ON THE INTEGRAL INEQUALITY OF SOME TRIGONOMETRIC FUNCTIONS IN $\mathbb{R}^n$

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ABSTRACT. In this note, we prove the inequality

$$\Bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\cos\bigg(\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\bigg)dx_1dx_2\cdots dx_n\Bigg|\leq \frac{\Big|\prod_{i=1}^{n}|b_i|-|a_i|\Big|}{|\Re(\langle a,b\rangle)|}$$

and

$$\Bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sin\bigg(\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\bigg)dx_1dx_2\cdots dx_n\Bigg|\leq \frac{\Big|\prod_{i=1}^{n}|b_i|-|a_i|\Big|}{|\Im(\langle a,b\rangle)|}$$

under some special conditions.

1. Introduction

There is hardly a formal introduction to the concept of an inner product and associated space in the literature. The inner product space is usually a good place to go for a wide range of mathematical results, from identities to inequalities. In this situation, the best potential result is frequently obtained. The Cauchy-Schwartz inequality obtained in the case of the Hilbert space [1] is a good example. The notion of the local product and the induced local product space are introduced in this study. This space reveals itself to be a unique form of complicated inner product space. The following inequality is obtained by utilizing this space.

Theorem 1.1. *Let $\vec{a}=(a_1,a_2,\ldots,a_n),\vec{b}=(b_1,b_2,\ldots,b_n)\in\mathbb{C}^n$ and $\langle,\rangle$ denotes the inner product such that $0\neq\Re(\langle a,b\rangle)$ with $\Re(\langle a,b\rangle)\leq 1$, then we have*

$$\Bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\cos\bigg(\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\bigg)dx_1dx_2\cdots dx_n\Bigg|\leq \frac{\Big|\prod_{i=1}^{n}|b_i|-|a_i|\Big|}{|\Re(\langle a,b\rangle)|}$$

where $\Re(\cdot)$ denotes the real part of a complex number.

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Theorem 1.2. *Let $\vec{a}=(a_1,a_2,\ldots,a_n),\vec{b}=(b_1,b_2,\ldots,b_n)\in\mathbb{C}^n$ and $\langle,\rangle$ denotes the inner product such that $0\neq\Im(\langle a,b\rangle)$ with $\Im(\langle a,b\rangle)\leq 1$, then we have*

$$\left|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sin\left(\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\right)dx_1dx_2\cdots dx_n\right|\leq\frac{\left|\prod_{i=1}^{n}|b_i|-|a_i|\right|}{|\Im(\langle a,b\rangle)|}$$

where $\Im(\cdot)$ denotes the imaginary part of a complex number.

The concept of the local product and associated space is often thought of as a black box for quickly establishing a huge class of mathematical inequalities that are difficult to prove using traditional mathematical methods. It operates by traveling into the space and selecting appropriate sheets as functions that are present in the anticipated inequality, as well as satisfying some local requirements with the appropriate support. The local product and associated space could be useful for more than just demonstrating complex mathematical inequalities. As a bi-variate map that assigns any two vectors in a complex inner product space to a complex number, they could be fascinating in and of themselves. It's a unique subspace in many ways. The k^{th} local product space over a sheet $f:\mathbb{C}\longrightarrow\mathbb{C}$ is an inner product space equipped with the local product $\mathcal{G}_f^k(;)$ over a fixed sheet.

2. The local product and associated space

In this section, we introduce and study the notion of the **local product** and associated space.

Definition 2.1. Let $\vec{a},\vec{b}\in\mathbb{C}^n$ and $f:\mathbb{C}\longrightarrow\mathbb{C}$ be continuous on $\cup_{j=1}^{n}[|a_j|,|b_j|]$. Let $(\mathbb{C}^n,\langle,\rangle)$ be a complex inner product space. Then by the k^{th} local product of $\vec{a}$ with $\vec{b}$ on the sheet f, we mean the bi-variate map $\mathcal{G}_f^k:(\mathbb{C}^n,\langle,\rangle)\times(\mathbb{C}^n,\langle,\rangle)\longrightarrow\mathbb{C}$ such that

$$\mathcal{G}_f^k(\vec{a};\vec{b})=f(\langle\vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}f\circ\mathbf{e}\left((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^{k}}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\right)dx_1dx_2\cdots dx_n$$

where $\langle,\rangle$ denotes the inner product and where $\mathbf{e}(q)=e^{2\pi iq}$. We denote an inner product space with a k^{th} **local product** defined over a sheet f as the k^{th} local product space over a sheet f. We denote this space with the triple $(\mathbb{C}^n,\langle,\rangle,\mathcal{G}_f^k(;))$.

In certain ways, the k^{th} local product is a universal map induced by a sheet. To put it another way, a local product can be made by carefully selecting the sheet. We get the local product by making our sheet the constant function $f:=1$

$$\mathcal{G}_1^k(\vec{a};\vec{b})=\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}dx_1dx_2\cdots dx_n$$
$$=\prod_{i=1}^{n}|b_i|-|a_i|.$$

Similarly, if we take our sheet to be $f=\log$, then under the condition that $\langle\vec{a},\vec{b}\rangle\neq 0$, we obtain the induced local product

$$\mathcal{G}^k_{\log}(\vec{a};\vec{b})=2\pi\times(i)^{k+1}\frac{\log(\langle\vec{a},\vec{b}\rangle)}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[k]{\sum_{j=1}^{n}x_j^k}dx_1dx_2\cdots dx_n.$$

By taking the sheet $f=\mathrm{Id}$ to be the identity function, then we obtain in this setting the associated local product

$$\mathcal{G}^k_{\mathrm{Id}}(\vec{a};\vec{b})=\langle\vec{a},\vec{b}\rangle\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\Bigg(\frac{(i)^k\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\Bigg)dx_1dx_2\cdots dx_n.$$

Again, by taking the sheet $f=\mathrm{Id}^{-1}$ with $\langle a,b\rangle\neq 0$, then we obtain the corresponding induced k^{th} local product

$$\mathcal{G}^k_{\mathrm{Id}^{-1}}(\vec{a};\vec{b})=\frac{1}{\langle\vec{a},\vec{b}\rangle}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\Bigg(-\frac{(i)^k\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\Bigg)dx_1dx_2\cdots dx_n.$$

Also by taking the sheet $f=\log\log$, then we have the associated k^{th} local product

$$\mathcal{G}^k_{\log\log}(\vec{a};\vec{b})=\log\log(\langle\vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\log\Bigg(i\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\Bigg)dx_1dx_2\cdots dx_n.$$

Similarly by taking the sheet $f=\frac{1}{\log}$, we obtain the corresponding induced k^{th} local product of the form

$$\mathcal{G}^k_{\frac{1}{\log}}(\vec{a};\vec{b})=\frac{1}{\log(\langle a,b\rangle)}\times(||\vec{a}||^{k+1}+||\vec{b}||^{k+1})\times\frac{1}{2i^{k+1}\pi}\times\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\frac{1}{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}dx_1dx_2\cdots dx_n.$$

3. **Properties of the local product product**

In this section we study some properties of the local product on a fixed sheet.

Proposition 3.1. *The following holds*

(i) *If f is linear such that $\langle a,b\rangle=-\langle b,a\rangle$ then*

$$\mathcal{G}^k_f(\vec{a};\vec{b})=(-1)^{n+1}\mathcal{G}^k_f(\vec{b};\vec{a}).$$

(ii) *Let $f,g:\mathbb{R}\longrightarrow\mathbb{R}^+$ such that $f(t)\leq g(t)$ for any $t\in[1,\infty)$. Then $|\mathcal{G}_f(\vec{a};\vec{b})|\leq|\mathcal{G}_g(\vec{a};\vec{b})|$.*

Proof. (i) By the linearity of f, we can write

$$\begin{aligned}
\mathcal{G}_f^k(\vec{a};\vec{b}) &= f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} f\circ \mathbf{e}\bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n\\
&= f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} f\circ \mathbf{e}\bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n\\
&= f(-\langle b,a\rangle)(-1)^n\int\limits_{|b_n|}^{|a_n|}\int\limits_{|b_{n-1}|}^{|a_{n-1}|}\cdots\int\limits_{|b_1|}^{|a_1|} f\circ \mathbf{e}\bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n\\
&= (-1)^{n+1}f(\langle b,a\rangle)\int\limits_{|b_n|}^{|a_n|}\int\limits_{|b_{n-1}|}^{|a_{n-1}|}\cdots\int\limits_{|b_1|}^{|a_1|} f\circ \mathbf{e}\bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n\\
&= (-1)^{n+1}\mathcal{G}_f^k(\vec{b};\vec{a}).
\end{aligned}$$

(ii) Property (ii) follows very easily from the inequality $f(t)\leq g(t)$.

□

4. **The lower bound**

In this section we prove the upper bounds announced at the outset of the paper.

Theorem 4.1. *Let $\vec{a}=(a_1,a_2,\ldots,a_n),\vec{b}=(b_1,b_2,\ldots,b_n)\in\mathbb{C}^n$ and $\langle,\rangle$ denotes the inner product such that $0\neq\Re(\langle a,b\rangle)$ with $\Re(\langle a,b\rangle)\leq 1$, then we have*

$$\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\cos\bigg(\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\bigg)dx_1dx_2\cdots dx_n\bigg|\leq\frac{\left|\prod_{i=1}^{n}|b_i|-|a_i|\right|}{|\Re(\langle a,b\rangle)|}$$

where $\Re(\cdot)$ denotes the real part of any complex number.

Proof. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ and $\vec{a},\vec{b}\in\mathbb{C}^n$. We note that by taking the sheet $f=\Re$, then we obtain the associated local product

$$\mathcal{G}_{\Re}^{4s}(\vec{a};\vec{b})=\Re(\langle a,b\rangle)\times\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\cos\bigg(\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\bigg)dx_1dx_2\cdots dx_n \tag{4.1}$$

by taking $k = 4s$ for any $s \in \mathbb{N}$. Also by taking the sheet $f := 1$ to be the constant function, then we obtain in this setting the associated local product

$$\mathcal{G}_1^{4s}(\vec{a};\vec{b}) = \int_{|a_n|}^{|b_n|}\int_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int_{|a_1|}^{|b_1|} dx_1 dx_2 \cdots dx_n$$
$$= \prod_{i=1}^{n} |b_i| - |a_i|.$$

Since $\Re(z) \leq 1$ for any root of unity $z \in \mathbb{C}$, the claim inequality is a consequence by appealing to Proposition 3.1 and the requirement $0 \neq \Re(\langle a,b\rangle)$ with $\Re(\langle a,b\rangle) \leq 1$. □

Theorem 4.2. *Let $\vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$ and $\langle , \rangle$ denotes the inner product such that $0 \neq \Im(\langle a,b\rangle)$ with $\Im(\langle a,b\rangle) \leq 1$, then we have*

$$\left|\int_{|a_n|}^{|b_n|}\int_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int_{|a_1|}^{|b_1|} \sin\left(\frac{\sqrt[4s]{\sum_{j=1}^{n} x_j^{4s}}}{||\vec{a}||^{4s+1} + ||\vec{b}||^{4s+1}}\right) dx_1 dx_2 \cdots dx_n\right| \leq \frac{|\prod_{i=1}^{n}|b_i| - |a_i||}{|\Im(\langle a,b\rangle)|}$$

where $\Im(\cdot)$ denotes the imaginary part of any complex number.

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\vec{a}, \vec{b} \in \mathbb{C}^n$. We note that by taking the sheet $f = \Im$, then we obtain the associated local product

$$\mathcal{G}_{\Im}^{4s}(\vec{a};\vec{b}) = \Im(\langle a,b\rangle) \times \int_{|a_n|}^{|b_n|}\int_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int_{|a_1|}^{|b_1|} \sin\left(\frac{\sqrt[4s]{\sum_{j=1}^{n} x_j^{4s}}}{||\vec{a}||^{4s+1} + ||\vec{b}||^{4s+1}}\right) dx_1 dx_2 \cdots dx_n \tag{4.2}$$

by taking $k = 4s$ for any $s \in \mathbb{N}$. Also by taking the sheet $f := 1$ to be the constant function, then we obtain in this setting the associated local product

$$\mathcal{G}_1^{4s}(\vec{a};\vec{b}) = \int_{|a_n|}^{|b_n|}\int_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int_{|a_1|}^{|b_1|} dx_1 dx_2 \cdots dx_n$$
$$= \prod_{i=1}^{n} |b_i| - |a_i|.$$

Since $\Im(z) \leq 1$ for any root of unity $z \in \mathbb{C}$, the claim inequality is a consequence by appealing to Proposition 3.1 and the requirement $0 \neq \Im(\langle a,b\rangle)$ with $\Im(\langle a,b\rangle) \leq 1$. □

5. Data availability statement

The manuscript has no associated data.

6. Conflict of interest statement

The author declares no conflict of interest regarding this publication.

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