ON A CERTAIN INEQUALITY ON ADDITION CHAINS

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Abstract. In this paper we prove that there exists an addition chain producing $2^n - 1$ of length $\delta(2^n - 1)$ satisfying the inequality

$$\delta(2^n - 1) \leq 2n - 1 - 2\left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right\rfloor + \lfloor \frac{\log n}{\log 2} \rfloor$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

1. Introduction

An addition chain producing $n \geq 3$, roughly speaking, is a sequence of numbers of the form $1, 2, s_3, s_4, \ldots, s_{k-1}, s_k = n$ where each term is the sum of two earlier terms in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The number of terms in the sequence excluding n is the length of the chain. There are quite a number of addition chains producing a fixed number n. Among them the shortest is regarded as the shortest or optimal addition chain producing n. Nonetheless minimizing an addition chain can be an arduous endeavour, given that there are currently no efficient method for obtaining the shortest addition producing a given number. This makes the theory of addition chains an interesting subject to study. By letting $\iota(n)$ denotes the length of the shortest addition chain producing n, Arnold scholz conjectured the inequality

Conjecture 1.1 (Scholz)**.** The inequality holds

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

It has been shown computationally that the conjecture holds for all $n \leq 5784688$ and in fact it is an equality for all $n \leq 64$ [2]. Alfred Brauer proved the scholz conjecture for the star addition chain, an addition chain where each term obtained by summing uses the immediately subsequent number in the chain. By denoting the shortest length of the star addition chain by $\iota^*(n)$, it is shown that (See,[1])

Theorem 1.1. *The inequality holds*

$$\iota^*(2^n - 1) \leq n - 1 + \iota^*(n).$$

In this paper we study short addition chains producing numbers of the form $2^n - 1$ and the scholz conjecture. We adopt the method of **backtracking** to obtain an explicit upper bound for an addition chain producing $2^n - 1$.

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2. **Sub-addition chains**

In this section we introduce the notion of sub-addition chains.

Definition 2.1. Let $n \geq 3$, then by the addition chain of length $k-1$ producing n we mean the sequence

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term s_j $(j \geq 3)$ in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

with $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the i th **generator** of the chain for $2 \leq i \leq k$. We call a_i the **determiners** and r_i the **regulator** of the i th generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2 \leq i \leq k$.

Definition 2.2. Let the sequence $1, 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing n with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then we call the sub-sequence (s_{j_m}) for $1 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a **sub-addition** chain of the addition chain producing n. We say it is **complete** sub-addition chain of the addition chain producing n if it contains exactly the first t terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

3. **Addition chains of numbers of special forms**

In this section we study addition chains of numbers of special forms. We examine ways of minimizing the length of addition chains for numbers of the form $2^n - 1$.

Theorem 3.1. *There exists an addition chain producing $2^n - 1$ of length $\delta(2^n - 1)$ satisfying the inequality*

$$\delta(2^n - 1) \leq 2n - 1 - 2\left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right\rfloor + \lfloor \frac{\log n}{\log 2} \rfloor$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. First we construct the sequence

$$2^{n-1} + 2^{n-2} + \cdots + 2^{\lfloor \frac{n-1}{2} \rfloor} = 2^n - 2^{\lfloor \frac{n-1}{2} \rfloor}$$

$$2^{n-1} - 2^{\lfloor \frac{n-1}{2} \rfloor} + (2^{\lfloor \frac{n-1}{2} \rfloor - 1} + \cdots + 2^{\lfloor \frac{n-1}{2^2} \rfloor}) = 2^n - 2^{\lfloor \frac{n-1}{2^2} \rfloor}$$

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$$2^{n-1} + 2^{n-2} + \cdots + 2^{\lfloor \frac{n-1}{2^{k-1}} \rfloor - 1} + \cdots + 2^{\lfloor \frac{n-1}{2^k} \rfloor} = 2^n - 2^{\lfloor \frac{n-1}{2^k} \rfloor}.$$

We note that there are at most

$$\lfloor \frac{\log n}{\log 2} \rfloor$$

terms in this sequence. We consider the following regulators $2^{n-1}-2^{\lfloor \frac{n-1}{2} \rfloor}, 2^{\lfloor \frac{n-1}{2} \rfloor}-2^{\lfloor \frac{n-1}{2^2} \rfloor}, \ldots, 2^{\lfloor \frac{n-1}{2^{k-1}} \rfloor}-2^{\lfloor \frac{n-1}{2^k} \rfloor}$. We also note that there are at most

$$\lfloor \frac{\log n}{\log 2} \rfloor$$

such regulators. Next we adjoin it to the previously constructed sequence and we note that it contributes at most

$$\lfloor \frac{\log n}{\log 2} \rfloor$$

terms in this sequence. Next we examine how these regulators are produced. We note that we can write

$$2^{n-1}-2^{\lfloor \frac{n-1}{2} \rfloor} = \left(2^{\lfloor \frac{n-1}{2} \rfloor}-2^{\lfloor \frac{n-1}{2^2} \rfloor}\right)+\left(2^{n-1}+2^{\lfloor \frac{n-1}{2^2} \rfloor}-2^{\lfloor \frac{n-1}{2} \rfloor+1}\right)$$

We note that we can recast the latter term as

$$2^{n-1}+2^{\lfloor \frac{n-1}{2^2} \rfloor}-2^{\lfloor \frac{n-1}{2} \rfloor+1} = 2^{n-2}+\cdots+2^{\lfloor \frac{n-1}{2} \rfloor+1}+2^{\lfloor \frac{n-1}{2^2} \rfloor}. \tag{3.1}$$

It is observed that there are $n-1-\lfloor \frac{n-1}{2} \rfloor$ such distinct terms in (3.1). We see that these terms generate

$$n-2-\lfloor \frac{n-1}{2} \rfloor \tag{3.2}$$

distinct sub-addition chains. Let us include $n-1-\lfloor \frac{n-1}{2} \rfloor$ such distinct terms in (3.1) and corresponding

$$n-2-\lfloor \frac{n-1}{2} \rfloor \tag{3.3}$$

terms in the sub-addition chain they produce into the a priori constructed sequence. Again, we can write

$$2^{\lfloor \frac{n-1}{2} \rfloor}-2^{\lfloor \frac{n-1}{2^2} \rfloor} = \left(2^{\lfloor \frac{n-1}{2^2} \rfloor}-2^{\lfloor \frac{n-1}{2^3} \rfloor}\right)+\left(2^{\lfloor \frac{n-1}{2} \rfloor}+2^{\lfloor \frac{n-1}{2^3} \rfloor}-2^{\lfloor \frac{n-1}{2^2} \rfloor+1}\right).$$

We note that we can recast the latter term as

$$2^{\lfloor \frac{n-1}{2} \rfloor}+2^{\lfloor \frac{n-1}{2^3} \rfloor}-2^{\lfloor \frac{n-1}{2^2} \rfloor+1} = 2^{\lfloor \frac{n-1}{2} \rfloor-1}+\cdots+2^{\lfloor \frac{n-1}{2^2} \rfloor+1}+2^{\lfloor \frac{n-1}{2^3} \rfloor}.$$

We note that there are

$$\lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2^2} \rfloor$$

such distinct terms in the sum. It is observed that these terms generate

$$\lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2^2} \rfloor - 1$$

distinct terms of a sub-addition chain. Let us include the

$$\lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2^2} \rfloor$$

such distinct terms in the sum and their corresponding

$$\lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2^2} \rfloor - 1$$

induced sums, which forms a sub-addition chain into the previously constructed sequence. By iterating, we obtain

$$2^{\lfloor \frac{n-1}{2^{k-2}}\rfloor}-2^{\lfloor \frac{n-1}{2^{k-1}}\rfloor}=2^{\lfloor \frac{n-1}{2^{k-1}}\rfloor}-2^{\lfloor \frac{n-1}{2^{k}}\rfloor}+\bigg(2^{\lfloor \frac{n-1}{2^{k-2}}\rfloor}+2^{\lfloor \frac{n-1}{2^{k}}\rfloor}-2^{\lfloor \frac{n-1}{2^{k-1}}\rfloor+1}\bigg).$$

We note that the latter term can be recast as

$$2^{\lfloor \frac{n-1}{2^{k-2}}\rfloor}+2^{\lfloor \frac{n-1}{2^{k}}\rfloor}-2^{\lfloor \frac{n-1}{2^{k-1}}\rfloor+1}=2^{\lfloor \frac{n-1}{2^{k-2}-1}\rfloor}+\cdots+2^{\lfloor \frac{n-1}{2^{k-1}}\rfloor+1}+2^{\lfloor \frac{n-1}{2^{k}}\rfloor}$$

We note that there are

$$\lfloor \frac{n-1}{2^{k-2}}\rfloor-\lfloor \frac{n-1}{2^{k-1}}\rfloor$$

such distinct terms in the sum. It is observed that these terms generate

$$\lfloor \frac{n-1}{2^{k-2}}\rfloor-\lfloor \frac{n-1}{2^{k-1}}\rfloor-1$$

distinct terms of a sub-addition chain. Let us include the

$$\lfloor \frac{n-1}{2^{k-2}}\rfloor-\lfloor \frac{n-1}{2^{k-1}}\rfloor$$

such distinct terms in the sum and their corresponding

$$\lfloor \frac{n-1}{2^{k-2}}\rfloor-\lfloor \frac{n-1}{2^{k-1}}\rfloor-1$$

induced sums, which forms a sub-addition chain into the previously constructed sequence. In particular, we may obtain the total contribution by adding the numbers in the following chains

$$n-1-\lfloor \frac{n-1}{2}\rfloor$$

$$\lfloor \frac{n-1}{2}\rfloor-\lfloor \frac{n-1}{2^2}\rfloor$$

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$$\lfloor \frac{n-1}{2^{k-1}}\rfloor-\lfloor \frac{n-1}{2^{k}}\rfloor.$$

Since there are at most $\lfloor \frac{\log n}{\log 2}\rfloor$ regulators of the form $2^{\lfloor \frac{n-1}{2^{k-1}}\rfloor}-2^{\lfloor \frac{n-1}{2^{k}}\rfloor}$, it follows the total number terms induced is at most

$$n-1-\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor}}\rfloor$$

obtained by adding the numbers in the chain above. Similarly the corresponding number of terms in the sub-addition induced by these

$$n-1-\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor}}\rfloor$$

terms is at most

$$n-1-\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor}}\rfloor-\lfloor \frac{\log n}{\log 2}\rfloor.$$

Let us introduce the non-existent term $2^{n-1}+2^{n-2}-2^{\lfloor \frac{n-1}{2}\rfloor}$ into the sequence - which can be obtained by adding the term 2^{n-2} in the sequence to the regulator $2^{n-1}-2^{\lfloor \frac{n-1}{2}\rfloor}$ - then we obtain the term $2^{n}-2^{\lfloor \frac{n-1}{2}\rfloor}$ by adding one more time the term 2^{n-2}. It is easy to see that our construction yields an addition chain of length at most

$$\delta(2^n-1)\leq 2(n-1)-2\left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor}}\right\rfloor-\lfloor \frac{\log n}{\log 2}\rfloor+1+2\lfloor \frac{\log n}{\log 2}\rfloor$$

which completes the proof. □

Corollary 3.1. Let $\iota(n)$ denotes the length of the shortest addition chain producing n. Then the inequality holds

$$\iota(2^n-1)\leq 2n-1-2\left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor}}\right\rfloor+\lfloor \frac{\log n}{\log 2}\rfloor$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

4. Comments on a new direction and the backtracking method

The method of **backtracking** adopted in the above construction produces (by default) a relatively short tail - of logarithmic order - of an addition chain producing 2^n-1 and backtracks by constructing required terms of the addition chain. The resulting upper bound for the length of this addition chain may be miles away from what is believed to be the truth

Conjecture 4.1 (Scholz)**.** The inequality holds

$$\iota(2^n-1)\leq n-1+\iota(n).$$

The method of **backtracking** may be refined and trimmed down by consolidating with a new idea, that is not the purpose of this paper. At least it yields an explicit upper bound which was not attainable in our previous investigations.

[1].

References

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