from the unfortunate choice upon which the second solution is based) is that we are well advised to spend a little time considering the geometric significance of terms in our equations before automatically employing solution techniques that we have learned from previous subjects.

References

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[2] J. Smith, "For GA Newcomers: Demonstrating the Equivalence of Different Expressions for Vector Rotations", https://vixra.org/pdf/2206.0001v2.pdf, 3 June 2022.