On the equation $f(x)=x^2+e^{-2x}-1=0$

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Abstract

In this note we give solution of the equation $f(x)=x^2+e^{-2x}-1=0$.

Introduction

Recall that

$$\pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\right)$$

The Lagrange inversión theorem, also known as the Lagrange-Burmann formula, gives the Taylor series expansión of the inverse function of an analytic function.
Suppose z is defined as a function of w by an equation of the form

$$z=F(w)$$

Where F is analytic at a point a and $F'(a)\neq 0$. Then it is posible to invert or solve the equation for w, expressing it in the form

$$w=G(z)$$

Given by a power series

$$G(z)=a+\sum_{n=1}^{\infty}\frac{G_n\left(z-F(a)\right)^n}{n!}$$

where

$$G_n=\lim_{w\to 0}\frac{d^{n-1}}{dw^{n-1}}\left[\left(\frac{w-a}{F(w)-F(a)}\right)^n\right]$$

The theorem furter states that this series has a non-zero radius of convergence, i.e., $G(z)$ represents an analytic function of z in a neighbourhood of $z=F(a)$. This is also called reversion of series.

Results

Entry 1.

$$f(x)=x^2+e^{-2x}-1=0\Rightarrow\begin{cases}x=0\\x=\lambda=0.916562...\end{cases}\tag{1}$$

$$x>\lambda\Rightarrow f(x)>0\tag{2}$$

$$x<0\Rightarrow f(x)>0\tag{3}$$

$$0<x<\lambda\Rightarrow f(x)<0\tag{4}$$

Entry 2.

$$\lambda=\sqrt{1-e^{-2\sqrt{1-e^{-2\sqrt{1-...}}}}}\tag{5}$$

$$\lambda=\sqrt{\frac{2}{1+\coth\sqrt{\frac{2}{1+\coth\sqrt{\frac{2}{1+...}}}}}}\tag{6}$$

Entry 3.

$$c_0=1\quad,\quad c_n=-\sum_{k=1}^{n}\left(\frac{2^k}{(k+1)!}-\frac{2^{k-2}}{(k-1)!}\right)c_{n-k}\quad,n=1,2,3,...\tag{7}$$

$$c_n=\left\{1,-\frac{1}{2},\frac{7}{12},\frac{5}{24},\frac{209}{720},\frac{161}{480},\frac{4297}{12096},...\right\}\tag{8}$$

$$\lim_{n\to\infty}\frac{c_n}{c_{n+1}}=\lambda\tag{9}$$

Entry 4.

$$x_0=0\quad,\quad x_{n+1}=\tanh^{-1}\left(e^{-\operatorname{sech}x_n}\right)\Rightarrow\lim_{n\to\infty}x_n=\cosh^{-1}\frac{1}{\lambda}\tag{10}$$

Entry 5.

$$\pi=2\sin^{-1}(\lambda)+2\sin^{-1}\left(e^{-\lambda}\right)\tag{11}$$

$$\pi=2\sin^{-1}\left(\sqrt{1-e^{-2\sqrt{1-e^{-2\sqrt{1-...}}}}}\right)+2\sin^{-1}\left(e^{-\sqrt{1-e^{-2\sqrt{1-e^{-2\sqrt{1-...}}}}}}\right)\tag{12}$$

Entry 6.

$$\pi=4\tan^{-1}(\tanh\lambda)+4\tan^{-1}\left(1-\lambda^2\right)\tag{13}$$

$$\pi=4\tan^{-1}\left(\tanh\left(\sqrt{1-e^{-2\sqrt{1-e^{-2\sqrt{1-...}}}}}\right)\right)+4\tan^{-1}\left(e^{-2\sqrt{1-e^{-2\sqrt{1-e^{-2\sqrt{1-...}}}}}}\right)\tag{14}$$

Entry 7. For $f(x)=x^2+e^{-2x}-1=0$, $i=\sqrt{-1}$,we have

$$2\pi = -\int_0^{2\pi} \frac{e^{ix}}{f\left(e^{ix}/2\right)}\,dx \tag{15}$$

Entry 8. For $g\left(x\right)=\frac{x\left(x-e^{-2x}\right)}{x^2+e^{-2x}-1}$, $i=\sqrt{-1}$,we have

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} g\left(1+\frac{e^{ix}}{2}\right)e^{ix}\,dx = 0.916562... \tag{16}$$

Entry 9.

$$x_0 = 1 \quad , \quad x_{n+1} = \cos\left(\sin^{-1}\left(e^{-x_n}\right)\right) \Rightarrow \lim_{n\to\infty} x_n = \lambda \tag{17}$$

$$x_0 = 1 \quad , \quad x_{n+1} = \sin\left(\cos^{-1}\left(e^{-x_n}\right)\right) \Rightarrow \lim_{n\to\infty} x_n = \lambda \tag{18}$$

Entry 10.

$$x_0 = 0 \quad , \quad x_{n+1} = e^{-2\sqrt{1-x_n}} \Rightarrow \lim_{n\to\infty} x_n = \sqrt{1-\lambda^2} \tag{19}$$

Entry 11.

$$x_0 = 0 \quad , \quad x_{n+1} = \frac{e^{-2+2x_n}}{2-x_n} \Rightarrow \lim_{n\to\infty} x_n = 1-\lambda \tag{20}$$

Entry 12.

$$\lambda = \cos\theta \Rightarrow \cos\theta = W\left(\cot\theta\right) \tag{21}$$

where W is the Lambert W-function: $W\left(z\right)e^{W(z)} = z$.

Remark: $\theta = 0.41139821....$

Entry 13.

$$\lambda = W\left(\frac{2\sinh\lambda}{\lambda}\right) \tag{22}$$

$$x_0 = 1 \quad , \quad x_{n+1} = W\left(\frac{2\sinh x_n}{x_n}\right) \Rightarrow \lim_{n\to\infty} x_n = \lambda \tag{23}$$

where W is the Lambert W-function.

Entry 14.

$$\lambda = W\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\right) \tag{24}$$

where W is the Lambert W-function.

Entry 15. For $f\left(x\right)=x^2+e^{-2x}-1=0$, $i=\sqrt{-1}$,we have

$$\int_0^{2\pi} \frac{e^{ix}}{f\left(1+\frac{e^{ix}}{2}\right)} dx = \frac{2\pi}{\lambda - e^{-2\lambda}} \tag{25}$$

Entry 16. For $f(x) = x^2 + e^{-2x} - 1 = 0$, $i = \sqrt{-1}$,we have

$$\frac{1}{A} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ix}}{f\left(1+\frac{e^{ix}}{2}\right)} dx \Rightarrow \lambda = A + \frac{1}{2}W\left(2e^{-2A}\right) \tag{26}$$

where W is the Lambert W-function.

Entry 17.

$$x_0 = 1 \quad , \quad x_{n+1} = \cosh^{-1}\left(e^{\tanh x_n}\right) \Rightarrow \lim_{n\to\infty} x_n = \tanh^{-1}\lambda \tag{27}$$

Entry 18.

$$x_0 = 1 \quad , \quad x_{n+1} = \tanh\left(\sinh^{-1}\left(x_n e^{x_n}\right)\right) \Rightarrow \lim_{n\to\infty} x_n = \lambda \tag{28}$$

Entry 19.

$$x_0 = 1 \quad , \quad x_{n+1} = \sin\left(\tan^{-1}\left(x_n e^{x_n}\right)\right) \Rightarrow \lim_{n\to\infty} x_n = \lambda \tag{29}$$

Entry 20.

$$\pi = 2\sin^{-1}(\lambda) + 2\tan^{-1}\left(\frac{e^{-\lambda}}{\lambda}\right) \tag{30}$$

Entry 21.

$$\lambda = \tanh\theta \Rightarrow W(\sinh\theta) = \tanh\theta \tag{31}$$

where W is the Lambert W-function.

Entry 22.

$$x_0 = 1 \quad , \quad x_{n+1} = 1 - \frac{e^{-2x_n}}{1+x_n} \Rightarrow \lim_{n\to\infty} x_n = \lambda \tag{32}$$

Entry 23.

$$F(x) = \frac{e^{-2x}}{1+x} \Rightarrow \lambda = 1 - F\left(1 - F\left(1 - F\left(1 - \ldots\right)\right)\right) \tag{33}$$

Entry 24.

$$c_0 = 1 \quad , \quad c_n = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{c_{n-2k+1}}{2k} \quad , n = 1, 2, 3, \ldots \tag{34}$$

$$c_n = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{37}{96}, \frac{71}{192}, \ldots\right\} \tag{35}$$

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lambda \tag{36}$$

Entry 25.

$$\pi = 4\tan^{-1}\left(\lambda^2\right) + 4\sum_{n=0}^{\infty}\left(\frac{e^{-2\lambda}}{2}\right)^{n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n}{n-2k} \tag{37}$$

$$\pi = 4\tan^{-1}\left(e^{-2\lambda}\right) + 4\sum_{n=0}^{\infty}\left(\frac{\lambda^2}{2}\right)^{n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n}{n-2k} \tag{38}$$

Entry 26.

$$\pi = 4\tan^{-1}\left(\sqrt{1-\lambda^2}\right) + 2\sum_{n=0}^{\infty}\frac{(-1)^n E_n\,\lambda^{2n+1}}{(2n+1)!} \tag{39}$$

where

$$\left\{E_n : n = 0,1,2,3,\ldots\right\} = \left\{1,1,5,61,1385,50521,\ldots\right\} \tag{40}$$

Remark: E_n , Euler numbers.

Entry 27. $i = \sqrt{-1}$

$$\lambda = 1 - \frac{1}{2\pi}\int_0^{2\pi}\frac{\left(2e^{e^{ix}} + e^{2+ix} - 2e^2\right)e^{2ix}}{4e^{e^{ix}} - 4e^{2+ix} + e^{2+2ix}}\,dx \tag{41}$$

Entry 28.

$$\lambda = 1 - \sum_{n=0}^{\infty}\frac{c_n\,e^{-2n-2}}{(n+1)!} \tag{42}$$

where

$$c_n = \sum_{k=0}^{n}\frac{(n+k)!(n+1)^{n-k}}{(n-k)!k!2^{2k+1}} \quad , n = 0,1,2,3,\ldots \tag{43}$$

References

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