FERRARI'S METHOD

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Abstract
Solving quartics via Ferrari's method

Introduction: Ferrari´s method for quartic equation

1. The basic idea: We will reduce the main quartic equation in two quadratic equation and as method for solution of quadratic equation is known we can easily solve main equation.

2. Let the quartic equation is given as

$$f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \tag{1}$$

3. We use the fact that

$$M^2 - N^2 = 0 \Rightarrow (M+N)(M-N) = 0 \Rightarrow (M+N) = 0 \ or \ (M-N) = 0 \tag{2}$$

4. We start with

$$(ax^2 + 2bx + s)^2 - (2mx + n)^2 = 0 \tag{3}$$

for some s, m, n

$$(3) \Rightarrow (a^2x^4 + 4b^2x^2 + s^2 + 4abx^3 + 2asx^2 + 4bsx) - (4m^2x^2 + n^2 + 4mnx) = 0 \tag{4}$$

$$(4) \Rightarrow a^2x^4 + 4abx^3 + (4b^2 + 2as - 4m^2)x^2 + (4bs - 4mn)x + (s^2 - n^2) = 0 \tag{5}$$

By equation (1) we have

$$a \cdot f(x) = a^2x^4 + 4abx^3 + 6acx^2 + 4adx + ae = 0 \tag{6}$$

Comparing equation (5) and (6) we get

$$2as+4b^2-4m^2=6ac \quad , \quad 4bs-4mn=4ad \quad , \quad s^2-n^2=ae \tag{7}$$

So we have

$$as+2b^2-2m^2=3ac \quad , \quad bs-mn=ad \quad , \quad s^2-n^2=ae \tag{8}$$

Now we have

$$(8) \Rightarrow (bs-ad)^2=\left(\frac{as+2b^2-3ac}{2}\right)(s^2-ae) \tag{9}$$

Simplifying and solving this equation for one value of s with trial and error method or as it will be cubic equation in s we can use cardano method to find one real value of s, using that find value of m & n.

Then using (3) we can have two quadratic equations and hence we can solve the main quartic equation.

Main Example

$$x^4-3x^2-2x+1=0 \tag{10}$$

Roots

$$x_1=-\frac{1}{2}\sqrt{3+2s}-\frac{1}{2}\sqrt{3-2s-\frac{4}{\sqrt{3+2s}}} \tag{11}$$

$$x_2=-\frac{1}{2}\sqrt{3+2s}+\frac{1}{2}\sqrt{3-2s-\frac{4}{\sqrt{3+2s}}} \tag{12}$$

$$x_3=\frac{1}{2}\sqrt{3+2s}-\frac{1}{2}\sqrt{3-2s+\frac{4}{\sqrt{3+2s}}} \tag{13}$$

$$x_4=\frac{1}{2}\sqrt{3+2s}+\frac{1}{2}\sqrt{3-2s+\frac{4}{\sqrt{3+2s}}} \tag{14}$$

where

$$s=-\frac{1}{2}+\frac{1}{6}\left(135-6\sqrt{249}\right)^{1/3}+\frac{1}{6}\left(135+6\sqrt{249}\right)^{1/3} \tag{15}$$

On $u = x_3$

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) \qquad (16)$$

Let

$$u = x_3 = \frac{1}{2}\sqrt{3+2s} - \frac{1}{2}\sqrt{3 - 2s + \frac{4}{\sqrt{3+2s}}} \qquad (17)$$

We have

$$\pi = 4\sqrt{2u}\sum_{n=0}^{\infty}\left(-\frac{u}{2}\right)^n \binom{2n}{n}\sum_{k=0}^{[n/2]}\frac{2^{-4k}}{2n-4k+1}\binom{n}{2k}\binom{2n-4k}{n-2k}^{-1} \qquad (18)$$

$$\pi = 4\sum_{n=0}^{\infty}u^{n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n}{n-2k} + 12u^2 + 12\sum_{n=1}^{\infty}(-3)^{-n}u^{2n+2}\sum_{k=\left[\frac{n-1}{4}\right]}^{[n/2]}\frac{(-1)^k 3^{4k}}{2k+1}\binom{2k+1}{n-2k} \qquad (19)$$

Sequence for $u = x_3$

$$u_n = \sum_{m=0}^{[n/4]}\sum_{k=0}^{\left[\frac{n-2m}{2}\right]}(-1)^m 2^{n-2m-2k}3^{-m+k}\binom{k}{m}\binom{n-k-2m}{k}, \; n = 1,2,3,\ldots \qquad (20)$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = u = x_3 \qquad (21)$$

$$\{u_n : n \geq 0\} = \{1,2,7,20,60,178,529,1572,4671,13880,\ldots\} \qquad (22)$$

$$\left\{\frac{u_n}{u_{n+1}} : n \geq 0\right\} = \left\{\frac{1}{2},\frac{2}{7},\frac{7}{20},\frac{1}{3},\frac{30}{89},\frac{178}{529},\frac{529}{1572},\frac{524}{1557},\frac{4671}{13880},\ldots\right\} \qquad (23)$$

Remark: $[x] = floor(x)$, is the floor function.

References

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