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Fractional Electromagnetic Metamaterials

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Abstract

This work deals with a new application of fractional electrodynamics in the physics of metamaterials. In this work based on recent nonlocal effects observed in electromagnetic metamaterials (EMMs) we propose the idea of investigations of EMMs in the framework of fractional dynamics which contains intrinsically nonlocal operators. For this aim we propose our fractional Drude metamaterial model. We present two different cases of fractional order systems for EMMs.

Keywords: Fractional Electrodynamics, Electromagnetic Metamaterials, Nonlocality

1. Introduction

Metamaterials are materials that consist of an artificial structure and have unusual electromagnetic characteristics [1-3]. Recently nonlocal effects in EMMs have attracted attentions of researchers in this field [4-13]. Nonlocality can occur in space or in time and the best mathematical tool for describing nonlocal phenomena is fractional calculus. In recent years classical and quantum electrodynamics have been investigated in the framework of fractional electrodynamics [14-24]. The main advantage of using fractional operators is that they intrinsically incorporate nonlocality in the equations. In this work we want to use this powerful tool for describing electromagnetic characteristics of metamaterials. So in the next section we present fractional differential vector calculus (FDVC) as a mathematical tool and then in Sec. (3) we introduce our new fractional Drude metamaterial models.

2. Fractional Vector Calculus

Fractional calculus (FC) is very useful and powerful tool for modeling phenomena which exhibit space time nonlocality [25]. In this section we present fractional generalization of vector calculus which are needed for building our new model [15, 18, 19 and refs. therein]. For this purpose we use the well-known Caputo partial fractional derivatives for the time derivatives and the Riesz fractional derivatives for the space ones and defining the fractional Laplacian. Left (forward) and right (backward) Caputo (RL) partial fractional derivatives of order α_μ , β_μ (which are positive real or even complex numbers) of a real valued function f of $d+1$ real variables $x^0, x^1, ..., x^d$ with respect to x_μ are as follow:

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$$ {}_{a_\mu}^{C}\partial_\mu^{\alpha_\mu} f(x^0,...,x^d) = \frac{1}{\Gamma(n_\mu-\alpha_\mu)}\int_{a_\mu}^{x}\frac{\partial_{x^\mu}^{n_\mu} f(x^0,...,x^{\mu-1},u,x^{\mu+1},...,x^d)}{(x^\mu-u)^{1+\alpha_\mu-n_\mu}}du \quad \text{(left Caputo)} \tag{1} $$

$$ {}_{\mu}^{C}\partial_{b_\mu}^{\beta_\mu} f(x^0,...,x^d) = \frac{(-1)^{n_\mu}}{\Gamma(n_\mu-\beta_\mu)}\int_{x}^{b_\mu}\frac{\partial_{x^\mu}^{n_\mu} f(x^0,...,x^{\mu-1},u,x^{\mu+1},...,x^d)}{(u-x^\mu)^{1+\beta_\mu-n_\mu}}du \quad \text{(right Caputo)} \tag{2} $$

where $\partial_{x^\mu}^{n_\mu}$ is the ordinary partial derivative of integer order n with respect to the variable x and a_μ , b_μ are real number which define the domain [25 and refs therein]. Also for the Riesz fractional derivative of order α_μ we have:

$$ {}^{R}\partial^{\alpha_\mu} f(x^0,...,x^d) = -\frac{1}{2\cos(\pi\alpha_\mu/2)\Gamma(n_\mu-\alpha_\mu)}\partial_{x^\mu}^{n_\mu}\left(\int_{-\infty}^{x^\mu}\frac{f(x^0,...,x^{\mu-1},u,x^{\mu+1},...,x^d)}{(x^\mu-u)^{1+\alpha_\mu-n_\mu}}du + \int_{x^\mu}^{\infty}\frac{f(x^0,...,x^{\mu-1},u,x^{\mu+1},...,x^d)}{(u-x^\mu)^{1+\alpha_\mu-n_\mu}}du\right) \tag{3} $$

Using this definition, we can easily introduce fractional curl operator as:

$$ (\vec{\nabla}\times)^\alpha \vec{F} = \hat{e}_\rho \varepsilon_{\rho\mu\nu}\partial_\mu^\alpha F_\nu \tag{4} $$

We can also derive a relation for the double curl operation in the following form:

$$ (\vec{\nabla}\times)^\alpha((\vec{\nabla}\times)^\alpha \vec{F}) = \hat{e}_\rho \varepsilon_{\rho\mu\nu}\varepsilon_{\nu\kappa\sigma}\partial_\mu^\alpha\partial_\kappa^\alpha F_\sigma \tag{5} $$

3. Fractional Drude Metamaterial Model

In this section first we review different standard model for wave propagation in metamaterials and then we present our new model based on FC. The lossy Drude model [26 and the references therein] is a popular model for metamaterial and in frequency domain can be written in the form of:

$$ \varepsilon(\omega) = \varepsilon_0\left(1-\frac{\omega_{pe}^2}{\omega(\omega-i\Gamma_e)}\right) = \varepsilon_0\varepsilon_r \tag{6} $$

$$ \mu(\omega) = \mu_0\left(1-\frac{\omega_{pm}^2}{\omega(\omega-i\Gamma_m)}\right) = \mu_0\mu_r \tag{7} $$

where the parameters ε_0 and μ_0 are the vacuum electric permittivity and magnetic permeability , the parameters ω_{pe} and ω_{pm} are the electric and magnetic plasma frequencies, Γ_e and Γ_m are the electric and magnetic damping frequencies, and ω is a general frequency, respectively. Based on the above model for metamatrials we can obtain the governing equations for modeling the wave propagation in metamatrials described by the Drude model as follow:

$$\varepsilon_0 \partial_t \vec{E} + \vec{J} = \vec{\nabla} \times \vec{H} \tag{8}$$

$$\mu_0 \partial_t \vec{H} + \vec{K} = -\vec{\nabla} \times \vec{E} \tag{9}$$

$$\frac{1}{\varepsilon_0 \omega_{pe}^2} \partial_t \vec{J} + \frac{\Gamma_e}{\varepsilon_0 \omega_{pe}^2} \vec{J} = \vec{E} \tag{10}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \partial_t \vec{K} + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \vec{K} = \vec{H} \tag{11}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields, respectively, $\vec{J}$ and $\vec{K}$ are the polarization and magnetization current densities, respectively.

And for the so-called Lorentz model for modeling wave propagation in metamaterials, we have[26 and the references therein]:

$$\varepsilon(\omega) = \varepsilon_0 (1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{e0}^2 - i\Gamma_e \omega}) \tag{12}$$

$$\mu(\omega) = \mu_0 (1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{m0}^2 - i\Gamma_m \omega}) \tag{13}$$

where ω_{e0} and ω_{m0} are the electric and magnetic resonance frequencies, respectively.

The above model will result in the following Lorentz model equations for metamaterials:

$$\varepsilon_0 \partial_t \vec{E} + \vec{J} = \vec{\nabla} \times \vec{H} \tag{14}$$

$$\mu_0 \partial_t \vec{H} + \vec{K} = -\vec{\nabla} \times \vec{E} \tag{15}$$

$$\frac{1}{\varepsilon_0 \omega_{pe}^2} \partial_t \vec{J} + \frac{\Gamma_e}{\varepsilon_0 \omega_{pe}^2} \vec{J} + \frac{\omega_{e0}^2}{\varepsilon_0 \omega_{pe}^2} \vec{P} = \vec{E} \tag{16}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \partial_t \vec{K} + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \vec{K} + \frac{\omega_{m0}^2}{\mu_0 \omega_{pm}^2} \vec{M} = \vec{H} \tag{17}$$

where $\vec{P}$ and $\vec{M}$ are the polarization and magnetization vectors, respectively and polarization and magnetization current densities can be defined using them as:

$$\vec{J} = \frac{\partial \vec{P}}{\partial t} \quad , \quad \vec{K} = \frac{\partial \vec{M}}{\partial t} \tag{18}$$

And finally, we have the Drude-Lorentz model in which the permittivity is described by the Drude model, while the permeability is described by the Lorentz model i.e., we have [26 and the references therein]:

$$\varepsilon(\omega) = \varepsilon_0 (1 - \frac{\omega_p^2}{\omega(\omega + iv)}) \tag{19}$$

$$\mu(\omega) = \mu_0 (1 - \frac{F\omega_0^2}{\omega^2 + i\gamma\omega - \omega_0^2}) \tag{20}$$

where ω is the excitation angular frequency, ω_p is the effective plasma frequency, and $v \geq 0$ is the loss parameter, ω_0 is the resonant frequency, $\gamma \geq 0$ is the loss parameter, and $F \in (0,1)$ is a parameter depending on the geometry of the unit cell of the metamaterial. Based on the above model we can write the governing equations for the Drude-Lorentz model as following:

$$\varepsilon_0 \partial_t \vec{E} + \vec{J} = \vec{\nabla} \times \vec{H} \tag{21}$$

$$\mu_0 \partial_t \vec{H} + \vec{K} = -\vec{\nabla} \times \vec{E} \tag{22}$$

$$\frac{1}{\mu_0 \omega_0^2 F} \partial_t \vec{K} + \frac{\gamma}{\mu_0 \omega_0^2 F} \vec{K} + \frac{1}{\mu_0 F} \vec{M} = \vec{H} \tag{23}$$

$$\frac{1}{\varepsilon_0 \omega_p^2} \partial_t \vec{J} + \frac{v}{\varepsilon_0 \omega_p^2} \vec{J} = \vec{E} \tag{24}$$

In the following using the fractional vector differential vectors' definitions presented in previous section and the above governing equations for modeling the wave propagation in metamatrials described by the Drude model, we introduce new fractional Drude metamaterial model which contains intrinsically nonlocal operators instead of local ones as a model for description of electromagnetic properties of recent nonlocal metamaterials. For this purpose, we consider two different cases:

2.1 Case a: $0 < \alpha, \beta < 1$

For the first case we consider Maxwell's equations in a Drude metamaterial [26, 27] in the framework of fractional calculus as:

$$\varepsilon_0 \partial_t^\alpha \vec{E} + \vec{J} = (\vec{\nabla} \times)^\beta \vec{H} \tag{25}$$

$$\mu_0 \partial_t^\alpha \vec{H} + \vec{K} = -(\vec{\nabla} \times)^\beta \vec{E} \tag{26}$$

$$\partial_t^\alpha \vec{J} = \varepsilon_0 \omega_{pe}^{2\alpha} \vec{E} \tag{27}$$

$$\partial_t^\alpha \vec{K} = \mu_0 \omega_{pm}^{2\alpha} \vec{H} \tag{28}$$

In the case of low level fractionality [28, 29] i.e., when the order of fractional calculus is close to integer i.e., the case of $\alpha = n - \varepsilon$ where $\varepsilon \ll 1$ we have:

Case 1: Caputo fractional derivative of order $\alpha = 1 - \varepsilon$

For this case we have:

$$^{C}\partial_t^{1-\varepsilon} f(t) = \partial_t f + \varepsilon(\partial_t f(0) \ln(t) + \ldots) \tag{29}$$

Case 2: Caputo fractional derivative of order $\alpha = 2 - \varepsilon$

For this case we have:

$$^{C}\partial_t^{2-\varepsilon} f(t) = (\partial_t)^2 f + \varepsilon((\partial_t)^2 f(0) \ln(t) + \ldots) \tag{30}$$

Case 3: Riesz fractional derivative of order $\alpha = 2 - \varepsilon$

For this case we have:

$$^{R}\partial_{|X|}^{2-\varepsilon} f(x) = (\partial_{|x|})^2 f + \varepsilon(\gamma(\partial_{|x|})^2 f + \ldots) \tag{31}$$

where $\gamma = 0.577215664901532...$ Euler–Mascheroni constant.
Base on the above equations one can find that the fractional Drude metamaterial model is characterized by a dispersive permittivity and permeability given as:

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_{pe}^{2\alpha}}{\omega^{2\alpha}}\right), \ \mu(\omega) = \mu_0 \left(1 - \frac{\omega_{pm}^{2\alpha}}{\omega^{2\alpha}}\right) \tag{32}$$

We call systems of Eqs. (25-28) the fractional Drude metamaterial model of order α and β in which $0 < \alpha, \beta < 1$.

2.2 Case b: $1 < \alpha < 2$, $0<\beta<1$
System of Eqs. (25-28) can be rewritten in the higher fractional orders i.e., $1 < \alpha < 2$, $0<\beta<1$ and with the condition of low level fractionality Eqs. (29, 30), in which we get two sets of equations:

- the first pair of equations involving only the field variables ($\vec{E}$, $\vec{K}$) modeled by the system of fractional equations:

$$\partial_t^\alpha \vec{E} + c^2 (\vec{\nabla}\times)^\beta ((\vec{\nabla}\times)^\beta \vec{E}) + \omega_{pe}^2 \vec{E} = -c^2 (\vec{\nabla}\times)^\beta \vec{K} \tag{33}$$

$$\partial_t^\alpha \vec{K} + \omega_{pm}^{2\alpha} \vec{K} = -\omega_{pm}^{2\alpha} (\vec{\nabla}\times)^\beta \vec{E} \tag{34}$$

- the second ones involving the field variables ($\vec{H}$, $\vec{J}$) modeled by the system of fractional equations:

$$\partial_t^\alpha \vec{H} + c^2 (\vec{\nabla}\times)^\beta ((\vec{\nabla}\times)^\beta \vec{E}) + \omega_{pm}^{2\alpha} \vec{H} = c^2 (\vec{\nabla}\times)^\beta \vec{J} \tag{35}$$

$$\partial_t^\alpha \vec{J} + \omega_{pe}^{2\alpha} \vec{J} = \omega_{pe}^{2\alpha} (\vec{\nabla}\times)^\beta \vec{H} \tag{36}$$

We note that we derived these systems of decupled equations through the divergence conditions.

3. Summary
In recent years researchers have reported some nonlocal effects in electromagnetic metamaterials. In this work using the frame work of fractional dynamics we present a new fractional Drude metamaterial model which can play important role in the future research in this field. In this study we just try to derive the systems of equations which can be solved analytically or numerically by using different methods.

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