The Clifford-Yang Algebra, Noncommutative Clifford Phase Spaces and the Deformed Quantum Oscillator

Carlos Castro Perelman
Ronin Institute, 127 Haddon Place, Montclair, N.J. 07043, USA
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314; perelmanc@hotmail.com

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Abstract

Starting with a brief review of our prior construction of n-ary algebras in noncommutative Clifford spaces, we proceed to construct in full detail the Clifford-Yang algebra which is an extension of the Yang algebra in noncommutative phase spaces. The Clifford-Yang algebra allows to write down the commutators of the *noncommutative* polyvector-valued coordinates and momenta and which are compatible with the Jacobi identities, the Weyl-Heisenberg algebra, and paves the way for a formulation of Quantum Mechanics in Noncommutative Clifford spaces. We continue with a detail study of the isotropic $3D$ quantum oscillator in *noncommutative* spaces and find the energy eigenvalues and eigenfunctions. These findings differ considerably from the ordinary quantum oscillator in commutative spaces. We find that QM in noncommutative spaces leads to very different solutions, eigenvalues, and uncertainty relations than ordinary QM in commutative spaces. The generalization of QM to noncommutative Clifford (phase) spaces is attained via the Clifford-Yang algebra. The operators are now given by the generalized angular momentum operators involving polyvector coordinates and momenta. The eigenfunctions (wave functions) are now more complicated functions of the polyvector coordinates. We conclude with some important remarks.

Keywords : Strings; Branes; Clifford algebras; n-ary algebras; Noncommutative Geometry.

1 Introduction : Noncommutative Clifford Space Coordinates and n-ary Algebras

After decades of string theory research its physical foundation is still unknown and the question *what* is string theory remains unanswered. General relativity is based on the principle of equivalence and general coordinate covariance. It is desirable to decipher the principle governing string theory. We have learned that string theory not only involves one-dimensional extended objects but higher dimensional ones, p and D-branes. Furthemore, the quantization of membranes and higher dimensional extended objects has been extremely difficult due to the intrinsic nonlinearity. The aim of this work is an attempt to bridge these conceptual obstacles by introducing Clifford spaces (C-spaces) [1].

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [1] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds. Namely, C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing [1].

Given $\mathbf{X} = X_M \Gamma^M$, a Clifford-valued coordinate associated to Clifford space (C-space), it admits the following expansion in terms of the Clifford algebra generators in D-dimensions : $\mathbf{1}, \gamma^\mu, \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \cdots, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_D}$

$$\mathbf{X} \ = \ x \ \mathbf{1} \ + \ x_\mu \ \gamma^\mu \ + \ x_{\mu_1\mu_2} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \ + \ x_{\mu_1\mu_2\mu_3} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \ + \ ...... \ +$$
$$x_{\mu_1\mu_2\mu_3......\mu_D} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \ \gamma^{\mu_3}....... \ \wedge \gamma^{\mu_D} \tag{1.1}$$

The numerical combinatorial factors can be omitted by imposing the ordering prescription $\mu_1 < \mu_2 < \mu_3 \cdots < \mu_D$. In order to match physical units in each term of (1.1) a length scale parameter must be suitably introduced in the expansion in eq-(1.1). In [1] we introduced the Planck scale as the expansion parameter in (1.1), and which was set to unity, when one adopts the units $\hbar = c = G = 1$.

The commuting scalar, vectorial, antisymmetric tensorial coordinates x, x_μ, $x_{\mu_1\mu_2} = -x_{\mu_2\mu_1}, \cdots, x_{\mu_1\mu_2\cdots\mu_D}$ are the scalar, vector, bivector, trivector, $\cdots$ components of the polyvector-valued coordinates in C-space. The $x_{\mu_1\mu_2}$ bivector (antisymmetric tensor of rank 2) corresponds to an oriented area element. The trivector $x_{\mu_1\mu_2\mu_3}$ (antisymmetric tensor of rank 3) corresponds to an oriented volume element, and so forth.

A *noncommutative* extension of these polyvector-valued coordinates was developed in [3]. In this introduction, we briefly review such procedure to prepare the groundwork for the construction of the Clifford-Yang algebra and its relevance in noncommutative Clifford phase spaces involving polyvector coordinates and momenta.

We begin firstly by writing the commutators $[\Gamma_A, \Gamma_B]$. For $pq = odd$ one has [2]

$$[\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ ]\ =\ 2\gamma_{b_1 b_2 .....b_p}^{a_1 a_2 ......a_q}\ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]}\ +\ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 ....a_4}_{[b_1 ....b_4}\ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]}\ -\ ...... \tag{1.2}$$

for $pq = even$ one has

$$[\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ ]\ =\ -\ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]}\ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1 ....a_3}_{[b_1 ....b_3}\ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]}\ +\ ...... \tag{1.3}$$

The anti-commutators for $pq = even$ are

$$\{\ \gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}\ \}\ =\ 2\gamma_{b_1 b_2 .....b_p}^{a_1 a_2 ......a_q}\ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]}\ +\ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 ....a_4}_{[b_1 ....b_4}\ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]}\ -\ ...... \tag{1.4}$$

and the anti-commutators for $pq = odd$ are

$$\{\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ \}\ =\ -\ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]}\ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1 ....a_3}_{[b_1 ....b_3}\ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]}\ +\ ...... \tag{1.5}$$

Let us write down the *noncommutative* algebra associated with the noncommuting polyvector-valued coordinates in $D = 4$ and which can be obtained from the Clifford algebra by performing the following replacements (and relabeling indices)

$$\gamma^\mu \leftrightarrow X^\mu, \quad \gamma^{\mu_1 \mu_2} \leftrightarrow X^{\mu_1 \mu_2}, \quad ....\ \gamma^{\mu_1 \mu_2 .....\mu_n} \leftrightarrow X^{\mu_1 \mu_2 ....\mu_n}. \tag{1.6}$$

When the spacetime metric components $g_{\mu\nu}$ are *constant*, from the replacements (1.6), and using the Clifford algebraic relations (1.2-1.5) (after one relabels indices), one can then construct the following *noncommutative* algebra among the polyvector-valued coordinates in $D = 4$, and *obeying* the Jacobi identities, given by the relations [3]

$$[\ X^{\mu_1},\ X^{\mu_2}\ ]\ =\ X^{\mu_1}\ X^{\mu_2}\ -\ X^{\mu_2}\ X^{\mu_1}\ =\ 2\ X^{\mu_1 \mu_2}. \tag{1.7}$$

As mentioned above, in most of the remaining commutators a suitable length scale parameter must be introduced in order to match units. We shall set this length scale (let us say the Planck scale) to *unity*. Secondly, by choosing the C-space coordinates to behave like anti-Hermitian operators we avoid the need to introduce i factors in the right hand side of (1.7), since the commutator of two anti-Hermitian operators is anti-Hermitian.

The other commutators are

$$[\ X^{\mu_1\mu_2},\ X^{\nu}\ ]\ =\ 4\ (\ g^{\mu_2\nu}\ X^{\mu_1}\ -\ g^{\mu_1\nu}\ X^{\mu_2}\ ). \qquad (1.8)$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu}\ ]\ =\ 2\ X^{\mu_1\mu_2\mu_3\nu},\quad [\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu}\ ]\ =\ -8\ g^{\mu_1\nu}\ X^{\mu_2\mu_3\mu_4}\ \pm ...... \qquad (1.9)$$

$$[\ X^{\mu_1\mu_2},\ X^{\nu_1\nu_2}\ ]\ =\ -8\ g^{\mu_1\nu_1}\ X^{\mu_2\nu_2}\ +\ 8\ g^{\mu_1\nu_2}\ X^{\mu_2\nu_1}\ + \\ 8\ g^{\mu_2\nu_1}\ X^{\mu_1\nu_2}\ -\ 8\ g^{\mu_2\nu_2}\ X^{\mu_1\nu_1}. \qquad (1.10)$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2}\ ]\ =\ 12\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\nu_2}\ \pm\ .... \qquad (1.11)$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}\ ]\ =\ -36\ G^{\mu_1\mu_2\ \nu_1\nu_2}\ X^{\mu_3\nu_3}\ \pm\ ...... \qquad (1.12)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]\ =\ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ \pm ...... \qquad (1.13)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]\ =\ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ +\ 16\ g^{\mu_1\nu_2}\ X^{\mu_2\mu_3\mu_4\nu_1}\ -\ .... \qquad (1.14)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3}\ ]\ =\ 48\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4}\ -\ 48\ G^{\mu_1\mu_2\mu_4\ \nu_1\nu_2\nu_3}\ X^{\mu_3}\ +\ ..... \qquad (1.15)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3\nu_4}\ ]\ =\ 192\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ -\ .... \qquad (1.16)$$

where

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}\ =\ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_n\nu_n}\ +\ \textit{signed permutations} \qquad (1.17a)$$

etc......The metric components $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ in C-space can also be written as a determinant of the $n \times n$ matrix $\mathbf{G}$ whose entries are $g^{\mu_I\nu_J}$

$$det\ \mathbf{G}_{n\times n}\ \equiv\ \frac{1}{n!}\ \epsilon_{i_1 i_2.....i_n}\ \epsilon_{j_1 j_2....j_n}\ g^{\mu_{i_1}\nu_{j_1}}\ g^{\mu_{i_2}\nu_{j_2}}\ ....\ g^{\mu_{i_n}\nu_{j_n}}. \qquad (1.17b)$$

$i_1, i_2, ..., i_n \subset I = 1, 2, ..., D$ and $j_1, j_2, ..., j_n \subset J = 1, 2, ..., D$. One must also include in the C-space metric G^{MN} the (Clifford) scalar-scalar component

G^{00} (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component $G^{\mu_1\mu_2.....\mu_D\ \nu_1\nu_2......\nu_D}$ (that could be related to the axion field).

One must emphasize that when the spacetime metric components $g_{\mu\nu}$ are *no* longer *constant*, the noncommutative algebra among the polyvector-valued coordinates in $D = 4$, does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background $g_{\mu\nu} = \eta_{\mu\nu}$.

N-ary algebras have been known for some time since Nambu introduced his bracket (a Jacobian) in the study of branes and the generalizations of Hamiltonian mechanics based on Poisson brackets. We shall recall next [3] how polyvector valued coordinates admit a very natural interpretation in terms of $n-$ary commutators of vector-valued coordinates.

The ternary commutator for noncommuting coordinates is defined as

$$[X^1, X^2, X^3] \ = \ X^1\ [X^2, X^3] \ + \ X^2\ [X^3, X^1] \ + \ X^3\ [X^1, X^2] \ =$$

$$\frac{1}{2}\ \{\ X^1,\ [X^2, X^3]\ \}\ +\ \frac{1}{2}\ [\ X^1,\ [X^2, X^3]\ ]\ +\ cyclic\ permutations \qquad (1.18)$$

Due to the Jacobi identities, the terms

$$\frac{1}{2}\ [\ X^1,\ [X^2, X^3]\ ]\ +\ cyclic\ permutations\ =\ 0. \qquad (1.19)$$

so that the ternary commutators become

$$[X^1, X^2, X^3]\ =\ \frac{1}{2}\ \{\ X^1,\ [X^2, X^3]\ \}\ +\ cyclic\ permutations. \qquad (1.20)$$

After using the relations

$$[X^2, X^3]\ =\ 2\ X^{23}, \quad \{\ X^1,\ X^{23}\ \}\ =\ 2\ X^{123}. \qquad (1.21)$$

one gets finally

$$[X^1, X^2, X^3]\ =\ 2\ X^{123}\ +\ cyclic\ permutations\ =\ 6\ X^{123}. \qquad (1.22)$$

since $X^{123} = X^{231} = X^{312} = -X^{132} = ...$

After using the above noncommutative algebraic relations, after some laborious but straightforward algebra, one arrives by recursion at the most general n-ary commutator given by

$$[\ X^1,\ X^2,\ ....,\ X^n\ ]\ =\ n!\ X^{123.....n}. \qquad (1.23)$$

for all $n = 2, 3, \cdots, D$ [3].

The immediate consequence of the n-ary algebra of the noncommutative polyvector-valued coordinates, associated with a quantum extension of the classical C-space, is that one must extend the usual formulation of Quantum Mechanics involving ordinary commutators of operators to one requiring n-ary commutators. In other words, quantizing the classical Nambu-Poisson mechanics [7].

The momentum analog of eq-(1.23) is

$$[\ P^1,\ P^2,\ ....,\ P^n\ ]\ =\ n!\ P^{123.....n}. \tag{1.24}$$

where $P^{123.....n}$ is the polyvector-valued momentum conjugate to the polyvector-valued coordinate $X^{123.....n}$ in C-space. However, if one wishes to implement a Weyl-Heisenberg algebra among the polyvector coordinates and momenta one runs into difficulties if one must satisfy the Jacobi identities, and the generalized Jacobi identities (Nambu fundamental identities) associated with the n-ary brackets (commutators).

For example, if one has the Jacobi identity

$$[\ P^{12},\ [X^1, X^2]\ ]\ +\ [\ X^1,\ [X^2, P^{12}]\ ]\ +\ [\ X^2,\ [P^{12}, X^1]\ ]\ =\ 0 \tag{1.25}$$

one would arrive at $[P^{12}, [X^1, X^2]] = 2[P^{12}, X^{12}] = 0$, when $[X^2, P^{12}] = [P^{12}, X^1] = 0$, which is problematic since P^{12} is the bivector canonical conjugate momentum to the bivector coordinate X^{12}, and hence it should *not* have a vanishing commutator. One would have to $modify$ the Weyl-Heisenberg algebra of all the coordinates and momenta accordingly in order to satisfy all the (generalized) Jacobi identities, which is a very difficult task.

For these reasons in the next section we shall follow a different route and construct what we coin the Clifford-Yang algebra that does not have these problems. It allows to write down the commutators of the *noncommutative* polyvector coordinates and momenta which are compatible with the Jacobi identities, and the Weyl-Heisenberg algebra, and paves the way for a formulation of Quantum Mechanics in Noncommutative Clifford spaces.

In section **3** we study in detail the isotropic $3D$ quantum oscillator in *noncommutative* spaces and find the energy eigenvalues and eigenfunctions. These findings differ considerably from the ordinary quantum oscillator in commutative spaces. QM in noncommutative spaces leads to very different solutions, eigenvalues, and uncertainty relations than ordinary QM in commutative spaces. The generalization of QM in noncommutative (phase) spaces to noncommutative Clifford (phase) spaces is attained via the Clifford-Yang algebra described in section **2**. The operators are given by the generalized angular momentum operators involving polyvector coordinates and momenta. The eigenfunctions (wave functions) are now functions of the polyvector coordinates. We conclude with some final remarks.

2 The Clifford-Yang Algebra and Noncommutative Clifford Phase Spaces

The idea of a Quantum Spacetime where the spacetime coordinates do not commute was proposed early on by Heisenberg and Ivanenko as a way to eliminate infinities from Quantum Field Theory. Snyder published the first concrete example [5] of a noncommutative algebra involving the spacetime coordinates, and it was generalized shortly after by Yang [6], to include noncommuting momentum variables as well. We learnt from General Relativity that the Poincare algebra cannot be implemented on a curved spacetime, but only on its flat tangent space (Minkowski spacetime). The momentum operators don't commute on a curved spacetime. And vice versa, by Born's principle of reciprocity [12], the coordinate operators do not commute on a curved *momentum* space. This prompted the formulation of Quantum Mechanics and Quantum Field Theory in Noncommutative spacetimes (also called Noncommutative QFT), and which might cast some light in the formulation of Quantum Gravity by encoding both key aspects of a curved and a noncommuting spacetime (a curved noncommuting spacetime).

In [13] we suggested that Born's Reciprocal Relativity Theory in Phase spaces is the arena to implement a space-time-matter unification. More precisely : quantum matter curves noncommuting spacetime, and vice versa, noncommuting spacetime curves quantum matter (quantum momentum space) as a result of the back-reaction of quantum spacetime on quantum matter. We believe that it is this Born's reciprocity principle that holds important clues to quantize gravity (geometry) in curved phase spaces within the context of Finsler geometry.

It was shown in [14] that the radial spectrum associated with a fuzzy sphere in a *noncommutative* phase space characterized by the Yang algebra, leads *exactly* to a Regge-like spectrum $GM_l^2 = l = 1, 2, 3, \ldots$, for *all* positive values of the angular momentum l, and which is consistent with the extremal quantum Kerr black hole solution that occurs when the outer and inner horizon radius coincide $r_+ = r_- = GM$.

Given a flat $6D$ spacetime with coordinates $Y^A = \{Y^1, Y^2, Y^3, Y^4, Y^5, Y^6\}$, and a metric $\eta_{AB} = diag(-1, +1, +1, \ldots, +1)$, the Yang algebra [6] can be derived in terms of the $so(5,1)$ Lorentz algebra generators described by the angular momentum/boost operators

$$J^{AB} = -(Y^A\ \Pi^B\ -\ Y^B\ \Pi^A)\ =\ i\ Y^A \frac{\partial}{\partial Y_B}\ -i\ Y^B \frac{\partial}{\partial Y_A} \tag{2.1}$$

where $\Pi^A = -i(\partial/\partial Y_A)$ is the conjugate momentum variable to Y^A. Their commutators are

$$[Y^A, Y^B]\ =\ 0,\ [\Pi^A, \Pi^B]\ =\ 0,\ [Y^A, \Pi^B]\ =\ i\,\eta^{AB},\ A, B = 1, 2, 3, 4, 5, 6 \tag{2.2}$$

The coordinates Y^A commute. The momenta Π^A also commute, and Y^A, Π^B obey the Weyl-Heisenberg algebra in $6D$.

Adopting the units $\hbar = c = 1$, the correspondence among the noncommuting $4D$ spacetime coordinates X^μ, the noncommuting momenta P^μ, and the Lorentz $so(5,1)$ algebra generators leading to the Yang algebra [6] is given by

$$X^\mu \ \leftrightarrow\ L_P\ J^{\mu 5}\ =\ -\,L_P\ (Y^\mu\ \Pi^5\ -\ Y^5\ \Pi^\mu)$$

$$P^\mu\ \leftrightarrow\ \frac{1}{\mathcal{L}}\ J^{\mu 6}\ =\ -\,\frac{1}{\mathcal{L}}\ (Y^\mu\ \Pi^6\ -\ Y^6\ \Pi^\mu),\quad \mu,\nu = 1,2,3,4 \tag{2.3}$$

and which requires the introduction of an ultra-violet cutoff scale L_P given by the Planck scale, and an infra-red cutoff scale $\mathcal{L}$ that can be set equal to the Hubble scale R_H (which determines the cosmological constant). It is very important to emphasize that despite the introduction of two length scales $L_P, \mathcal{L}$ the Lorentz symmetry is not lost. This is one of the most salient features of the Snyder [5] and Yang [6] algebras.

One must include also the remaining $so(5,1)$ generators

$$\mathcal{N}\ \equiv\ J^{56} = -(Y^5\,\Pi^6 - Y^6\,\Pi^5),\ \ J^{\mu\nu}\ =\ -(Y^\mu\,\Pi^\nu - Y^\nu\,\Pi^\mu),\ \ \mu,\nu = 1,2,3,4 \tag{2.4}$$

One can then verify that the Yang algebra is recovered after imposing the above correspondence (2.3)

$$[X^\mu,\ X^\nu]\ =\ -\,i\ L_P^2\ J^{\mu\nu},\ \ [P^\mu,\ P^\nu]\ =\ -\,i\ (\frac{1}{\mathcal{L}})^2\ J^{\mu\nu},\ \ \eta^{55}\ =\ \eta^{66} = 1 \tag{2.5}$$

$$[X^\mu,\ J^{\nu\rho}]\ =\ i\ (\eta^{\mu\rho}\ X^\nu\ -\ \eta^{\mu\nu}\ X^\rho) \tag{2.6}$$

$$[P^\mu,\ J^{\nu\rho}]\ =\ i\ (\eta^{\mu\rho}\ P^\nu\ -\ \eta^{\mu\nu}\ P^\rho\ ) \tag{2.7}$$

$$[X^\mu,\ P^\nu]\ =\ -\,i\ \eta^{\mu\nu}\ \frac{L_P}{\mathcal{L}}\ \mathcal{N},\ \ [J^{\mu\nu},\mathcal{N}] = 0 \tag{2.8}$$

$$[X^\mu,\ \mathcal{N}]\ =\ i\ L_P\mathcal{L}\ P^\mu,\ \ [P^\mu,\ \mathcal{N}]\ =\ -\,i\ \frac{1}{L_P\mathcal{L}}\ X^\mu \tag{2.9}$$

and where the $[J^{\mu\nu}, J^{\rho\sigma}]$ commutators are the same as in the $so(3,1)$ Lorentz algebra in $4D$. They are of the form

$$[\ J^{\mu_1\mu_2},\ J^{\nu_1\nu_2}\ ]\ =\ -\,i\ \eta^{\mu_1\nu_1}\ J^{\mu_2\nu_2}\ +i\ \eta^{\mu_1\nu_2}\ J^{\mu_2\nu_1}\ +$$
$$i\ \eta^{\mu_2\nu_1}\ J^{\mu_1\nu_2}\ -i\ \eta^{\mu_2\nu_2}\ J^{\mu_1\nu_1},\ \ \hbar = c = 1 \tag{2.10}$$

The generators are assigned to be Hermitian so there are i factors in the right-hand side of eq-(2.10) since the commutator of two Hermitian operators is anti-Hermitian. The $4D$ spacetime metric is $\eta_{\mu\nu} = diag(-1,1,1,1)$.

Given the above correspondence (2.3), one can *extend* it further to the higher grade polyvector coordinates and momenta as follows

$$X^{\mu\nu}\ \leftrightarrow\ L_P^2\ J^{\mu\nu 5},\ \ J^{\mu\nu 5}\ \equiv\ -\,(Y^{\mu\nu}\ \Pi^5\ -\ Y^5\ \Pi^{\mu\nu}) \tag{2.11}$$

$$P^{\mu\nu}\ \leftrightarrow\ \frac{1}{\mathcal{L}^2}\ J^{\mu\nu 6},\ \ J^{\mu\nu 6}\ \equiv\ -\,(Y^{\mu\nu}\ \Pi^6\ -\ Y^6\ \Pi^{\mu\nu}) \tag{2.12}$$

$$X^{\mu\nu\rho}\ \leftrightarrow\ L_P^3\ J^{\mu\nu\rho 5},\ \ J^{\mu\nu\rho 5}\ \equiv\ -\,(Y^{\mu\nu\rho}\ \Pi^5\ -\ Y^5\ \Pi^{\mu\nu\rho}) \tag{2.13}$$

$$P^{\mu\nu\rho} \leftrightarrow \frac{1}{\mathcal{L}^3} J^{\mu\nu\rho 6}, \quad J^{\mu\nu\rho 6} \equiv -(Y^{\mu\nu\rho} \Pi^6 - Y^6 \Pi^{\mu\nu\rho}) \tag{2.14}$$

$$X^{\mu\nu\rho\tau} \leftrightarrow L_P^4 J^{\mu\nu\rho\tau 5}, \quad J^{\mu\nu\rho\tau 5} \equiv -(Y^{\mu\nu\rho\tau} \Pi^5 - Y^5 \Pi^{\mu\nu\rho\tau}) \tag{2.15}$$

$$P^{\mu\nu\rho\tau} \leftrightarrow \frac{1}{\mathcal{L}^4} J^{\mu\nu\rho\tau 6}, \quad J^{\mu\nu\rho\tau 6} \equiv -(Y^{\mu\nu\rho\tau} \Pi^6 - Y^6 \Pi^{\mu\nu\rho\tau}) \tag{2.16}$$

The correspondence in eqs-(2.11-2.16) is just the natural extension of the correspondence in eq-(2.3). Working with *dimensionless* generalized angular momenta allows to simply match the physical units in eqs-(2.11-2.16) by introducing numerical factors involving suitable powers of $L_P, \mathcal{L}$ as shown.

For example, the way to have a *dimensionless* $J^{\mu\nu 5} = -(Y^{\mu\nu} \Pi^5 - Y^5 \Pi^{\mu\nu})$ is by inserting suitable powers of a length λ_l and momentum scale λ_p parameter as follows $J^{\mu\nu 5} = -[(\lambda_l)^{-1} Y^{\mu\nu} \Pi^5 - Y^5 \Pi^{\mu\nu}(\lambda_p)^{-1})]$. Another example is $J^{\mu\nu\rho 5} = -[(\lambda_l)^{-2} Y^{\mu\nu\rho} \Pi^5 - Y^5 \Pi^{\mu\nu\rho}(\lambda_p)^{-2}]$, and so forth. By setting the parameters $\lambda_l = 1$ and $\lambda_p = 1$, it won't be necessary to explicitly write them down in all of our expressions. λ_l can be identified with the Planck length L_P, and λ_p with the Planck mass in units of $\hbar = c = G = 1$. However, we shall retain L_P explicitly in our fundamental expressions as a book-keeping device.

From the correspondence in eqs-(2.3, 2.11-2.16) one can then read-off the relevant commutators. For convenience, we shall *omit* the *signs* in the right hand side of most of the commutators. One can always incorporate the signs from the defining generalized angular momentum algebra in the $6D$ space. Therefore, the commutators involving the *noncommuting* polyvector coordinates and momenta of the Clifford phase space associated with the Clifford algebra in $4D$ spacetime are then given by

$$[X^\mu, X^\nu] \sim i L_P^2 J^{\mu\nu}, \quad [P^\mu, P^\nu] \sim i \mathcal{L}^{-2} J^{\mu\nu}, \tag{2.17}$$

The *noncommuting* bivector coordinates obey

$$[X^{\mu_1\mu_2}, X^{\nu_1\nu_2}] \sim i L_P^4 \eta^{55} J^{\mu_1\mu_2|\nu_1\nu_2}, \quad J^{\mu_1\mu_2|\nu_1\nu_2} \equiv -(Y^{\mu_1\mu_2} \Pi^{\nu_1\nu_2} - Y^{\nu_1\nu_2} \Pi^{\mu_1\mu_2}) \tag{2.18}$$

$Y^{\mu_1\mu_2}$ is a bivector coordinate associated with the $Cl(5,1)$ algebra of the $6D$ flat spacetime. $\Pi^{\mu_1\mu_2} = -i(\partial/\partial Y_{\mu_1\mu_2})$ is the corresponding bivector momentum conjugate. Their commutators are

$$[Y^{\mu_1\mu_2}, Y^{\nu_1\nu_2}] = 0, \quad [\Pi^{\mu_1\mu_2}, \Pi^{\nu_1\nu_2}] = 0, \quad [Y^{\mu_1\mu_2}, P^{\nu_1\nu_2}] = i\, \eta^{\mu_1\mu_2|\nu_1\nu_2} \tag{2.19}$$

and from eq-(1.17b) one has that the generalized metric involving bivector indices is

$$\eta^{\mu_1\mu_2|\nu_1\nu_2} = \eta^{\nu_1\nu_2|\mu_1\mu_2} = \eta^{\mu_1\nu_1} \eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2} \eta^{\mu_2\nu_1} \tag{2.20}$$

The *noncommuting* bivector momenta obey

$$[P^{\mu_1\mu_2}, P^{\nu_1\nu_2}] \sim i\mathcal{L}^{-4}\ \eta^{66}\ J^{\mu_1\mu_2|\nu_1\nu_2} \tag{2.21}$$

Similarly,

$$[X^{\mu_1\mu_2\mu_3}, X^{\nu_1\nu_2\nu_3}] \sim iL_P^6\ \eta^{55}\ J^{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} \tag{2.22}$$

$$J^{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} \equiv -(Y^{\mu_1\mu_2\mu_3}\ \Pi^{\nu_1\nu_2\nu_3} - Y^{\nu_1\nu_2\nu_3}\ \Pi^{\mu_1\mu_2\mu_3}) \tag{2.23}$$

$Y^{\mu_1\mu_2\mu_3}$ is a trivector coordinate associated with the $Cl(5,1)$ algebra of the $6D$ flat spacetime. $\Pi^{\mu_1\mu_2\mu_3} = -i(\partial/\partial Y_{\mu_1\mu_2\mu_3})$ is the corresponding trivector momentum conjugate. Their commutators are

$$[Y^{\mu_1\mu_2\mu_3}, Y^{\nu_1\nu_2\nu_3}] = 0, \quad [\Pi^{\mu_1\mu_2\mu_3}, \Pi^{\nu_1\nu_2\nu_3}] = 0,$$
$$[Y^{\mu_1\mu_2\mu_3}, P^{\nu_1\nu_2\nu_3}] = i\ \eta^{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} \tag{2.24}$$

$$\eta^{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} = \eta^{\nu_1\nu_2\nu_3|\mu_1\mu_2\mu_3} = \eta^{\mu_1\nu_1}\ \eta^{\mu_2\nu_2}\ \eta^{\mu_3\nu_3} \pm \cdots \tag{2.25}$$

where the terms $\cdots$ in the right hand side are obtained from permutations of indices.

The *noncommuting* trivector momenta obey

$$[P^{\mu_1\mu_2\mu_3}, P^{\nu_1\nu_2\nu_3}] \sim i\mathcal{L}^{-6}\ \eta^{66}\ J^{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} \tag{2.26}$$

The commutator

$$[X^{\mu_1\mu_2\mu_3\mu_4}, X^{\nu_1\nu_2\nu_3\nu_4}] \sim iL_P^8\ \eta^{55}\ J^{\mu_1\mu_2\mu_3\mu_4|\nu_1\nu_2\nu_3\nu_4} \to 0 \tag{2.27}$$

vanishes since in $4D$ the generator

$$J^{\mu_1\mu_2\mu_3\mu_4|\nu_1\nu_2\nu_3\nu_4} \equiv Y^{\mu_1\mu_2\mu_3\mu_4}\ \Pi^{\nu_1\nu_2\nu_3\nu_4} - Y^{\nu_1\nu_2\nu_3\nu_4}\ \Pi^{\mu_1\mu_2\mu_3\mu_4} \to 0 \tag{2.28}$$

vanishes identically.

$Y^{\mu_1\mu_2\mu_3\mu_4}$ is a quadvector coordinate associated with the $Cl(5,1)$ algebra of the $6D$ flat spacetime. $\Pi^{\mu_1\mu_2\mu_3\mu_4} = -i(\partial/\partial Y_{\mu_1\mu_2\mu_3\mu_4})$ is the corresponding quadvector momentum conjugate. Their commutators are

$$[Y^{\mu_1\mu_2\mu_3\mu_4}, Y^{\nu_1\nu_2\nu_3\nu_4}] = 0, \quad [\Pi^{\mu_1\mu_2\mu_3\mu_4}, \Pi^{\nu_1\nu_2\nu_3\nu_4}] = 0,$$
$$[Y^{\mu_1\mu_2\mu_3\mu_4}, P^{\nu_1\nu_2\nu_3\nu_4}] = i\ \eta^{\mu_1\mu_2\mu_3\mu_4|\nu_1\nu_2\nu_3\nu_4} \tag{2.29}$$

$$\eta^{\mu_1\mu_2\mu_3\mu_4|\nu_1\nu_2\nu_3\nu_4} = \eta^{\nu_1\nu_2\nu_3\nu_4|\mu_1\mu_2\mu_3\mu_4} = \eta^{\mu_1\nu_1}\ \eta^{\mu_2\nu_2}\ \eta^{\mu_3\nu_3}\ \eta^{\mu_4\nu_4} \pm \cdots \tag{2.30}$$

The quadvector momenta commutator also vanishes in $4D$

$$[P^{\mu_1\mu_2\mu_3\mu_4}, P^{\nu_1\nu_2\nu_3\nu_4}] \ \sim \ i\mathcal{L}^{-8} \ \eta^{66} \ J^{\mu_1\mu_2\mu_3\mu_4|\nu_1\nu_2\nu_3\nu_4} \to 0 \tag{2.31}$$

The modified Weyl-Heisenberg algebra is

$$[X^\mu, P^\nu] \ = \ \frac{L_P}{\mathcal{L}} \ [J^{\mu 5}, J^{\nu 6}] \ = \ - \, i \ \frac{L_P}{\mathcal{L}} \ \eta^{\mu\nu} \ \mathcal{N}, \ \ \mathcal{N} \equiv J^{56} \tag{2.32}$$

$$[X^{\mu_1\mu_2}, P^{\nu_1\nu_2}] \ \sim \ i\frac{L_P^2}{\mathcal{L}^2} \ \eta^{\mu_1\mu_2|\nu_1\nu_2} \ \mathcal{N} \tag{2.33}$$

$$[X^{\mu_1\mu_2\mu_3}, P^{\nu_1\nu_2\nu_3}] \ \sim \ i\frac{L_P^3}{\mathcal{L}^3} \ \eta^{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} \ \mathcal{N} \qquad (2.34$$

$$[X^{\mu_1\mu_2\mu_3.\mu_4}, P^{\nu_1\nu_2\nu_3\nu_4}] \ \sim \ i\frac{L_P^4}{\mathcal{L}^4} \ \eta^{\mu_1\mu_2\mu_3\mu_4|\nu_1\nu_2\nu_3\nu_4} \ \mathcal{N} \tag{2.35}$$

The commutators where the polyvector coordinates are exchanged for polyvector momenta, and vice versa, are of the form

$$[X^\mu, \ \mathcal{N}] \ = \ i \ L_P\mathcal{L} \ P^\mu, \ \ [P^\mu, \ \mathcal{N}] \ = \ - \, i \ \frac{1}{L_P\mathcal{L}} \ X^\mu \tag{2.36}$$

$$[\mathcal{N}, X^{\mu_1\mu_2}] \ = \ L_P^2 \ [J^{56}, J^{\mu_1\mu_2 5}] \sim iL_P^2\mathcal{L}^2 \ \eta^{55} \ P^{\mu_1\mu_2} \tag{2.37}$$

$$[\mathcal{N}, P^{\mu_1\mu_2}] \ = \ \mathcal{L}^{-2} \ [J^{56}, J^{\mu_1\mu_2 6}] \sim iL_P^{-2}\mathcal{L}^{-2} \ \eta^{66} \ X^{\mu_1\mu_2}, \quad \cdots \tag{2.38}$$

and so forth.

The commutator of the Clifford scalar coordinate X with the Clifford scalar momentum P is simply $[X, P] = -iJ^{56} = -i\mathcal{N}$. X, P are chosen to be dimensionless. This results from the correspondence between the $4D$ Clifford scalars X, P and the $6D$ coordinates and momenta given by

$$X \ \leftrightarrow \ - \, (Y \ \Pi^5 \ - \ Y^5 \ \Pi), \ \ P \ \leftrightarrow \ - \, (Y \ \Pi^6 \ - \ Y^6 \ \Pi) \tag{2.39}$$

where Y, Π are the Clifford scalar coordinate and scalar momentum associated with the $Cl(5,1)$ algebra in $6D$, and obeying $[Y, \Pi] = i$.

From the correspondence (2.11-2.16) one can write many other commutators, but to simplify matters we shall stop here since the number of combinations is very large. All the commutators have the same structural form of a generalized angular momentum algebra as follows

$$[J^{A(r_1)|B(r_2)}, \ J^{C(s_1)|D(s_2)}] \ = \ -i \, \eta^{A(r_1)|C(s_1)} \ J^{B(r_2)|D(s_2)} + i \, \eta^{A(r_1)|D(s_2)} \ J^{B(r_2)|C(s_1)} +$$

$$i \ \eta^{B(r_2)|C(s_1)} \ J^{A(r_1)|D(s_2)} \ - i \ \eta^{B(r_2)|D(s_2)} \ J^{A(r_1)|C(s_1)}, \ \ \hbar = c = 1 \tag{2.40}$$

where the grades of the polyvector indices $A(r_1)B(r_2), C(s_1), D(s_2)$ appearing in the generators are r_1, r_2, s_1, s_2, respectively. The shorthand notation for $J^{a_1a_2\cdots a_{r_1}|b_1b_2\cdots b_{r_2}}$ is $J^{A(r_1)|B(r_2)}, \cdots$. The generalized metric tensor $\eta^{A|C} = 0$ if the grade of A is *not* equal to the grade of C. Similarly, $\eta^{A|D} = 0$ if the grade of A is *not* equal to the grade of $D, \cdots$. Also, $\eta^{\mu 5} = \eta^{\mu 6} = 0$ since the $6D$ metric is diagonal. The commutators (2.40) will ensure that the Jacobi identities are satisfied.

The *noncommutative* polyvector coordinates and momenta described by the algebraic relations in this section are the defining relations of what we may call the Clifford-Yang algebra. It is a *novel* algebra to our knowledge. The Clifford-Yang algebra displays Born's Reciprocity. The commutators enjoy a coordinate/momentum symmetry (reciprocity).

Reinstating $\hbar$ which was set to unity, the modified uncertainty relations due to the Yang algebra are obtained from the Robertson-Schrodinger inequalities

$$\Delta X^\mu \, \Delta P_\nu \;\geq\; \frac{1}{2} \, |\langle \, [X^\mu, \; P_\nu] \, \rangle| \;\Rightarrow\; \Delta X^\mu \, \Delta P_\nu \;\geq\; \frac{\hbar L_P}{2\mathcal{L}} \, \delta^\mu_\nu \, |\langle \, J_{56} \, \rangle| \;=\; \frac{|\tilde{m}|\hbar L_P}{2\mathcal{L}} \, \delta^\mu_\nu \tag{2.41}$$

after evaluating the expectation values with respect to the normalized eigenfunctions of J_{56} (rotations in the $Y_5 - Y_6$ plane) given by $\frac{1}{\sqrt{2\pi}} e^{i\tilde{m}\phi}$, with $\tilde{m}$ an integer that differs from the quantum number m corresponding to the $Y_{lm}(\theta, \varphi)$ spherical harmonics associated with the three-dim angular momentum operators. The eigenfunctions of the angular momentum operators $\mathbf{J}^2_{S^N}$ associated with the N-dim sphere S^N are given in terms of N angles $\theta_1, \theta_2, \cdots, \theta_N$, and N quantum numbers $l_1, l_2, l_3, \cdots, l_N$, and were studied long ago by [8]. In 5-spatial dimensions one would require in general to study the eigenfunctions of the angular momentum operators in S^4 which are highly nontrivial.

The uncertainty relations involving the bivector coordinates $X^{\mu_1\mu_2}, P_{\nu_1\nu_2}$, with $\mu_1 < \mu_2; \nu_1 < \nu_2$, is given by

$$\Delta X^{\mu_1\mu_2} \; \Delta P_{\nu_1\nu_2} \;\geq\; \frac{1}{2} \, |\langle \, [X^{\mu_1\mu_2}, \; P_{\nu_1\nu_2}] \, \rangle| \;\Rightarrow$$

$$\Delta X^{\mu_1\mu_2} \; \Delta P_{\nu_1\nu_2} \;\geq\; \frac{|\tilde{m}|\hbar^2 L_P^2}{2\mathcal{L}^2} \; \delta^{\mu_1\mu_2}_{\nu_1\nu_2} \tag{2.42}$$

$$\delta^{\mu_1\mu_2}_{\nu_1\nu_2} \;=\; \delta^{\mu_1}_{\nu_1} \; \delta^{\mu_2}_{\nu_2} \;-\; \delta^{\mu_1}_{\nu_2} \; \delta^{\mu_2}_{\nu_1} \tag{2.43}$$

and so forth with the higher grade polyvectors coordinates and momenta.

Therefore, one learns that due to the *noncommutative* coordinates and momenta, the uncertainty relations (2.41) $differ$ from the ones in standard QM. This is one of the salient features of the Yang algebra, and by extension, to the polyvector coordinates and momenta of the Clifford-Yang algebra. In the next section we shall study the isotropic quantum oscillator in noncommutative spaces.

3 The Quantum Oscillator in Noncommutative Spaces

In [14] we discussed two approaches in the evaluation of the areal spectrum in $3D$ and associated with noncommutative coordinates that we labeled as operators as $\mathbf{x_i}; i = 1,2,3$. One approach was to write the operator $L_P^{-2} \sum_{i=1}^{i=3} \mathbf{x_i x^i}$ (in Planck units) as the difference $\sum_{i,j=1}^{i,j=4} \mathbf{J}_{ij}^2 - \sum_{i,j=1}^{i,j=3} \mathbf{J}_{ij}^2$ of the total orbital angular momentum squared in $D = 4$ and $D = 3$. So the eigenvalues can be obtained from the difference between the quadratic Casimirs of $SO(4)$ and $SO(3)$ given by $C_2[SO(4)] - C_2[SO(3)] = l_3(l_3+2) - l_2(l_2+1)$, where l_3 is the orbital angular momentum quantum number of the three-sphere S^3, and l_2 is the orbital angular momentum quantum number of the two-sphere S^2. In the very special case when $l_3 = l_2$ the difference $C_2[SO(4)] - C_2[SO(3)]$ is given by l_2 and such that $\sum_{i=1}^{i=3} \mathbf{x_i x^i} = l_2 L_P^2$ turns out to be *linear* in the angular momentum quantum number of the two-sphere $l_2 = l$.

However there is a subtlety because the eigenfunctions of the angular momentum operators associated with S^2 and S^3 are not the same. The eigenfunctions of the angular momentum operators $\mathbf{J}^2_{S^2}$ associated with S^2 are the spherical harmonics $Y_{lm}(\theta,\varphi)$ and which can be rewritten as $Y_{l_2 l_1}(\theta_2,\theta_1)$

$$Y_{l_2 l_1}(\theta_2,\theta_1) \ \equiv \ Y_{lm}(\theta,\varphi) \ = \ (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \ P_l^m(cos\theta) \ e^{im\varphi} \quad (3.1)$$

with $l_1 = m, l_2 = l; \theta_2 = \theta, \theta_1 = \varphi$ and where $P_{lm}(cos\theta)$ are the associated Legendre ploynomials.

The eigenfunctions of the angular momentum operators $\mathbf{J}^2_{S^3}$ associated with S^3 are given in terms of three angles $\theta_1 = \varphi, \theta_2 = \theta, \theta_3 = \xi$ and three quantum numbers l_1, l_2, l_3, obeying $l_3 \geq l_2 \geq |l_1|$, as follows [8]

$$Y_{l_1 l_2 l_3}(\theta,\varphi,\xi) \ = \ Y_{l_1 l_2}(\theta,\varphi) \sqrt{\frac{2l_3+2}{2} \ \frac{(l_3+l_2+1)!}{(l_3-l_2)!}} \ \sqrt{sin\xi} \ P_{l_3+\frac{1}{2}}^{-(l_2+\frac{1}{2})}(cos\xi) \quad (3.2)$$

where $P_{l_3+\frac{1}{2}}^{-(l_2+\frac{1}{2})}(cos\xi)$ is the associate Legendre function of the first kind that can be written in terms of the hypergeometric function ${}_2F_1$ as

$$P_{l_3+\frac{1}{2}}^{-(l_2+\frac{1}{2})}(cos\xi) \ \equiv \ \frac{1}{\Gamma(1+l_2+\frac{1}{2})} \ (\frac{1-cos\xi}{1+cos\xi})^{\frac{1}{2}(l_2+\frac{1}{2})} \ \times$$

$${}_2F_1\left(-(l_3+\frac{1}{2}), \ (l_3+\frac{1}{2})+1; \ 1+(l_2+\frac{1}{2}); \ \frac{1-cos\xi}{2}\right) \quad (3.3)$$

Note that because $Y_{l_1 l_2 l_3}(\theta,\varphi,\xi)$ factorizes $Y_{l_1 l_2}(\theta,\varphi) F_{l_3 l_2}(\xi)$, it can be seen also as an eigenfunction of $\mathbf{J}^2_{S^2}$ (the angular momentum operator associated with

S^2) because $\mathbf{J}^2_{S^2} Y_{l_1 l_2 l_3}(\theta, \varphi, \xi) = l_2(l_2+1) Y_{l_1 l_2 l_3}(\theta, \varphi, \xi)$ due to the factorization property and the trivial fact that $\mathbf{J}^2_{S^2}$ does not act on the extra angle ξ.

Therefore one has

$$(\sum_{i=1}^{i=3} \mathbf{x_i x^i})\ Y_{l_1 l_2 l_3} \ = \ L_P^2\ (\mathbf{J}^2_{S^3} - \mathbf{J}^2_{S^2})\ Y_{l_1 l_2 l_3} \ = \ L_P^2\ [l_3(l_3+2) - l_2(l_2+1)] Y_{l_1 l_2 l_3} \tag{3.4a}$$

giving $L_P^2 l_2 Y_{l_1 l_2 l_3}$ for the right hand side in the special case when $l_3 = l_2$. Since $4\pi r^2$ is the area of a sphere, when the coordinates are noncommutative, we can label $\mathbf{r}^2$ as the square of the radial operator, and the area spectrum of the quantum sphere is $4\pi L_P^2[l_3(l_3+2) - l_2(l_2+1)]$. The areal spectrum becomes *linear* in the angular momentum when $l_3 = l_2 = l$, so the areas are quantized in multiples of the Planck area, not unlike the Schwarzschild black hole horizon areas quantized in bits of Planck areas [15],

Repeating this whole procedure for the momentum, the spectrum of the noncommutative momenta that we labeled as operators by $\mathbf{p_i}$, and that have a correspondence with the angular momentum J_{i5}, is given by

$$(\sum_{i=1}^{i=3} \mathbf{p_i p^i})\ Y_{l_1 l_2 \tilde{l}_3} \ = \ \mathcal{L}^{-2}\ (\mathbf{J}^2_{\tilde{S}^3} - \mathbf{J}^2_{S^2})\ Y_{l_1 l_2 \tilde{l}_3} \ =$$

$$\mathcal{L}^{-2}\ [\tilde{l}_3(\tilde{l}_3+2) - l_2(l_2+1)]\ Y_{l_1 l_2 \tilde{l}_3} \tag{3.4b}$$

where the 3-spheres are not the same $S^3 \neq \tilde{S}^3$, because S^3 lives in the (Y_1, Y_2, Y_3, Y_4) space, while $\tilde{S}^3$ lives in the (Y_1, Y_2, Y_3, Y_5) space. However, because there is an S^2-overlap of S^3 and $\tilde{S}^3$ in the (Y_1, Y_2, Y_3) space, as result of this overlap one should have functions depending only on 4 angles, $Y_{l_1 l_2 l_3}(\theta, \varphi, \xi)$, and $Y_{l_1 l_2 \tilde{l}_3}(\theta, \varphi, \tilde{\xi})$, given by $\theta, \varphi, \xi, \tilde{\xi}$, instead of 6 angles. For this reason the quantum numbers describing the momentum eigenfunctions are $l_1, l_2, \tilde{l}_3$. The areal momentum (3.4b) is quantized in units of $\mathcal{L}^{-2}$. Setting $\mathcal{L}$ to the Hubble scale, one finds that the areal momentum is quantized in bits of a minimal areal momentum. Likewise, the areas were quantized in bits of a minimal Planck area.

One could try to exploit the factorization property $Y_{l_1 l_2 l_3}(\theta, \varphi, \xi) = Y_{l_1 l_2}(\theta, \varphi) F_{l_3 l_2}(\xi)$ in the study of the noncommuting $3D$ isotropic oscillator, involving the noncommuting spatial momenta and coordinates $\mathbf{P}_i, \mathbf{X}_i; i = 1, 2, 3$. Let us look for the energy eigenvalues and eigenfunctions associated with the following Hamiltonian operator

$$(\ \frac{\mathbf{P}_i^2}{2m}\ + \frac{m\omega^2}{2}\ \mathbf{X}_i^2\ )\Psi \ = \ E\ \Psi, \quad i = 1, 2, 3 \tag{3.5}$$

where m, ω is the mass and frequency of the oscillator. After using the correspondence of the previous section between the noncommuting coordinates and momenta with the angular momentum operators, eq-(3.5) becomes

$$\left(\frac{1}{2m\mathcal{L}^2} \ (Y_i\partial_5 - Y_5\partial_i)^2 \ + \frac{m\omega^2}{2} \ L_p^2 \ (Y_i\partial_4 - Y_4\partial_i)^2 \ + \ E \right) \Psi(Y_1, Y_2, Y_3, Y_4, Y_5) \ = \ 0 \tag{3.6}$$

Eq-(3.6) can also be written as

$$\left(\frac{1}{2m\mathcal{L}^2} \ (J_{i5})^2 \ + \frac{m\omega^2}{2} \ L_p^2 \ (J_{i4})^2 \ - \ E \right) \ \Psi(Y_1, Y_2, Y_3, Y_4, Y_5) \ = \ 0 \tag{3.7}$$

Let us look for spherically symmetric solutions in 5 spatial dimensions by introducing the radial coordinate $r^2 = (Y_1)^2 + (Y_2)^2 + (Y_3)^2 + (Y_4)^2 + (Y_5)^2$, and the 4 angles associated with S^4, to be of the form

$$\Psi \ = \ \Xi(r) \ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \tag{3.8}$$

where the eigenfunctions $Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4)$ of the angular momentum operators in S^4 can be found in [8]. $\Xi(r)$ governs the radial part that is not affected by the angular momentum operators. One may set $\Xi(r) = C$ to a constant which can be fixed from the normalization condition of the wave function $C^2 \int Y_{l_1 l_2 l_3 l_4} Y^*_{l_1 l_2 l_3 l_4} d\Omega_4 \ = \ 1$ resulting from an integration over the four-dim solid angle Ω_4 in S^4. One may note that the 4 angles $\theta_1, \cdots, \theta_4$ encode already a functional dependence on the full 5 cartesian coordinates $Y_1, \cdots, Y_5$. For example, in $3D$ one has $tan(\varphi) = \frac{y}{x}; cos(\theta) = \frac{z}{r}$.

After rewriting

$$\sum_{i=1}^{i=3} \ (\ J_{i4}^2 + J_{i5}^2 \ ) \ = \ \sum_{i=1}^{i=3} \ (J_{i4}^2 + J_{i5}^2) \ + \ \sum_{i,j=1}^{i,j=3} J_{ij}^2 \ + \ J_{45}^2 \ -$$

$$\sum_{i,j=1}^{i,j=3} J_{ij}^2 \ - \ J_{45}^2 \ = \ \mathbf{J}^2_{S^4} - \mathbf{J}^2_{S^2} \ - \ \mathbf{J}^2_{S^1} \tag{3.9}$$

and setting $m^2\mathcal{L}^2 = \omega^2 L_P^2 = 1$, eq-(3.7) can be recast in the form

$$\frac{m}{2} \ (\ \mathbf{J}^2_{S^4} \ - \ \mathbf{J}^2_{S^2} \ - \ \mathbf{J}^2_{S^1} \ - \frac{2E}{m} \ ) \ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \ = \ 0 \tag{3.10}$$

given in terms of the angular momentum operators corresponding to S^4, S^2, S^1 respectively. S^1 is spanned by the Y_4, Y_5 coordinates and associated to rotations in the $Y_4 - Y_5$ plane. Similarly, S^2 is spanned by the Y_1, Y_2, Y_3 coordinates, and S^4 is spanned by the 5 spatial coordinates $Y_1, Y_2, \cdots, Y_5$.

A set of *commuting* generalized orbital angular momentum operators in N-dimensional polar coordinates can be defined, and their eigenvalues and simultaneous eigenfunctions can be determined by the use of results known from the *factorization* method of solving eigenvalue problems [8]. Basically, if a polar coordinate system in N-dim is known then a polar coordinate system in

$N+1$-dim can be constructed by iteration. Also, if two polar coordinate systems are known in N_1, N_2 dimensions a third polar coordinate systems can be constructed in $N_1 + N_2$ dimensions [8]. From this construction one learns that

$$[\mathbf{J}^2_{S^4},\ \mathbf{J}^2_{S^2}] \ = \ 0, \ \ [\mathbf{J}^2_{S^4},\ \mathbf{J}^2_{S^1}] \ = \ 0, \ \ [\mathbf{J}^2_{S^2},\ \mathbf{J}^2_{S^1}] \ = \ 0 \tag{3.11}$$

When the angles $\theta_1, \theta_2, \theta_3, \theta_4$ are arranged consistently with the nested set of the spheres $S^1 \subset S^2 \subset S^3 \subset S^4$, due to the *factorization* property of $Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4)$, as we saw earlier in the case of $Y_{l_1 l_2 l_3}(\theta, \varphi, \xi)$ in eq-(3.2), then one has that $Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4)$ is a common eigenfunction of the angular momentum operators but with different eigenvalues

$$\mathbf{J}^2_{S^4}\ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \ = \ l_4(l_4+3)\ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \tag{3.12a}$$

$$\mathbf{J}^2_{S^2}\ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \ = \ l_2(l_2+1)\ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \tag{3.12b}$$

$$\mathbf{J}^2_{S^1}\ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \ = \ (l_1)^2\ Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) \tag{3.12c}$$

Therefore, from eqs-(3.10, 3.12) one obtains the energy eigenvalues

$$E_{l_4 l_2 l_1} \ = \ \frac{m}{2}\ [\ l_4(l_4+3) \ - \ l_2(l_2+1) \ - \ (l_1)^2], \ \ l_4 \geq l_2 \geq |l_1| \tag{3.13}$$

To ensure that $E_{l_4 l_2 l_1} \geq 0$ one has to choose the values of l_4, l_2, l_1 appropriately.

The result (3.13) was obtained in the very special case when $m^2 \mathcal{L}^2 = \omega^2 L_P^2 = 1$. When these conditions are not met then one cannot rewrite eq-(3.7) in the form described by eq-(3.10) and this complicates matters. To find the eigenvalues and eigenfunctions in the more general case is a more difficult task. In this case one would have a radial dependence as found in [14]. The conditions $m^2 \mathcal{L}^2 = \omega^2 L_P^2 = 1$ select a minimal mass for m given by the inverse of the infrared cut-off scale (Hubble radius), and select a Planck energy for the value of the frequency ω ($\hbar = c = 1$ units). The Yang algebra captures both physics in the ultra-violet (small scales) and in the infra-red (large scales), a key property that a successful theory of Quantum Gravity must have.

Compare the noncommutative isotropic $3D$ oscillator described by the differential equation (3.6) involving 5 variables $Y_1, Y_2, \cdots, Y_5$ with the ordinary isotropic $3D$ oscillator in QM involving a differential equation in 3 variables associated with the 3 commuting coordinates $[\mathbf{X}_i, \mathbf{X}_j] = 0$, and commuting momenta $[\mathbf{P}_i, \mathbf{P}_j] = 0$, with $[\mathbf{X}_i, \mathbf{P}_j] = i\hbar\delta_{ij}$. The energy eigenvalues are $E = (n_1 + n_2 + n_3 + \frac{3}{2})\hbar\omega$, with n_1, n_2, n_3 integers, and the eigenfunctions factorize into products of Gaussians and Hermite polynomials. In spherical coordinates the solutions are given by [9], [11]

$$\Psi_{klm} \ = \ N_{kl}\ r^l\ e^{-\kappa r^2}\ L_k^{(l+\frac{1}{2})}\ (2\kappa r^2)\ Y_{lm}(\theta, \varphi) \tag{3.14}$$

where N_{kl} is a normalization constant. $\kappa = \frac{M\omega}{2\hbar}$; M is the mass of the particle. $L_k^{(l+\frac{1}{2})}(2\kappa r^2)$ are the generalized Laguerre poynomials, with k a non-negative integer. $Y_{lm}(\theta, \varphi)$ are the $3D$ spherical harmonics.

The energy eigenvalue is

$$E \ = \hbar\ \omega\ \left(2k + l + \frac{3}{2} \right) \tag{3.15}$$

and is usually described by the single quantum number $n \equiv 2k + l$ which is associated with the *radial* quantum number k and l. Whereas the energy eigenvalues found in eq-(3.13) depend only on the angular quantum numbers l_4, l_2, l_1, since there is no radial dependence of the wave function.

The degeneracy at every level is

$$\sum_{l=\cdots,n-2,n} (2l+1) \ = \ \frac{(n+1)(n+2)}{2} \tag{3.16}$$

where the sum starts from 0 or 1, according to whether n is even or odd. This amounts to the dimensionality of a symmetric representation of $SU(3)$ [10], [11] the relevant degeneracy group. For details of the isotropic N-dimensional quantum oscillator we refer to [11].

4 Conclusion

We found that QM in noncommutative spaces leads to very different solutions, eigenvalues, and uncertainty relations than ordinary QM in commutative spaces. The generalization of QM in noncommutative (phase) spaces to noncommutative Clifford (phase) spaces is attained via the Clifford-Yang algebra described in section **2**. The operators are given by the generalized angular momentum operators involving polyvector coordinates and momenta as shown in (2.11-2.16). The eigenfunctions (wave functions) are functions of the polyvector coordinates $Y, Y_i, Y_{ij}, Y_{ijk}, \cdots$. The differential equations required to solve are more complicated than the ones described above. The relativistic case requires adding the temporal coordinates leading to a formulation of QFT in Clifford spaces.

We avoided the need to use star products and symplectic Clifford Algebras developed by Crumeyrolle in our treatment of noncommutative Clifford phase spaces. Choosing the $6D$ metric $\eta_{AB} = diag(-1,1,1,1,1,-1)$ leads to the conformal algebra $so(4,2) \sim su(2,2)$. The Clifford-Yang algebra does not change much, one just needs to take into account that $\eta^{66} = -1$.

In [16] we extended Born's principle of reciprocity [12] to the case of curved spacetimes and constructed a $deformed$ Born reciprocal general relativity theory in curved spacetimes (without the need to introduce star products) as a local gauge theory of the deformed Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the $deformed$ Weyl-Heisenberg group. The Hermitian metric is complex-valued with symmetric and nonsymmetric components and there are two different complex-valued Hermitian Ricci tensors.

The relevance of this work [16] is that it bears many similarities with the construction of the Yang algebra by invoking higher dimensions and the algebras $so(5,1), so(4,2)$. This is because the semi-direct product of $U(1,3)$ with the *deformed* Weyl-Heisenberg group can be embedded into a $U(1,4)$ group as shown in [16]. It is warranted to explore further relations between the noncommutative Born reciprocal relativity theory and the Yang algebra.

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