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THE CASE FOR PROCOPIU'S QUANTUM

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The general traits of the Planck's quantization procedure for the case of light are considered. The Procopiu's quantization is presented as a result of this quantization procedure, only applied not to light, but to matter structures. Using its guidance, the general possibility of quantization in matter is discussed, by analogy with the quantization in light. A dynamics is then constructed from the point of view of the classical natural philosophy, revealing invariants that are analogous to Planck's constant. The Procopiu's quantum is then justified, based on this dynamics. Rules are recognized for quantization in general, as a physically justifiable theory, to be applied in the case of matter, and to describe the coexistence of matter and light.

Keywords: Planck's quantization, Planck's Quantum, Procopiu's quantization, Procopiu's quantum, Thomson's natural philosophy, fundamental dynamics within matter, resonator, instanton

No matter how far we come, our parents are always in us.
Brad Meltzer

1. Introduction

Max Planck's discovery of the fact that the light admits quantization for its energy has opened a real possibility of conceiving that the same might be the case with the matter itself. However, in the case of matter the things get complicate, by a couple of interrelated details. The first – and, actually, the main of these details – is that the matter is not a simple physical structure like the light itself. Whereby, secondly, it follows a subordinate aspect, as it were, in that the concept of energy can hardly be defined for matter as precisely as in the case of light. Fact is that all the attempts to apply the quantization in matter, following the primeval example of quantization in the case of light, have touched one or another of its physical aspects, but only through the concept of energy. Unfortunately, this route is not quite as flawless as it is needed for quantization, for the simple reason that the concept of energy is quite uncertain for a complex physical structure. And this lack of certainty can be made rather precise on two, again mingling, levels: first, on a *qualitative* theoretical level, and then, as a consequence, at the corresponding *quantitative* level, which, for quantization is essential. In the case of light, the Maxwellian electromagnetic theory brought a substantial clarification of the concept of energy, allowing a quantitatively more precise estimation of this physical magnitude, necessary in describing the light as a thermodynamical process.

This is not to say that the concept of energy in the case of light was absolutely clarified, and that the quantization is an absolute method, as unfortunately appears to be considered in the modern physics. All we want to say is that, in the case of matter, the quantization is a *scale dependent* physical manifestation of the world, that, again fortunately or unfortunately, depends on the quantitative definition of the energy, which is only available mechanically for a limited number of physical systems. The things get further complicated if the electrostatics and electrodynamics enter the stage. To wit: it was quite clear, even from a classical point of view, i.e., at the daily scale of things, that in the case of matter the energy cannot be precisely defined. The following excerpt from an old work shows clearly what are the kinds of imprecision in the classical definition of energy:

> … For example, Weber assumes that the reciprocal action of two electric molecules depends not only on the distance between them, but also on their velocity and acceleration. Should the material points attract each other by a similar law, then U *would depend on velocity*, *and could even contain a term proportional to the square of velocity*.
>
> Then how are we to discern, among the terms proportional with the square of velocity, those of T (*the kinetic energy*, *n*/*a*) from those of U (*the potential energy*, *n*/*a*)? Therefore, how are we to distinguish the two parts of the energy?
>
> Even more, *how do we define the energy itself*? We have no reason to take as definition T + U instead of any other function of T + U, when the property characterizing T + U of being the sum of two terms of particular form disappeared.
>
> And this is not even all of it, because we will have to account not only for the mechanical energy proper, but also for some other forms of energy, heat, chemical energy, electric energy etc. The principle of conservation of energy must then be written
>
> T + U + Q = const.,
>
> where T would represent the sensible kinetic energy, U the potential energy, depending only on the positions of the bodies, Q the internal molecular energy in thermal, chemical or electric forms.
>
> Everything would go just fine, should these terms be perfectly distinct, i.e., should T be proportional to the square of velocities, should U be independent of these velocities and of the state of bodies, should Q be independent of the velocities and positions of the bodies and depend only on their internal state.
>
> The expression of energy could then be decomposed in just one single way into three terms of this form.
>
> But it is not so; consider some electrified bodies: the electrostatic energy due to their reciprocal action *will obviously depend on their charge*, i.e., on their state; but it *will depend equally well on positions*. If these bodies are in motion, they will interact electrodynamically and the electrodynamic energy *will depend not only on their state and positions*, *but also on their velocities*.
>
> We have therefore *no means to select the terms which belong to T, to U and to Q*, in order to separate the three parts of the energy.” [(Poincaré, 1897); our translation and emphasis.]

We took this extensive excerpt from a work classical – at least in our opinion – but otherwise largely ignored today, in order to illustrate that the present situation in theoretical physics is by no means any different from what it was a century or so ago. The actual status of the field and gauge theories – based, in a way or another, upon a method of Lagrangian construction, for instance, that introduces an *a priori* structure for matter analyzing subsequently its feasibility – just proves this conclusion, but there is a subtle change in emphasis. Indeed, in those old times, the uncertainty in the definition of energy, as revealed by our experience at the daily scale of things, would prevent the precise quantitative definition of the energy at the *very same* daily scale, as Poincaré shows in the above excerpt. On the other hand, today, on the ground of the daily scale-based experience, we seek the definition of energy for the *microscopic scale* of the world we perceive. And the truth is that in the microscopical world not only the energy, but the very physical structure of the world is, by and large, a product of our imagination. This fact was basically unrecognized in the old times, when one used to talk freely about molecules for instance, as about any real things that falls under our senses. And our spirit was taken by surprise, so to speak, when the involvement of imagination became critical, as soon as physics inched its way into getting the transfer of quantization from the realm of light into the realm of ponderous matter.

The quantization needs, indeed, the concept of energy in order to be accomplished, but, as the above excerpt from Poincaré plainly shows, according to our experience the physics has not such a precise concept at its disposal in order to properly operate the procedure. At least it does not have it but in quite a few simple cases, where the energy either can be sharply defined from a mechanical point of view, or it can be defined only statistically, as an ensemble average or as a standard deviation on a statistical ensemble. Only this last procedure of quantization is efficient in extending for matter in general the Planck's original quantization *condition* from the case of light, for that is basically a statistical procedure. Its application, though, raises a few fundamental problems in understanding the Planck statistics, and finding its expression inside matter.

Stefan Procopiu's quantization, the subject-matter of this work, is an outcome of the method of quantization of this last kind [60], [61], and has a few particular physical and mathematical pointers that may shed some light even on the modern theoretical physics at large. The most important one of these aspects is that the Procopiu's quantization is completely analogous to Planck's quantization. In this respect it is unique among the methods of quantization in matter. This analogy was the principal incentive of the present work, which is mainly dedicated to inferring, based on it, of what, in hindsight, appear to be *the physical laws of quantization*. The points of difference also show our criteria of analogy: the first and foremost of these is *the statistics*. Although the statistics in the two cases are described by the same general family of probability distributions, in the case of Planck's quantization the distribution is referring to a *discrete statistical variate*, while in the case of Procopiu's quantization we have a *continuous statistical variate*. However, the statistics has the same general properties in both cases, apparently imposed by the natural requirements of quantization process. On the other hand, the matter *is present* for the quantization of Planck – in the form of resonators – while the light *is missing* in the case of Procopiu, where the main quantization unit is the Weiss magneton [75] which, in hindsight, proved to be quite unreliable in such a capacity [62].

The present work is assembled along the following lines: in order to bring the discussion to a proper fruition, we first need to promote a few points of principle for the existing quintessential Planck's quantization case of light. This, obviously, requires the presentation in some detail of the main arguments of Planck's quantization procedure. The presentation is conducted along an obvious path that led to the idea of quantum, so those points of principles are surfacing themselves as milestones on this path, so to speak. Then we need to describe the correspondent in matter of the procedure of quantization. For this, Procopiu's quantization is typical and suggests a proposal, in view of its close analogy with Planck's model, as we already mentioned. Thus, the Procopiu's quantization is presented, with special emphasis on the points of analogy. The general principles of quantization are then extracted as part of the classical Newtonian natural philosophy. That, in fact, in its historical unfolding the physics followed this path objectively, with no exception, is shown here by the case of the planetary Kepler motion in two theoretical instances: the classical *Bertrand theorem* and the modern *regularization procedures*. It is in view of this last observation that we need to ask one question that during the development of the present work became hotter and hotter, almost burning white, if we may be allowed such a secular expression: *was the classical physics completely devoid of* *quantization*? Regarding the answer that we propose to this question, we just need to notice here that the Procopiu's magneton answers to a classical necessity regarding the existence of general gauge fields, a necessity that could not be fulfilled but by the modern gauge fields of the Yang-Mills type.

2. Essentials of Light Quantization and a Case for Matter

The Planck's quantization was a theory apparently without precedent in the existing natural philosophy, even though, based on the results of the present work, we have strong reasons to argue against that: the quantization seems to have been with us from the very beginning of the modern science, i.e., from the times of Galilei, and especially Newton. However, in those historical times the universe of knowledge was too small to include the light in the structure of its microscopic world, so that its planetary model – the one physical structure called for accommodating the light at that scale – was only a concern of finite scale of the space at our disposal. Fact is that the presence of light in the world was bound to radically change our way of thinking, insofar as the light is conspicuously a transcendent phenomenon: it comes to us from afar, which suggests the *transfinite* space scale of a universe. It is produced in microscopic processes, which suggests the *infrafinite* space scale of the microscopic world. These two worlds are by and large figments of our imagination, inasmuch as we cannot reach too far within either one of them. They are only rationally structured based on our experience at the *finite* space scale, where our senses dominate the natural philosophy, and where the light is, with a word of the wise… the life of man. In broad strokes, the scientific attitude accommodating such a transcendental existence of light was at a crossroad by the end of the 19th century, and Max Planck was the one called for indicating a new path that led us to the physics of today: the quantum physics.

In building the quantum statistics for blackbody radiation Max Planck had at his disposal the two formulas for the spectral density of radiation, corresponding to the limit cases of high and low temperatures. Both these cases were *theoretically* ratified, in the sense that they would satisfy the *Wien's displacement law*, which was, and still is in fact, the only theoretical criterion for the choice of the correct physical radiation laws. They are currently known as the *Rayleigh-Jeans* and *Wien laws of radiation*. The story that follows is basically due to Max Born, who created it long after the founding of the Planck's quantization procedure for light, with the precise purpose of delineating the Einstein's contribution to the subject [80].

2.1. THE PLANCK'S LAW OF RADIATION ACCORDING TO MAX BORN

According to Born, Planck's first line of reasoning can be linked to the properties of the Gaussian probability distribution, therefore even without mentioning anything else related to a specifically used statistics. Because, the Planck's quantization is, indeed, all about a specifically used statistics. This fact was discovered and considered in detail by Einstein in his repeated attempts to properly assess the quantization procedure for light, in order to be correctly applied for the case of matter [23], [24], [25]. Even by this moment in the course of the present work, we can add that Planck's is a statistics based on *exponential distributions* with densities given by *quadratic variance function*. Used by statisticians every now and then, incidentally we might say, under this name, it was finally systematized as a concept only late in the last century [51].

It is worth going into a little detail along Max Born's path, inasmuch as it gives us, indeed, the clear overall idea of this important gnoseological process. Born's starting point is, like Planck's, the spectral energy density of the radiation field as a function of temperature, with its two extreme cases:

$$u(\beta) = \begin{cases} \beta^{-1} \, for \, T \to \infty \\ or \\ u_0 e^{-\varepsilon_0 \beta} \;\; for \;\; T \to 0 \end{cases} \tag{2.1.1}$$

Here $\beta \equiv (kT)^{-1}$, with k the Boltzmann constant, T the absolute temperature and ε_0 an energy that must be proportional with the frequency of light in order to satisfy the requirement of Wien's displacement law. Then Born proceeds to the first of Planck's steps, which was to study the *entropy* of such a system, that he assimilated with a system of oscillators. The rationale of this association should have been like this: the entropy is classically related to the heat exchanged, at equilibrium, between two thermodynamical systems, or between a thermodynamic system and the universe containing it. And here we have the very heat in the form of thermal radiation! Thus, nothing more natural, than considering the heat as represented by the energy of the radiation field. According to Born, Planck's intention has been facilitated by the important discovery that the coefficient $u'(\beta)$ – a prime denoting the derivative of function with respect to its variable, as usual – has a simple statistical meaning, which can be derived starting from the equilibrium thermodynamics. Indeed, as the radiation represents the heat exchanged *in equilibrium*, in the classical formula defining the thermodynamical entropy:

$$dS = \frac{\delta Q}{T} \tag{2.1.2}$$

we only have to identify the amount of heat (δQ) with the differential (du) of the density of radiation energy. In so doing, equation (2.1.2) can be rewritten as

$$dS = k\beta \cdot du \quad \therefore \quad S'(u) = k\beta \tag{2.1.3}$$

At this point one must recall the Einstein's procedure of transferring the concepts from thermodynamics to statistics, whereby one identifies the *thermodynamic entropy* with the *statistical entropy* as related to the probability by the Boltzmann relation:

$$S = -k \cdot \ln P \quad \therefore \quad P = e^{-S/k} \tag{2.1.4}$$

Here two arguments must be used: first, the energy as carried by radiation is in fact characterized by fluctuations and, as these fluctuations take place at equilibrium, the entropy must be maximum according to classical precepts. Then the entropy, as a function of energy density can be expanded around the equilibrium value u_0, thus providing an approximative form

$$\begin{aligned} S(u) &= S(u_0) + S'(u_0) + \frac{1}{2}S''(u_0)\cdot(\Delta u)^2 + \ldots \\ &\approx S(u_0) + \frac{1}{2}S''(u_0)\cdot(\Delta u)^2 \end{aligned} \tag{2.1.5}$$

and the Boltzmann formula reveals (although, approximately) a Gaussian describing the fluctuations of the physical system represented by the thermal radiation. We write it in the normalized form necessary for statistical purposes, as:

$$P_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{2.1.6}$$

where $X \equiv \Delta u$ and σ^2 is the *statistical variance* of this process which, in view of equation (2.1.3), can be written as:

$$\sigma^2 \equiv \left(S''(u)\right)^{-1} = u'(\beta) \tag{2.1.7}$$

provided *u* is a continuous function of temperature. In other words, the first derivative of the energy density with respect to the inverse temperature is, in fact, a statistic describing this system: the *variance* of a normal distribution characterizing, although approximately, as we have seen, the fluctuations of energy of the field representing the thermal radiation.

In this connection, it is worth recalling that, for the archetypal ideal gas of the classical thermodynamics, it was the absolute temperature that played such a role, not its reciprocal. It is quite significant that only the *statistical role* has *not* changed here: for both statistics, the classical and the new one, it is the *variance* that plays the part of a physical statistic, even though it has different physical meanings for the two cases. Still significant from a physical point of view, is the fact that the variance of the distribution remains, in both instances, connected to the absolute temperature, as defined by the classical ideal gas.

The equation (2.1.7) was the statistical meaning apparently revealed by Max Planck, and the reason he insisted upon a close consideration of the equation (2.1.1), which in view of this discovery can be rewritten as [80]:

$$u(\beta) = \begin{cases} -u^2 \ for \ T \to \infty \\ or \\ -\varepsilon_0 u \ for \ T \to 0 \end{cases} \qquad (2.1.8)$$

Here ε_0 is a proportionality constant, introduced here for reasons of dimensional homogeneity. Actually, Planck worked with the entropy directly [54]. As mentioned above, the approach based on consideration of energy density is the mark of an Einstein style approach of the problem, but it closely parallels Planck's own way. The reason we assume this approach will be shortly apparent. For now, recalling the equation (2.1.7), which shows that we are looking, in fact, at the *variance of a Gaussian process*, the equation (2.1.8) can be interpreted as representing two *copies* of the light as a thermodynamical process, for the cases of high and low temperatures. If these two processes are *statistically independent*, and the two components are assumed to coexist, the variance of the compound process is the sum of the two component variances, so one can infer that, in general, the law of radiation could be represented by the following differential equation:

$$u'(\beta) = -u^2 - \varepsilon_0 u \tag{2.1.9}$$

This equation is the cornerstone of the Planck's method of quantization, and its solution becomes part and parcel of the method. One has an immediate *particular* solution

$$u(\beta) = \frac{\varepsilon_0}{e^{\varepsilon_0 \beta} - 1} \tag{2.1.10}$$

From this moment on everything is recorded history. Mention should be made, though, that the *general* solution of equation (2.1.9) depends on one more arbitrary constant of integration. The choice of this constant leads to what later became known as the Bose-Einstein and Fermi-Dirac statistics, to be applied to the two different spin particles.

However, limiting our discussion strictly to equation (2.1.10), it has an outstanding physical interpretation unveiling a *statistical ensemble* that has it for *a mean*. This physical interpretation is the key point that started the process of establishing of the modern concept of quantum. It goes on describing an ensemble of *harmonic oscillators*, each one having the energy an integer multiple of ε_0, of which we know nothing but that it is an energy – to make the equation (2.1.9) physically meaningful – proportional to the frequency – to make the equation (2.1.10) physically meaningful, inasmuch as the density of thermal spectrum must satisfy to Wien's displacement law. The partition function of this ensemble can be written as

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-n\varepsilon_0 \beta} = \frac{1}{1 - e^{-\varepsilon_0 \beta}} \tag{2.1.11}$$

so that $u(\beta)$ from equation (2.1.10) appears indeed as the mean over this ensemble of oscillators, as we said, for we have:

$$u(\beta) \equiv -\frac{\partial}{\partial \beta} \ln Z(\beta) \tag{2.1.12}$$

The presence of quantum is here suggested by the random variable characterizing this distribution in equation (2.1.11), which is a natural integer. The photon, as a carrier of the energy ε_0, enabling the interpretation of the thermal light as a physical system, enters the stage only later. However, taking in consideration the idea of interpretation in order to delineate a physical system, it seems that the modern theoretical physics relies predominantly upon what has been left behind along the way towards quantum mechanics, that started from this very beginning. Let us, therefore, see what is going on here, by revealing some well-known facts from a new angle.

2.2. A First View to Planck's Constant

According to previous plot set up by Max Born everything is based on the normal distribution. Moreover, it seems that the Planck's pursuit was, indeed, for *statistically independent processes*. One can easily be induced into thinking that the two processes might not be *statistically independent* and then, assuming that we have definitely two *Gaussian processes* representing the two limit cases of radiation, they are in a general statistical relationship. One may further assume that the general radiation should actually be a linear combination between the two processes, but we limit here the line of reasoning to just the *sum* of the two processes. The general bivariate normal distribution is given by

$$P_{XY}(x,y) = \frac{\sqrt{ac-b^2}}{2\pi}\exp\left\{-\frac{1}{2}\left(ax^2+2bxy+cy^2\right)\right\} \tag{2.2.1}$$

In terms of the *variances* σ_x, σ_y of the two processes and their *correlation coefficient* r, the coefficients a, b, c can be written as

$$a^{-1} = \sigma_x^2(1-r^2),\ b^{-1} = -\sigma_x\sigma_y(1-r^2),\ c^{-1} = \sigma_y^2(1-r^2) \tag{2.2.2}$$

Now we can write the probability density of the compound process (X+Y) having the values ξ, which is given by

$$P_{X+Y}(\xi) = \sqrt{\frac{ac-b^2}{2\pi(a+c-2b)}}\exp\left\{-\frac{1}{2}\frac{ac-b^2}{a+c-2b}\xi^2\right\} \tag{2.2.3}$$

i.e., a Gaussian of zero mean, having the variance $(a+c-2b)/(ac-b^2)$ or, in terms of variances and correlation coefficient of the two component processes, as introduced in equation (2.2.2):

$$\sigma_\xi^2 = \sigma_x^2 + \sigma_y^2 + 2r\sigma_x\sigma_y \tag{2.2.4}$$

Still maintaining the Planck's philosophy above, represented by equation (2.1.7), instead of equation (2.1.9) we should have

$$u'(\beta) = -u^2 - \varepsilon_0 u - 2r\sqrt{\varepsilon_0}\cdot u\sqrt{u} \tag{2.2.5}$$

This equation can be integrated to give

$$\beta\varepsilon_0 = \ln\left(1+2rw+w^2\right) - 2\frac{r}{\sqrt{1-r^2}}\tan^{-1}\frac{1-rw}{w\sqrt{1-r^2}},\quad w^2 \equiv \frac{\varepsilon_0}{u} \tag{2.2.6}$$

Here something is immediately obvious, which shows that our focus might have been misplaced by the mirage of a quick interpretation already at hand, when using the mathematical fact represented by equations (2.1.11) and (2.1.12). Namely the energy ε_0, which has been introduced from dimensional considerations, according to the Wien's displacement law, and which has *subsequently* been explained as a quantum of energy to be carried by an invented particle, must have been, in fact, *a priori* explained.

Indeed, it is the case to take notice of a fact of concern here: when considering the radiation as a thermodynamical system, a contradiction creeps into our reasoning. This can be relegated to the fact that the formula (2.1.2), used in calculating the entropy of radiation, is an *equilibrium formula* from thermodynamical point of view. In other words, we cannot physically conceive one and the same system having two parts existing at different temperatures, without exchanging energy between them. In fact, according to Born, the equation (2.1.9) was taken by Planck only as an *interpolation* between the two extreme cases, which is a purely mathematical trick. Still, if the system at a given temperature is described by the equation (2.1.9), this means that in the interaction with the environment it behaves partly as being at high temperature and partly as being at low temperature: the temperature plays a dual role, and even this needs to be physically explained. Planck himself opted, as well known, for the idea of *resonator*, whose existence is not only allowed, but even imposed we should say, by the established Kirchhoff's laws of equilibrium radiation.

However, the Kirchhoff's laws *do not ask* specifically for an oscillator physical structure, as Planck assumed for the resonator. This issue is still in suspension even today, and here is one scenario of possible explanations: the idea of particle – specifically, the photon – resides in the *correlation* between the two fundamental 'sub-processes' of the light, considered as a thermodynamical process. It is quite significant then – mostly for a source of inspiration of this idea – that Einstein established the properties of quantum only *based on only one* of the two parts of the light process [22], which may be taken as a particular case of this general explanation. That one part represented a single temperature from the two temperatures involved here, and thus was 'legal', as it were, from the point of view of equilibrium thermodynamics. In general, however, i.e., considering the whole process representing the light, a particle becomes a means of transition between the light at the high temperature and the light at the low temperature. In modern terms this can be translated thus: insofar as the light is considered as the *modus essendi* of the *vacuum*, the particle accomplishing its interpretation can just as well be seen as the *modus essendi* of the *vacuum tunneling* process. Fact is, that the multiplicity of vacuum – the existence of an infinity of vacua simultaneously – seems to be an already settled issue, at least from a modern theoretical point of view [34].

In the statistical context of the present section of our work, this conclusion can be ascertained in the limit where the statistics depends exclusively on the correlation coefficient, as follows: in the good old fashion of statistical mechanics, we correlate the energy ε_0 with an exponential factor, which can play the role of a partition function over a certain ensemble, and which can be easily extracted from equation (2.2.6) as

$$e^{-\beta\varepsilon_0} = \frac{1}{1+2rw+w^2}\cdot\exp\left\{2\frac{r}{\sqrt{1-r^2}}\tan^{-1}\frac{w\sqrt{1-r^2}}{1-rw}\right\} \tag{2.2.7}$$

The left-hand side of this equation is a *Boltzmann factor* that represents a thermal ensemble for the energy ε_0, having the mean β. The odd thing here is that the right-hand side also depends on ε_0. However, this dependence occurs through the intermediary of the ratio w, which allows us to say that a statistical interpretation depends, in fact, on a sort of ε_0*-content* of the density of energy of the thermal radiation. This conclusion sounds quite normal: an experimentalist knows exactly how to characterize the radiation depending on its density. Should this density be of the order of ε_0 then $w \approx 1$, and the right-hand side of equation (2.2.7) does not depend but on the correlation coefficient between the two processes:

$$e^{-\beta\varepsilon_0} = \frac{1}{2(1+r)}\cdot\exp\left\{2\frac{r}{\sqrt{1-r^2}}\tan^{-1}\sqrt{\frac{1+r}{1-r}}\right\} \tag{2.2.8}$$

This is the limit we are seeking for. Almost four decades ago, H. Ioannidou tried to explain the quantum through the *correlation of ensembles* associated with oscillators, based on the uncertainty relation [33]. The attempt has been forgotten, probably due to the connection it suggested with an idea of contingency. However, that connection seems to be sound, for here it is again, in equation (2.2.8), which can be taken as a relationship between the quantum and the correlation coefficient of the two statistical processes representing the whole thermodynamical radiation. Further on, one can speculate that if the correlation of the two processes is faint, which is the Einstein's case, then

$$\varepsilon_0 = (ln2) \cdot kT \tag{2.2.9}$$

independently of any other consideration. Thus, in this limit, the 'quantum', and therefore the frequency, is directly proportional to the temperature, i.e., it can be expressed by the intermediary of a statistic thermodynamically established. This fact has been discussed at length by Louis de Broglie [8], who identified the action with the entropy, based on the idea of *cycles of phase*. As we shall see, such cycles are a necessity from the point of view of characterization of the phase.

In the general case, though, when the ε_0-content units and the correlation of the two processes are both arbitrary, it helps noticing that the right-hand side of (2.2.7) is the generating function of a particular class of *Pollaczek polynomials* [17]. Specifically, we can write (2.2.7) in the form

$$e^{-\beta\varepsilon_0} = \sum_{n=0}^{\infty} P_n^1(r) \cdot w^n \tag{2.2.10}$$

The orthogonality relation of the polynomials involved here is given by

$$\int_{-1}^{1} \varrho(r) P_m^a(r) P_n^a(r) dr = \frac{\pi}{2} \frac{m+1}{m+a+1} \delta_{mn} \tag{2.2.11}$$

with the weight function ρ given by

$$\varrho(r) = \sin\theta \cdot e^{(2\theta-\pi)\tan\theta} \cdot \left| \Gamma(1 + i \cdot \tan\theta) \right|^2, \quad r \equiv \cos\theta, \quad \theta \in [0, \pi] \tag{2.2.12}$$

where Γ is the *Euler function* of the first kind, generalization of the factorial. Should we agree to interpret ε_0 as the energy of a *photon*, as historically has been the case, then the formula (2.2.10) would be the source of constructions of some modern quantum states related to the coherence properties of radiation.

2.3. GETTING OVER THE GAUSSIAN: PLANCK'S NEW STATISTICS

The contradiction still remains, though, and it cannot be removed, at least not from thermodynamical point of view, as long as we do not settle on an independent statistical meaning of ε_0, and, perhaps, even a true statistical meaning for the temperature. Indeed, as long as one and the same thermodynamical system is characterized by two different temperatures and there are reasons to assume that it is physically unique, there is room for contradiction with the second principle of thermodynamics. In a specific way, such a contradiction flared up in the last half of the 20th century in the form of the problem of *zero point energy* [11], [12]. That problem extended conceptually the old observation that the connection between the Planck's formula (2.1.10) and the classical meaning of the temperature requires the consideration of one half quantum in order to agree with each other [26].

In this specific case, the *concept per se* was the one liberating physics from the authority of the second principle of thermodynamics: the existence of *zero point radiation*, i.e., an electromagnetic radiation having a spectrum depending exclusively on frequency, without the intervention of the temperature. As to the *classical meaning* of the temperature, it is that of a *statistic*: the *variance of the velocity field* of the gas molecules, as measured by the kinetic energy, considered itself as a statistical variate on an ensemble of classical material points. Our point of view is that the application of the absolute temperature in order to describe the radiation as a thermodynamical system, revealed the fact that it is not a *sufficient statistic* [50]. This would mean, first and foremost, that in describing such physical systems as the radiation from a thermodynamical point of view requires more than the heuristic point of view represented by the Gaussian statistic.

Fact is that by accepting the Boltzmann's and Gibbs' views regarding the relation between probability and entropy, physics placed itself at the disposal of a statistical science based on the so-called class of *exponential families of distributions*. In the case of one statistical variable X such a family, having the elementary probability given by

$$F_{\xi}(dx) = [Z(\xi)]^{-1} \cdot e^{\xi x} \cdot \nu(dx) \quad (2.3.1)$$

is known as *natural exponential family*. Here ξ is the parameter scanning the family, while $\nu(dx)$ is a Stieltjes measure of the domain of statistical variable X. The parameter ξ, usually related to the measurement of the variate X, is connected to the mean of the ensembles characterized by (2.3.1) through equation

$$m(\xi) = \frac{d}{d\xi}\ln Z(\xi) \quad (2.3.2)$$

a relation used explicitly before in order to interpret u as a mean over an ensemble of oscillators.

Now, the previous stride of reasoning may be flawed by the fact that the differential equation (2.1.9) looks, indeed, very much like an interpolating equation, having no substance of physical principle involved in its derivation. Its formal inference based on

Gaussian distribution in the previous section might thus be tarnished not only as being too particular, if not very approximate, but, as we have seen, on the very physical grounds. The only fact to start with here is that one can get the equation for the fluctuations starting from physical considerations upon the field sustaining the fluctuations, and the result is a quadratic polynomial for the variance. This proves to be essential from a certain point of view. Indeed, if we limit the considerations to natural exponentials, we then have necessarily

$$m'(\xi) = V(\xi) \tag{2.3.3}$$

where $V(\xi)$ is the variance function of the family of exponential distributions. If this function depends on the parameter ξ in such a way that *it can be arranged into a quadratic polynomial in the mean* $m(\xi)$ of the family of distributions, then we have a particular case of exponentials, nowadays termed as exponential distributions with *quadratic variance functions* [51]. These distributions cover just about everything we use today in the realm of physics and engineering: Binomial, Negative Binomial, Poisson, Gaussian, Gamma, and Generalized Hyperbolic Secant. The first three of these are distribution for discrete statistical variates, while the last three are referring to continuous variates.

Then, one can notice the advantage of the Born's approach to Planck's problem even if we do not use the Gaussian approach: it is referring, in a way, to all of the six distributions enumerated, because all of them have the Gaussian as a limit case ([51], Table 1). Therefore, *in principle*, the statistics of radiation has to be characterized by such a quadratic relationship between the mean and the variance of an ensemble representing the radiation. The corresponding distribution should not be *necessarily* a Gaussian, even though in limiting situations it can be reduced to a Gaussian, as most of the known statistical distributions can indeed. In this case the whole problem of radiation can be treated in its utmost generality, for the two limiting processes are quite different by their nature, albeit quadratic variance type processes both.

To wit, take the Planck's case: the equation (2.1.9) which is the characteristic of Born's line of reasoning can, by a simple and natural metamorphosis, be put into one of the Morris' canonical forms devised for the generic quadratic variance function distributions ([51], Table 1):

$$m'(\theta) = \frac{m^2}{r} + m \tag{2.3.4}$$

Here m, which is, statistically speaking, the mean of the distribution, is assumed to also be the measured density of spectral energy, and we used the notations

$$\theta = -\beta\varepsilon_0, \quad r = \frac{\varepsilon_0}{u_0} \tag{2.3.5}$$

where u_0 is an *arbitrary* constant energy density. Here, the parameter *r is not a correlation coefficient* anymore, but we still use the same letter to denote the parameter as in the original work. Then, according to Morris' scheme of describing such distributions, Planck's result is a *Negative Binomial* distribution [$NB(r, p)$] with the density of probability given by

$$P_X(x;r,p) = \frac{\Gamma(x+r)}{x!\Gamma(r)} p^x q^r \tag{2.3.6}$$

where the variable X is discrete, with values $x = 1, 2, \ldots, p \equiv e^{\theta}$ is a probability (the 'Boltzmann factor' mentioned earlier) and $q = 1 - p$ is the complementary probability. The mean of this distribution is provided by:

$$m = \frac{1}{u_0} \frac{\varepsilon_0}{e^{\beta\varepsilon_0} - 1} \tag{2.3.7}$$

which is Planck's formula, written, however, so as to make the occurrence of unit u_0 explicit: $u = mu_0$. Let us try to find some limit distributions for $NB(r, p)$.

First, we are interested, for instance, in those probabilities p close to *one*, describing highly probable events. As we have

$$V(m) = \frac{m^2}{r} + m \tag{2.3.8}$$

and by definition

$$p \equiv \frac{m}{m+r} \tag{2.3.9}$$

one can directly write

$$V(m) = \frac{m^2}{r \cdot p} \xrightarrow{p \to 1} \frac{m^2}{r} \tag{2.3.10}$$

According to Morris' classification, this represents a *Gamma* distribution from his Table 1, having the density

$$P_X(x;r,\lambda) = \left(\frac{x}{\lambda}\right)^{r-1} \cdot \frac{e^{-\frac{x}{\lambda}}}{\lambda \cdot \Gamma(r)}, \quad \lambda \equiv -\frac{1}{\theta} = \frac{1}{\beta\varepsilon_0} \tag{2.3.11}$$

for values X real and positive. In case of $r = 1$, i.e., $\varepsilon_0 = u_0$, this distribution is the classical exponential. In general, for finite r, this limit distribution of radiation is referring to the case where the mean energy of the ensemble characterizing the heat radiation is high i.e., we have $m \rightarrow \infty$. The temperature of this state cannot be but high too, in order to make up for the condition $p \sim 1$. At arbitrary but finite densities of radiation energy this limit distribution is also characteristic for $r \rightarrow 0$, which comes down to $\varepsilon_0 << u_0$, no matter of the relationship between ε_0 and kT, i.e., no matter of the value of the mean m. In other words, the process can be a Gamma process in *classical* as well as in *quantum* case, depending on the unit we choose for the measurement of the radiation spectrum.

Another limiting case of the general *Negative Binomial* distribution characterizing the Planck process, is the opposite one, i.e., that where the probability p is very *small*. According to the definition of p this happens for low temperatures, so that $\theta \rightarrow -\infty$, and is realized by an ensemble of very low energy density. In equation (2.3.8) the linear term prevails so that, according to Morris' classification the process is a *Poisson* one from his Table 1, and is characterized by the probability density

$$P_X(x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad \lambda \equiv e^{-\beta\varepsilon_0}, \quad m = \lambda \tag{2.3.12}$$

for the variable $x = 0, 1, 2, \ldots$ For arbitrary energy densities*, the Poisson limit distribution can also be realized when* $r \rightarrow \infty$, i.e., when $\varepsilon_0 >> u_0$, no matter of the relationship between ε_0 and kT. Again, just as before, the process can be a Poisson process in *classical* as well as in *quantum* case, depending on the unit we choose for the measurement of the radiation spectrum.

The previous limiting cases of the Negative Binomial Distribution recommend it as universal in the case of light, of course. However, the actual physical experience of the last century shows that the Negative Binomial is a universal distribution in a much larger acceptance [15]: it seems to be also the natural distribution for the fundamental phenomena occurring in just about any of the intimate matter structures. If so, then the quantization in matter is only a normal course of the natural philosophy and should be entirely analogous to the quantization in light. There remain, though, a few problems of detail, related to the statistics really involved in the case of matter. As the Table 1 of [51] shows, *all* of the quadratic variance function distributions have the Gaussian as a limit law. As we said, in view of this circumstance, the Gaussian cannot be more than a heuristic tool in a physical argument, just as it was to Planck himself. The weight of the argument in a physical statistics must fall on the *idea of variance*, involving therefore one or more of the other five distributions from the Morris' table. As we will see presently, the Procopiu's quantization seems to indicate this very prospect for the case of matter.

We cannot close this case without highlighting once more that quantization interpretation of the statistic of light is simply due to an accident: Planck's formula (2.1.10) can also be produced by the 'mainstream' *physical statistical* argument represented by equations (2.1.11) and (2.1.12). If physics does not enter the play, the statistics *per se* seems to have no suggestion of quantum in producing the formula, as its deduction by Planck, or even the deduction of equation (2.3.7) shows it. Thus, even though the scheme involving the equations (2.1.11) and (2.1.12) may seem a happy 'brain wave', as it were, the fact of the matter is that the right *procedure of quantization* must contain it necessarily, in order to enact the concept of quantum. Otherwise, we are not able to say if the quantization procedure is physically correct or not. Whence the importance of having such a scheme ingrained in the statistical procedure from the very beginning. This was the case of Procopiu's quantization procedure.

2.4. Getting Over the Discreteness: Procopiu Quantization

Like anyone in the field of theoretical physics at the beginning of the 20th century, Stefan Procopiu was mainly interested in two issues connected to the application of quantization for the case of matter: the structure of the Planck's constant [60], and the structure of the physical unit carrying the quantum of energy i.e., the analogous of the Planck's resonator. He calls it a *magneton* [61], and associates it with the *Weiss magneton* [75]. Mention should be made that this kind of magneton stayed also in the views of Einstein and Stern on the occasion of introducing the half quantum of energy to the natural philosophical awareness [26]. The present section of our work follows closely the general structure of Procopiu's own work, only skipping the original details. On the other hand, such details will appear later along our proceedings here, but only as acquired via theoretical physics in its historical development during the last century. They are due to a natural philosophy entirely analogous to Newtonian natural philosophy, but constructed here for the case of matter exclusively.

Regarding the blackbody radiation, Procopiu takes the Planck's law of radiation in the old form expressed in terms of the wavelength of light, which we write in the differential form of spectral radiance:

$$E_\lambda d\lambda = \frac{c_1}{\left(e^{\frac{c_2}{\lambda T}} - 1\right)} \cdot \frac{d\lambda}{\lambda^5} \tag{2.4.1}$$

In connection with this formula, Procopiu concentrates on a fundamental issue, which, in his mind appears closely correlated with the intimate structure of the matter. Quoting:

> If one would find for *h* a physical meaning, the constants c_1 and c_2 *would not depend but on physical quantities*.
>
> This interpretation will be satisfactory, when the oscillator *emits energy in discrete parts*, this energy *not depending on the nature of the oscillator* but only on its own period. When this period is large, the emitted energy is small, and vice versa.
>
> Therefore, *the problem is to find a mechanical or electrodynamical model for the oscillator*, whether it resides within the molecule of O, H, N, etc. or it represents itself the molecule, the atom or the electron. ([60]; our translation and emphasis)

Obviously, by 'oscillator' here, Procopiu means 'resonator': otherwise, an 'electrodynamical model' for oscillator does not seem to make much sense. In order to understand the problem, we need to translate it in the language of the §2.1., in order to be able to connect it with the Planck's formula (2.1.10). Transcribing (2.4.1) in frequency, it becomes

$$E_\nu d\nu = const \cdot \frac{\varepsilon_0}{\left(e^{\beta\varepsilon_0} - 1\right)} \cdot \nu^2 d\nu \equiv const \cdot u \cdot \nu^2 d\nu \tag{2.4.2}$$

where we used the symbolistic of the §2.1., with $u \equiv u(\beta)$ given by equation (2.1.10). This equation contains *two* factors including the differential, out of which only one was provided by the Planck's statistics before, as $u(\beta)$. The other factor, namely $\nu^2 d\nu$ comes from a Euclidean *geometry of the frequency*: the frequency ν is considered here as the magnitude of a three-dimensional frequency vector, uniformly distributed in the frequency space over the blackbody radiation realm, whose structure is assumed to be that of an ensemble of plane waves.

The previous excerpt is referring only to $u(\beta)$, though, which incorporates the two constants targeted by Procopiu, by replacing them equivalently with the *Planck's constant* and the *Boltzmann's constant*:

$$\varepsilon_0 = h \cdot \nu \quad and \quad \beta^{-1} = k \cdot T \tag{2.4.3}$$

However, in the interest of the correctness, it is necessary to always remember that the experimental results on light are expressed by the spectral density (2.4.1) or (2.4.2), which, when judging the Planck's result (2.1.10) may not mean too much. When we try to apply the quantization outside the realm of light, though, for instance into matter, the consideration of (2.4.1) or (2.4.2) makes all the difference. They tell us what the frequency is in the case of Planck's quantization: a *three-dimensional vector* uniformly distributed over an ensemble of plane waves. If this condition is not satisfied, the quantization may not work properly, and needs amendments. A case in point may be illuminating, mostly because it is quintessential when it comes to quantization in matter.

The Louis de Broglie's quantization of matter, by assigning a ponderous particle to the Planck's quantum [7], worked only under condition of a *plane wave* in phase with the motion of the ponderous particle in question. This assignment was completely made in the spirit of the initial statistics of frequencies in the case of thermal light. However, the point of association here is the definition of phase of the particle: it is assumed to be its uniform motion. Then, the relativistic precepts can be applied to this motion, and this opens Pandora's box, if we may be allowed, again, the sin of a secular term. For, when this is done, the results show that as long as the frequency is defined by time intervals, de Broglie's quantization does not work. It works properly only if the particle does not move. In the case of motion, there is a group of waves that represents the particle, not a single plane wave, as in the case of light. However, in the cases of more complex structures, the plane wave does not even work anymore, so we need to look either for some other kinds of waves, or for a different approach to quantization, etc., etc.

Coming back to our present discourse, Procopiu tried first to identify the material units suspected as being liable of representing fundamental 'bricks', so to speak, for the construction of matter in general, and then calculated their energy. His leitmotif is constituted by the molecular magnetic properties of matter, especially because the fresh discovery of the electrons encouraged, in the epoch, a certain reality of the classical Ampère currents within the realm of matter. The fundamental idea was here that of *frequency*: an Ampère current is liable to have a definite frequency which can be retrieved from the light it emits as electromagnetic radiation, according to Hertz theory. Quoting:

> … Thus, an electron that revolves around a molecule a certain number of times, can *render molecular magnetism phenomena observable*.
>
> But a molecule has *a central positive nucleus*, and around it the negative electrons circulate.
>
> Assume that the positive center is at rest, and that an electron revolves around it. Their electric charges are equal and of different signs, $\pm e$. The electron will prompt a *convection current*, and, according to Ampère, *a magnet.*
>
> Imagine that this revolving electron *can be identified with the Planck resonator*, that is it will *absorb or emit energy discontinuously*. This is not possible but only if the electron is removed from the molecule, thus making *some magnetic phenomenon disappear*, or under condition that a foreign electron is introduced into the molecule, *thus making such a phenomenon happen*. If in these conditions, the electron prompting the magnetism can be taken as a Planck resonator, one can apply the relation $\varepsilon = h \cdot \nu$, which connects its energy ε, to the frequency ν. That is, when ε, the revolving electron energy, and ν, its frequency, are known, one can have h. ([60]; our translation and emphasis)

The method of calculating the Planck's constant is thus quite simple: calculate the energy of the fundamental structure, and the pulsation of the electron making it up, and then, their *ratio* is Planck's constant. At the time when Procopiu took the challenge, there seemed to be an established value of the Weiss magneton, independent of the nature of substances used in establishing this value, of 1.64×10^{-21} erg/Gauss. Procopiu took it and calculated the energy and frequency of such a magneton, after which he calculated the Planck's constant. The result is:

$$h = 4\pi \cdot M \cdot \frac{m}{e} \quad or \quad h = 1.73 \times 10^{-27}\ erg \cdot sec \tag{2.4.4}$$

where M is the value of Weiss magneton, m is the mass of the electron and e its charge. The epoch's value of the Planck's constant was of 6.55×10^{-27} erg·sec, which shows that the Procopiu's evaluation is way too far from the truth. But he had another chance: the physical structure of the magneton, as an Ampère current, involved 'a positive center at rest, and an electron that revolves around it', as he states in the previous excerpt. So, the chance is to evaluate the energy as the static potential energy of the orbiting electron. He did it, and based on the same principle of calculating the Planck's constant he found:

$$h = \pi \cdot r \cdot \frac{e^3}{M \cdot c} \quad or \quad h = 6.64 \times 10^{-27}\ erg \cdot sec \tag{2.4.5}$$

where epoch's value for r was about 10^{-9} cm. This value is closer to the target, but *in hindsight* it cannot be satisfactory, knowing the unreliability of the Weiss magneton value [62]. However, as we shall show here, there may be an explanation involving a principle for such a blatant improvement in the numerical result, when calculating the Planck's constant this way, i.e., by considering the electrostatic energy.

Be it as it may, fact is that Procopiu himself was discontented with both results so that he tried one more way of evaluation, and this was radically different. Quoting:

> I *also tried* to find a relation for h, starting from one of the most general phenomena for the molecules, the *diamagnetism*.
>
> Consider a molecule *having the shape of a sphere* – which is true at least for the monoatomic gases. In order that this molecule possesses a magnetic moment, i.e., a quantity that characterizes a magnet, *it must be assumed that this molecule has inside arrangements which manifest themselves as magnets, or even magnets*. Long time ago Ampère has imagined that the magnetism of a molecule issues from an electric current displacing on the surface of the molecular sphere. ([60]; our translation and emphasis)

The work per se on this new settlement of ideas, was deferred to another paper [61], and this one, we think, has a great importance of principle. The numerical results are, again, quite qualitative: the Planck's constant is taken this time as known numerically and based on this the value of magneton is calculated and compared with values existing in the specialty literature. But this is not the point we want to make: the fact of the matter is that *Procopiu finds a relation identical to Planck's* (2.1.10) or, better, to equation (2.4.2) here, but for a *continuous univariate distribution* from the same class with the Negative Binomial Distribution of Planck: the *quadratic variance function exponential distribution*.

We are talking about the *Generalized Hyperbolic Secant* distribution. In the process of quantization, the scheme involving equations like (2.1.11) and (2.1.12) is incorporated into method and does not have to appear as a lucky guess based entirely on mathematical contingencies. Moreover, the statistic is referring explicitly to the physical unit carrying the magneton, just as the Planck's statistics is referring to the physical unit carrying the photon. This would mean that regarding the procedure of quantization per se, but mostly the quantization in the realm of matter, it is not important the distribution, as much as the *family of distributions*: they must be, indeed, *quadratic variance function distributions*. Moreover, it seems that within matter, unlike the light, the quantization should go mainly for *continuous distributions*. But let us see the case in point.

Following the Langevin's theory of the magnetism [37], Procopiu's quantization replicates the equations (2.1.11) and, implicitly, (2.1.12), based on the fact that a resonator – the physical structure producing or absorbing the light – must be an *ensemble of magnetons*, therefore appropriate for statistical treatment. To wit, Procopiu gets for the mean energy of an ensemble of magnetons entering the structure of a resonator, instead of equation (2.1.10) of Planck, the equation of Langevin:

$$u(\beta) = -\varepsilon_0\left(coth\left(\beta\varepsilon_0\right) - \frac{1}{\beta\varepsilon_0}\right),\quad \varepsilon_0 \equiv \frac{Me}{r^2} \tag{2.4.6}$$

This mean is prone to a classical statistical treatment using the partition function, for we then can infer right away:

$$u(\beta) = \frac{\partial}{\partial\beta}\ln Z(\beta),\quad Z(\beta) \equiv \frac{\beta\varepsilon_0}{\sinh\left(\beta\varepsilon_0\right)} \tag{2.4.7}$$

No question, we have here a perfect parallel with the case of Planck: the mean (2.4.6) is first inferred by a statistics having a physical content, and then shown to be calculable by a partition function. It is just that, in the Planck's case, the statistics having a physical content was invented ad hoc, while here it was undertaken from Langevin's theory. Based on the analogy carried only this far, Procopiu identifies ε_0 with a Planck quantum, in spite of the fact that there is no frequency involved here. This is why he tries to find regularity in some spectral series in order to rustle up a frequency, based on which to calculate the magneton *M*. As we said, the numerical results are only qualitative, but this does not diminish by any means the merit of the method.

Indeed, we have to ask: should a resonator structure be identical with the light structure? The analogy with Planck's theory goes to details. In fact, we may not even be interested on the provenance of the function (2.4.6) as long as we know that it is obtainable from a known partition function by the recipe from equation (2.4.7). Here too, ε_0 is an energy that does not depend on the temperature, just like the $(h\cdot\nu)$ in the statistics of blackbody radiation. However, a major problem still remains for us to solve: is the function (2.4.6) exactly of the type (2.1.10) or of the type (2.4.2), involving two statistics?! And while trying to answer this question, we discover an essential fact: Procopiu's quantization procedure is centered on a statistics based on a quadratic variance function distribution, just like Planck's statistics. The analogy, therefore, goes to details!

Indeed, it is the very formula (2.4.7) that tells us which case is to be considered here. Fact is that not quite $u(\beta)$ from (2.4.6), but $Z(\beta)$ from (2.4.7) involves *a priori* two identical statistics, of the Hyperbolic Secant type. It is time to show that here too, we have a formula analogous to (2.1.11), not quite for $Z(\beta)$ but for its logarithm, which is actually sufficient for physics' tasks. In order to do this, we follow closely [69] (see also [35], pp. 113 – 163). Indeed, we have the identity:

$$\frac{x}{\sinh(x)} \equiv \frac{1}{\prod_0^\infty \cosh(2^{-r}x)} \quad \therefore \quad \ln\frac{x}{\sinh(x)} = -\sum_0^\infty \ln\{\cosh(2^{-r}x)\} \tag{2.4.8}$$

so that

$$\frac{\partial}{\partial x}\ln\frac{x}{\sinh(x)} = -\sum_0^\infty 2^{-r}\{\tanh(2^{-r}x)\} \tag{2.4.9}$$

Thus, the theory can go just like in the Planck's case taking the quantum as $2\cdot\varepsilon_0$, and the hyperbolic tangent replacing the exponential from (2.1.11). Provided, of course, the hyperbolic tangent has the necessary physical properties. But this is not momentarily our concern, for the formula we are looking for, is, again, in front of us: the one representing the *convolution property*:

$$\frac{2x}{\sinh(x)} = \int_{-\infty}^{\infty}\frac{dy}{\cosh y\cdot\cosh(x-y)} \tag{2.4.10}$$

This is showing that the distribution used by Procopiu is of the type from equation (2.4.2): there is a 'dummy' distribution, analogous to the a priori distribution over the frequencies in the case of blackbody radiation, and a main distribution, analogous to

(2.1.11). Just like in the Planck's case, both these distributions are here of the same type: quadratic variance function distributions, but of continuous type, namely Hyperbolic Secant. The problem is only to establish their physical meaning, and the meaning of the convolution from the equation (2.4.10). As there is no physical theory allowing an interpretation leading to ensembles supporting these distributions, such a theory has to be constructed by starting from new principles.

This would mean that, inasmuch as there is no principle involved in the identification of the *Procopiu*'*s quantum* with *Planck*'*s quantum*, they *should be taken as different*. In fact, one is referring explicitly to the structure of the resonator while the other is referring to the structure of light. Then, just as a theory of light has been developed based on the existence of quantum, so it has to be developed a theory of matter based on the existence of a *different* quantum. Therefore, we need also a quantization for the resonator, a fact obvious even from the times of Planck himself. The way this quantization was achieved was by ad hoc involving the Planck's constant only [9], [65]. Procopiu himself does not make exception. However, if there is a frequency involved in the interpretation of Procopiu's quantum, this must be constructed from the fundamental physical characteristics of the matter, just like the physical frequency of a harmonic oscillator: this is the whole morale of the identification of the resonator with an oscillator. As it turns out, a quantum theory of matter has, indeed, always been with us even from the times of Newton.

3. THE NATURAL PHILOSOPHY OF QUANTIZATION IN MATTER

Two issues have to be discussed, starting with this chapter of our work. First, is the *physics of quantization in matter*, based on which, according to previous analysis, the quantum must be constructed. Secondly, the *structure of the quantum* itself, and its connection with the *replacements*, within matter, of the frequency from the case of light. We are following thus the idea to find in matter the analogous of the Planck's constant from the case of light. The place to start in constructing a theory is that of any physical theory whatsoever: the classical natural philosophy.

3.1. THE THOMSON'S PHYSICAL THEORY

Joseph John Thomson's work specifically addressed to the problem of quantization in matter is the only one known in history that can be taken in the capacity of a natural philosophy. Its fundamental point of view is, just like the one of the prototypical Newtonian natural philosophy, *the presence of forces in matter*. And just like in the case of that precursor natural philosophy the accent here falls, indeed, upon the presence of forces determining a trajectory of motion via the dynamics instituted by Newton. However, unlike that forerunner, this new natural philosophy had to account for the presence of radiation in the world it describes. Then, the idea surfaced that the quantization in the case of matter is a natural physical law regulating the interaction of the matter with radiation: Planck's quantization in the case of light should be an expression of this physical law, by exacting the necessity of inventing the resonators. The Thomson addition to the classical natural philosophy, however, seems to point out naturally to their *dynamical* necessity. To wit: the ancestor classical natural philosophy appears to inherently hold facts, even mathematical facts, that make the quantization in matter a natural law. It is this aspect of classical physics that we shall pursue in the remaining part of the present work.

In hindsight, one can precisely label the Thomson's dynamics as a Newtonian dynamics, whereby the controlling forces had to cope with the idea of *open orbits* in the case of Kepler problem. It is well known, indeed, that the foundation of Newtonian natural philosophy is provided by an image of heavens where the Kepler systems are routine, if we may say so. In order to elucidate this point in connection with the existence of the light as a fundamental phenomenon in the world, we need to appeal to the *classical theorem of Joseph Bertrand*, which proves that the only forces admitting *closed orbits* for the dynamical system that describes the classical Kepler problem are the central Newtonian forces and the central elastic forces [6]. Therefore, one can say that the closed orbits are only connected with these two classes of forces. In this case then, the problem can be aroused as to what is, theoretically speaking, beyond the closed orbits, inasmuch as the observations show that they are, in fact, not quite a routine. In order to get an answer, let us show the case in detail.

Joseph Bertrand proves these statements by taking the classical Binet's equation in the form

$$\frac{d^2u}{d\phi^2} + u = \dot{a}^{-2}F(u), \;\; F(u) \equiv r^2 f(r) \tag{3.1.1}$$

where the magnitude of the central force determining the Newtonian dynamics that controls the motion has the magnitude $f(r)$. The polar coordinates (r,ϕ) of the mobile material point are reckoned with respect to the center of force, and the dependent variable u is defined by equation $u \cdot r = 1$; further on, $\dot{a}$ denotes the area rate from the second of the Kepler's laws. The equation (3.1.1) has the immediate integral

$$W = \left(\frac{du}{d\phi}\right)^2 + u^2 - \dot{a}^{-2}\varpi(u), \;\; \varpi'(u) \equiv 2F(u) \tag{3.1.2}$$

where a prime means derivative with respect to the independent variable, as usual. Now, in order to have a closed orbit the function $u(\phi)$ must vary between two limits, α and β in Bertrand's notations, and this imposes definite requirements on the period of central angle ϕ of the motion. Bertrand describes this period in connection with the dynamical properties of the motion, using the differential of angular variable which results from the equation (3.1.2), written in the form

$$d\phi = \pm \frac{du}{\sqrt{W + \dot{a}^{-2}\varpi(u) - u^2}} \tag{3.1.3}$$

Then, notice that the two extremes of u correspond to the extremes of r at the vertices of the orbit, where it cuts its axes of symmetry. In such points we have $u'(\phi) = 0$, so that α and β should satisfy the algebraic equation

$$W + \dot{a}^{-2}\varpi(u) - u^2 = 0 \tag{3.1.4}$$

which ensues from (3.1.2) under this condition. The two equations corresponding to α and β provide the constants from (3.1.3):

$$\dot{a}^{-2} = \frac{\beta^2 - \alpha^2}{\varpi(\beta) - \varpi(\alpha)}, \;\; W = \frac{\alpha^2\varpi(\beta) - \beta^2\varpi(\alpha)}{\varpi(\beta) - \varpi(\alpha)} \tag{3.1.5}$$

and these two equations show that the condition

$$q\pi = \int_{\alpha}^{\beta} du\left(W + \dot{a}^{-2}\varpi(u) - u^2\right)^{-½} \tag{3.1.6}$$

must be valid no matter what values α and β may have. Here they represent a minimum and the maximum that follows immediately after it, or vice versa, and q must be a *rational number*. This is a purely geometrical condition defining the closed plane curves. Specifically, the equation (3.1.6) is an expression of the geometrical theorem saying that when a plane curve admits two axes of symmetry, it is a closed curve if, and only if, the angle between the two axes is a rational multiple of π. The equations from (3.1.5) secure the conclusion that the condition (3.1.6) should be valid for any values that α and β may happen to have.

Now, the most common among the cases one practically meets in astronomy, is that of an orbit with very small eccentricity. According to Bertrand, two things happen in such a case. First, the integral from equation (3.1.6) can be *approximately* carried through, with the result

$$q^2 = \frac{\varpi'(\alpha)}{\varpi'(\alpha) - \alpha\varpi''(\alpha)} \tag{3.1.7}$$

with the usual notation for the derivative of a function with respect to its argument. Secondly, this result can be turned into a differential equation, having the remarkable solution

$$\varpi'(\alpha) = A\alpha^{1-q^{-2}} \tag{3.1.8}$$

where A is an integration constant. In view of equations (3.1.1) and (3.1.2) this solution means

$$f(r) = \frac{A}{2} r^{-3+q^{-2}} \tag{3.1.9}$$

Starting from this point, by a reasoning which, in our opinion, involves a space *scale transition*, because it asks for a further convenient integration and differentiation, Bertrand concludes to the existence of *only two* possible values of q: *1* and *1/2*, so that the only central forces having closed orbits are those of magnitudes

$$f(r) = \frac{A}{2} r^{-2} \ \ and \ \ f(r) = \frac{A}{2} r \tag{3.1.10}$$

Therefore, the only forces responsible for this dynamics, cannot be but central forces, with the magnitude depending only on the distance between center of force and the position of action, in a special way: inversely proportional with the square of that distance, and directly proportional to it. The first kind of forces is that of the Newtonian forces, like the gravitational and electric or magnetic Coulomb ones, while the second kind is given by elastic isotropic forces. Thus, only these two categories of forces can have closed orbits according to classical Newtonian dynamics.

This theorem shows that the dynamics with Newtonian and elastic forces – that is, those forces that laid the foundation of the classical dynamics – excludes from the very beginning the possibility of description of one of the best documented phenomena in astronomy: the *planetary perihelion advance*. One possibility of incorporating the description of this conspicuous phenomenon into the classical theory is to assume the existence of some other kinds of forces for the fundamental dynamics, not necessarily central. Newton himself, though, imprinted a particular custom to the classical natural philosophy, insofar as he persevered on central forces in order to solve the issue, in a manner that, by today's standards may be deemed as a 'perturbation method': he added to the force inversely proportional to the square of distance, that describes the planet's motion as a dynamical problem, a force which is still central but weaker than it at the location of its action. To wit: a central force of the same type as those from equation (3.1.10), and generated by the same center of force, but with magnitude inversely proportional to the *third power* of distance [(Newton, 1974), Book I, Propositions IX & XLIV]. By today's standards, such a perturbed Newtonian force generates a Kepler motion with a different area rate according to the second of the Kepler's laws (see [45], and the literature cited there). Therefore, if the area constant itself varies in a certain way in this dynamics, a family of rotated ellipses is generated having different orientations in the plane of motion. Notably, if the force of perturbation with the inverse cube magnitude is acting *only by itself*, it generates an *open trajectory* of the kind of spirals [(Newton, 1974), Book I, Propositions IX].

Notice, however, in connection with the very Bertrand's result as described by us above, that the magnitude of the central force from the equation (3.1.9), which, being determined by a solution of the differential equation (3.1.7), would *not involve any space scale transition*. This means that this force would describe a dynamics in the very same same realm characterized by the condition (3.1.7). And in this realm the equation (3.1.9) has an interesting limiting case, when the angle between the two successive extrema of the distance of the mobile point from the center of force appears as *very large*, in a precise sense: it takes a great many turns of the trajectory in order to complete that angle ([82]; see Figure 2 of this reference in order to get an idea for the concept of 'turn'). Mathematically this means:

$$\lim_{q\to\infty} f(r) = \frac{A}{2} r^{-3} \tag{3.1.11}$$

This is exactly the magnitude of the central perturbative force invoked by Newton in explaining the perihelion advance. It would correspond, indeed, to a practically open orbit, having a very big number of revolutions around the center of force, between a maximum and the minimum following it. Such an orbit can practically count as an open orbit, something like a spiral indeed, as Newton himself noticed. The action of the force (3.1.11) can also be described by a succession of ellipses in rotation, as noticed by Newton too, but only in those cases where it acts *concurrently* with a Newtonian inverse quadratic force, *and issues from the same center*, in such a way that their magnitudes are linearly combined [(Newton, 1974), Book I, Propositions XLIII and XLIV].

It is this last case that attracted Joseph John Thomson's attention to the point where he took for granted that the dynamics within the realm of matter that would be proper for quantization – a manifold that we would like to call *Thomson's realm*, not just to honor the name of great theorist, but mainly to indicate, as we said, the space proper for quantization in the matter – is dictated by an equation like [70], [71]:

$$m\ddot{r} + \frac{Ae}{r^2} = \frac{Ce}{r^3} \tag{3.1.12}$$

Here A and C are two constants, e is the charge of the moving particle and m its inertial mass. This is a 'radial' dynamics generated by Newtonian forces and *perturbed* by the weaker forces of the kind (3.1.11) generated by the same center of force and acting in conjunction. It would appear as quite particular from the point of view of the Newtonian natural philosophy. However, we shall attempt to prove here that it is the most general force that would describe the dynamics in the world where the light dominates the phenomenology.

To start with, notice that the corresponding Binet equation of (3.1.12) is

$$\frac{d^2u}{d\phi^2} - \frac{Ce}{m\dot{a}^2}u + \frac{Ae}{m\dot{a}^2} = 0 \quad \therefore \quad \frac{d^2u}{d\phi^2} - \Lambda^2\left(u - \frac{1}{r_0}\right) = 0 \tag{3.1.13}$$

where the notations

$$\Lambda^2 \equiv \frac{Ce}{m\dot{a}^2} \quad \text{and} \quad r_0 \equiv \frac{C}{A} \tag{3.1.14}$$

have been used. The general solution of the second-order equation (3.1.13), can be written in the form

$$\frac{r_0}{r} = 1 + a \cdot \cosh(\Lambda\phi) + b \cdot \sinh(\Lambda\phi) \tag{3.1.15}$$

with a and b two constants of integration. Obviously, if r can be written as a quadratic form in a binary domain generated by the hyperbolic functions, then this equation can be interpreted as a family of conics in that binary domain. This is an important issue that came up in astronomy mostly in the second half of the last century, under the name of *regularization*, and we shall address it in due time. For the moment though, we concentrate in the Thomson's path to constructing this dynamics.

Thomson starts from the observation that r_0 is a 'static point', as it were: at that radial position, the two radial forces from the equation (3.1.12) are in equilibrium, and $u''(\phi) = 0$. It is thus interesting to see what is happening around this point, and this task is easy to carry out. Namely, Thomson considers small departures q from the equilibrium point r_0, for which the equation of motion can be written as:

$$m\ddot{q} + \frac{Ae}{(r_0 + q)^2} = \frac{Ce}{(r_0 + q)^3} \tag{3.1.16}$$

which, in the first order in q this becomes an equation of the harmonic oscillator:

$$\ddot{q} + \omega^2 q = 0, \quad mr_0^4\omega^2 \equiv Ce \tag{3.1.17}$$

It is only natural then to assume that any real perturbation – for a common instance say a collision event – of an electric corpuscle located at the equilibrium distance in an atom described by the above fields of forces, will lead to a harmonic motion described by equation (3.1.17) destined to recover the equilibrium of the two forces. Then, Thomson's contention is that the frequency of this harmonic *disturbance* is to be retrieved from the electromagnetic radiation emitted on the occasion of a collision. Applying to this radiation the Planck's quantization procedure, it should carry an energy amounting to:

$$w \overset{def}{=} \hbar\omega = \frac{\hbar}{r_0^2}\sqrt{\frac{Ce}{m}} \tag{3.1.18}$$

where $\hbar$ is, in today's terms, the *reduced Planck's constant*. We have here an identification of the frequency with an expression in terms of fundamental constants of the matter, while still holding the Planck's constant as fundamental for the case of matter. However, the Thomson's theory adds on important observation: such fundamental constants must characterize some equilibria. For once, the free particle in this dynamics is due to an equilibrium of forces. On the other hand, the specific dependence of this energy on the equilibrium position where *the radiation is assumed to be created*, indicates that it cannot originate but in the field of inverse cube forces. Indeed, their potential energy would be, according to its classical definition:

$$\int_{r_0}^{\infty} \frac{Ce}{r^3} dr = \frac{1}{2} \frac{Ce}{r_0^2} \tag{3.1.19}$$

and a comparison with (3.1.18) proves our statement. Therefore, in order to have a *quantization via the energy*, like in the Planck's case, the inverse cubic force *must be universal* in the Thomson's realm: it is not just a perturbation, inasmuch as its existence is mandatory for quantization. Then the constant C of the universal third inverse power forces can be evaluated by the conservation law of energy, from which Thomson gets

$$C = \frac{(2\hbar)^2}{em} \tag{3.1.20}$$

which is about 2.05×10^{-19} SI units. The next point in order now, is to theoretically assess this new universal force, to which Thomson assigned only centrifugal properties, intuitively obvious by the concept of inertia. As it turns out, from this assessment, the case seems to be more involved than exacting the existence of the centrifugal forces: the whole structure of a quantum in the Thomson realm, depends on these universal forces. In fact, it is the Newtonian force that turns out to be incidental, however not in the sense of its weakness.

3.2. THOMSON PREREQUISITES: THE INVERSE CUBIC FORCE

To start in our assessment, one must notice that there is not any reason to assume the universal validity of the central force (3.1.11), while knowing that, according to classical natural philosophy, the Newtonian field of force should be also universally present in the charged matter. In other words, the classical *planetary* atom, for instance – if not the classical planetary model at all – would be compromised with the dynamics described by equation (3.1.12). At this point Thomson invokes one of the conspicuous properties of the forces in the classical dynamics: they are acting in *limited spaces*. From this point of view, they resemble Planck's light, which can only be physically studied and, therefore, quantized as such, within Wien-Lummer enclosures [76]. And, we must add, that from this point of view, the Newtonian forces are invariant to a special dimensional gauging [79]. To be specific, dynamically, the Newtonian forces act only along the orbits, exactly the way the Coulomb forces act statically, i.e., only along the lines of force. Then Thomson contends that within the space of an atom, for instance, the inverse quadratic forces act only inside some *tubes of flux*, centered around the lines of electric force, while outside such tubes the moving particle is under the exclusive control of the universal inverse cubic forces.

On the other hand, there is a result due to Paul Langevin, according to which there cannot be a purely radial motion unless the central force controlling the dynamics of motion has a magnitude of the form (3.1.11) (see [37], pp. 97 – 102). In other words, the Kepler orbits are 'frozen by default', as it were, in the framework of the classical dynamics with Newtonian forces. Then the quantization of such orbits appears to be just a convenient manner of expression of the classical fact that the Newtonian forces are defined only along with the orbits. And if this quantization is accepted in the way of Niels Bohr for instance [9], then the forces (3.1.11) are the natural forces in the realm of matter, while the Newtonian forces become subordinate ([77]; see also [50], Chapter 7). Thomson's contention is that this should be a fact that takes precedence over any kind of quantization within matter. All that counts now, is to be able to put in order two apparently contradicting facts: the *confinement* of the action of Newtonian forces and the lack of confinement of the action of the inverse cubic force. Quoting:

> If a corpuscle at P were *inside one of the tubes of attractive force* inside the atom, it could be removed to an infinite distance: (1) *by moving it gradually outwards and keeping it inside the tube the whole way*. If the attractive force on unit charge at a distance r from the centre is A/r^2, the work required to remove the corpuscle in this way from r to an infinite distance is Ae/r. The corpuscle could however be moved to an infinite distance in another way (2) *by moving it sideways out of the tube at P and then moving it outside the tube to an infinite distance*; this later process will

> absorb no work as the attractive force vanishes outside the tube. By the conservation of energy, the work required must be the same, whether we adopt the process (1) or (2); hence *the work required to move the corpuscle sideways* out of tube at *P*, must be equal to *Ae*/*r*. ([71]; our emphasis)

In what is left from this very section we shall be occupied with that 'motion sideways' that seems to be counter-intuitive, even paradoxical we should say, in a classical central field: the Thomson's assumptions would be, indeed, capable to save the day, provided one is able to argue how the moving particle manages to get out of a flux tube.

In order to do this, we first need to characterize the forces, and not just anyway, but in matter. Thus, define a generic reference frame which, while independent of any particular geometry, still retains a geometrical spirit. We present here a case of 'second degree' vectors in Cartesian coordinates, that can be very well taken as a Cartesian reference frame in the general case. Indeed, a reference length geometrically described as a distance *r*, can be taken in constructing three unit vectors in the matrix form, as follows:

$$\hat{e}_1 = \begin{pmatrix} 2\frac{x^2}{r^2} - 1 \\ 2\frac{xy}{r^2} \\ 2\frac{xz}{r^2} \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 2\frac{xy}{r^2} \\ 2\frac{y^2}{r^2} - 1 \\ 2\frac{yz}{r^2} \end{pmatrix}, \quad \hat{e}_3 = \begin{pmatrix} 2\frac{xz}{r^2} \\ 2\frac{yz}{r^2} \\ 2\frac{z^2}{r^2} - 1 \end{pmatrix} \tag{3.2.1}$$

where (x, y, z) are *any* three real numbers satisfying the quadratic condition $r^2 = \sum x^2$. These vectors form an orthonormal reference frame, in the Euclidean sense, as one can easily verify. A short elaboration on the possible origin and realm of this reference frame, from a continuous mechanics' point of view, should be salutary for what we have to say further in this work.

The classical Poisson equation was, from its very beginning [59], taken to mean the preponderance of *matter* over the *field*. Both these concepts were naturally brought to human knowledge, but only along with the idea of apparently natural 'dominance', as it were, of the matter via the Newtonian theory of forces. There is, however, a renowned case whereby the Poisson's equation is taken out of this classical habit of *defining the field* when the *density of matter* is known, and used *into defining the density* of matter when *the field is known*. Considering this doctrine, we are compelled to find such an idea significant from yet another point of view: it is a construction that served to build an image of the electric ether, based on *considerations of statics*.

Indeed, the essence of ether, is well epitomized physically by the so-called *Maxwell stress system*, describing *a continuum* in a given space. This stress system was described by Clerk Maxwell in the Chapters 4 and especially 5 of the first volume of his classical treatise [47]. This system is mostly cited as an example of the failure to describe the ether as what is classically conceived as an isotropic incompressible medium (see especially [43], where the system of Maxwell stresses is presented from different mechanical perspectives in various places of the book). However, we think that it is still in position to straighten up some of our modern physical concepts, especially that of *static ensemble of equilibrium*, so necessary, for instance, to any theory of interpretation in the wave mechanics [19]. Maxwell himself did not seem to have used very much the system of stresses defined this way. In hindsight, this appears to have happened mostly because he seems to have been carried away by the electromagnetic image of the light, whereby the dynamics appears to be the essential working ingredient. By the same token, however, the subsequent neglect of the Maxwell stress system in physics may have been due to a deeper, objective reason, that can be assigned to the necessity of *interpretation* in physics, as aroused through the occurrence of the wave mechanics [19].

Maxwell's problem was to find the stresses induced by the action of forces in ether, in order to explain the *omnipresent* gravitational and electric forces. The attraction was represented in those times just as it is today, by the Newton's forces, which proved also to be valid for electricity, as Charles Coulomb would have long shown. On the other hand, the ether was considered, by default as it were, to be matter. Maxwell did not take into consideration these properties directly, but first translated them into a problem *involving the continua*, in order to describe the *local action*, not the action at distance: to find those stresses *statically equivalent* to a system of forces in general. Notice that these stresses had also to face, later though, the fact that the matter does not seem to be dragged by ether, which was proved experimentally toward the end of the 19th century. This circumstance too, indicating that the ether might not be matter after all, may have added to ignoring the case as inessential, inasmuch as neither the gravitation, for instance, nor the electric action could be subsequently explained as drag forces. This conclusion was even reinforced by

Henri Poincaré, who specifically showed that the electric matter of Lorentz is in default with respect to classical dynamics, inasmuch as it does not obey the classical principle of action and reaction (Poincaré, 1900). He even pushed this property into describing the forces of gravitation, and the forces of cohesion of matter in general, thus inventing the so-called *Poincaré stresses* for instance, in order to explain the material structure of an electron (Poincaré, 1906).

As mentioned above, the mathematics of a force generated by *matter within matter* was in those times, and still is today for that matter, expressed by the Poisson equation, which we rewrite here in the form:

$$\nabla^2 U(x,y,z) = 4\pi\varrho(x,y,z) \tag{3.2.2}$$

In this equation $U(x, y, z)$ represents the potential of the forces in the medium of density $\rho(x, y, z)$. If this medium is electrically active, then ρ is the density of electricity and U is an electric potential. Maxwell apparently took equation (3.2.2) as *defining the density* of the medium, rather than the potential, for the following good reason: he proved that the equation of equilibrium of a system of stresses is satisfied with the volumetric forces corresponding to a matter with density given by (3.2.2). Indeed, the equation of equilibrium of a continuous stress system in general – which, in its simplest form, asserts that the divergence of the second order stress tensor, $\boldsymbol{t}$ say, is given by the density of volume forces $\boldsymbol{f}$ – can be written as [43]:

$$\nabla\cdot\boldsymbol{t} + \boldsymbol{f} = \boldsymbol{0} \tag{3.2.3}$$

When this theory is specifically applied to the stress tensor $\boldsymbol{t}$ defined by the matrix

$$\boldsymbol{t} \equiv \frac{1}{8\pi}\begin{pmatrix} 2\left(\frac{\partial U}{\partial x}\right)^2 - (\nabla U)^2 & 2\frac{\partial U}{\partial x}\frac{\partial U}{\partial y} & 2\frac{\partial U}{\partial x}\frac{\partial U}{\partial z} \\ 2\frac{\partial U}{\partial x}\frac{\partial U}{\partial y} & 2\left(\frac{\partial U}{\partial y}\right)^2 - (\nabla U)^2 & 2\frac{\partial U}{\partial y}\frac{\partial U}{\partial z} \\ 2\frac{\partial U}{\partial x}\frac{\partial U}{\partial z} & 2\frac{\partial U}{\partial y}\frac{\partial U}{\partial z} & 2\left(\frac{\partial U}{\partial z}\right)^2 - (\nabla U)^2 \end{pmatrix} \tag{3.2.4}$$

the equilibrium equation is identically satisfied for a force density $\boldsymbol{f}$ given by

$$\boldsymbol{f} \overset{def}{=} \frac{1}{4\pi}(\nabla^2 U)\cdot\nabla U \tag{3.2.5}$$

In other words, the stress system (3.2.4) is *statically equivalent* to the system of volume forces (3.2.5) of the matter having a density given by Poisson's equation. Thus, the gravitation, for instance, can be conceived as a tension due to these stresses through ether; and likewise, the electric force. Poincaré's conclusion about Lorentz material system, which is based exclusively on electric forces, can be taken as showing that such stresses are insufficient to do the job they are called for, regardless the system of electric forces taken into consideration.

Obviously, the matrix having the vectors (3.2.1) as columns fits into this picture for a special potential:

$$U = r, \quad \nabla U = \frac{\boldsymbol{r}}{r}, \quad \nabla^2 U = \frac{2}{r} \tag{3.2.6}$$

and therefore, the field of static forces (3.2.5) equivalent to the stresses given by (3.2.4) in the sense of equilibrium equation (3.2.3) is given by the logarithmic force:

$$\boldsymbol{f} = \frac{1}{2\pi}\frac{1}{r}\frac{\boldsymbol{r}}{r} \tag{3.2.7}$$

This force represents a *Maxwell continuum* of the kind once 'criticized' by Eugenio Beltrami as being unrealistic according to human experience [5]. We should even add more beyond the Beltrami's conclusions: standing in the words of Poincaré, this continuum even qualifies as *fictitious* according to human experience. However, in a note of disagreement with Beltrami, we have to add that, far from being a drawback in the description of matter, this quality is actually its 'strength', as it were. In fact, the matter thus described cannot have any physical reality without an interpretation in the sense once defined for the necessities of the wave mechanics [19], and the key to such an interpretation stays in the existence of static forces (3.2.7) equivalent to this tensor, as described above.

By the same token, the conservative central forces (3.2.7) are defined by a logarithmic potential. For this very reason, we would like to designate them like we did above, as *logarithmic forces* for future references in the present work. They are essentially described by the equations:

$$U = \ln r, \quad \nabla U = \frac{\boldsymbol{r}}{r^2}, \quad \nabla^2 U = \frac{2}{r^2} \tag{3.2.8}$$

analogous to (3.2.6). They have a Maxwell field of stresses (3.2.4) equivalent to a volume field of forces of the kind assumed by J. J. Thomson:

$$\boldsymbol{f} = \frac{1}{2\pi}\frac{1}{r^3}\frac{\boldsymbol{r}}{r} \tag{3.2.9}$$

Therefore, the two forces (3.2.7) and (3.2.9) are connected: if the force (3.2.7) is the conservative force that creates a stress in space, this stress is equivalent to a local action of a force which is central and has a magnitude going inversely with the third power of distance. This localized force is, therefore, universal, and in this sense it is a *physical field defined all over the space comprising the matter*.

We need to insist a little more on the Maxwell system of forces, revealing some facts apparently not considered properly in the literature. The Maxwell definition involves two forces: the one *a priori defined* by a potential, a *conservative force*, as they call it, and the one *a posteriori defined*, as it were, by the system of forces equivalent to the stress generated by the conservative force. Of course, the a posteriori force can be itself a conservative force, as it happens in the two examples above. Moreover, one and the same force may happen to be of both kinds: a priori defined, according to classical dynamics for instance, it turns out to be equivalent with a stress system in a continuum. This happens to the case of logarithmic forces in the examples above: a priori defined as in equation (3.2.8), they are statically equivalent to the Maxwell stress of the a priori defined elastic forces from equation (3.2.6). We need to discern between the two occurrences of forces: this is the case that can best explain the *concept of field*, and especially of the kind called *gauge field*.

Fact is that the sideways displacements can be defined like a field too, just as universally as the Thomson's field force. Indeed, considering the displacements as differentials, in any position $\boldsymbol{r}$ with respect to a Euclidean frame, the displacements $d\boldsymbol{r}$ defined by equations

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y} \tag{3.2.10}$$

is a priori perpendicular to the position vector. Here x, y, z are the coordinates, i.e., the components of the position vector, so that the definition of these displacements is local but universal: in any position, the whole linear span of the vector of components

$$y-z,\quad z-x,\quad x-y \tag{3.2.11}$$

is perpendicular to the position vector $\boldsymbol{r}$. Indeed, for arbitrary a_1, a_2, a_3 we have

$$a_1(y-z)+a_2(z-x)+a_3(x-y)=\boldsymbol{a}\times\boldsymbol{r} \tag{3.2.12}$$

which proves the statement. Here $\boldsymbol{a}$ is the vector of components a_1, a_2, a_3 in the reference frame where the position vector is defined. Thus, the linear span (3.2.12) can be taken as the *transversal space* of the position vector $\boldsymbol{r}$.

Now, assuming that a particle may have the displacements (3.2.10), they describe a motion that can be further characterized as follows: the common value of the ratios from equation (3.2.10) is an abstract differential, of the nature of a phase, say $d\theta$, where θ is the metric on the unit sphere. Thus, the system (3.2.10) becomes

$$x' = y - z, \quad y' = z - x, \quad z' = x - y \tag{3.2.13}$$

where a prime means derivative with respect to θ. Differentiating two more times and using along the original system conveniently, results in identical equations for all three components. In other words, the components of the position vector giving the positions in the transversal space, are all solutions of the same third-order differential equation. Thus, we can infer that the vector is a solution of such an equation. It is:

$$\boldsymbol{r}''' + 3\boldsymbol{r}' = \boldsymbol{0} \tag{3.2.14}$$

The solution of this equation is given by the vector:

$$3\boldsymbol{r} = \boldsymbol{a} \cdot \cos\left(\theta\sqrt{3}\right) + \boldsymbol{b} \cdot \sin\left(\theta\sqrt{3}\right) + \boldsymbol{c} \tag{3.2.15}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are some constant vectors. Eliminating the phase, we end up with a geometrical characterization of the sideways motion. Namely, the motion is performed on some quadratic cones with their vertex in the origin of the reference frame:

$$\{(\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{r}\}^2 + \{(\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{r}\}^2 - \{(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{r}\}^2 = 0 \tag{3.2.16}$$

Thus, the *Thomson tubes* are primarily *cone surfaces*. The history of sideways motion goes back to Newton's times, who actually had to solve the same problem, but somehow in reverse. The force was, of course, a fact of social experience, but Newton 'invented' a related concept, in order to describe *in actuality* facts thought to have happened in the past of the universe and *imprinted in its structure*. The established structure that led to the construction of the force concept, made by Newton into the condition of 'imprint', was the Kepler motion, thought of as an *everlasting* material structure. This structure was presented by Newton as the mark of an accident in a possible evolution of the universe, therefore as an *expression of the memory* of such an event, we should say.

That possible evolution of the universe was an assumed evolution, of course, leading to an event imaginable by the sheer analogy with nowadays facts of experience on Earth:

the *free fall of the bodies* toward what, again, *is assumed* to be the unique center of the Earth. Quoting from a first letter to Bishop Richard Bentley:

> …it seems to me that *if* the matter of our sun and planets, and all the matter of the universe, were evenly scattered throughout all the heavens, and every *particle had an innate gravity* towards all the rest, and the whole space throughout which this matter was scattered was but *finite*; the matter on the outside of this space would, by its gravity, tend towards all the matter on the inside, and, by consequence, *fall down into the middle of the whole space* and there compose one great spherical mass. But *if the matter was evenly disposed throughout an infinite space*, it *could never convene into one mass*; but some of it would convene into one mass, and some into another, so as to make *an infinite number of great masses*, scattered at great distances from one to another, throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing the matter were of a lucid nature. But how the matter should divide itself into two sorts, and that part of it which is fit to compose a shining body should fall down into one mass and make a sun, and the rest which is fit to compose an opaque body should coalesce, not into one great body, like the shining matter, but into many little ones; or if the sun at first were an opaque body like the planets, or the planets lucid bodies like the sun, whilst he alone should be changed into a shining body, whilst all they continue opaque, or all they be changed into opaque ones, whilst he remains unchanged; I do not think explicable by mere natural causes, but am forced to ascribe it to the *counsel and contrivance of a voluntary Agent*. ([81], pp. 203 – 204, our Italics)

The structure of matter in this primary instance of natural philosophy is simple: it is made out of *particles*. Now, *if* these particles had innate gravity, then they would fall toward each other, insofar as the gravity is universal, and manifests itself attractively according to our experience. As one can see right away, the Newton's fundamental condition for the multiplicity of matter formations in the universe is *the infinity of space of matter's existence*. Only under this proviso can we consider escaping rationally from the condition of finiteness of the classical view of the universe.

It is according to this view, that the science discovered an apparently eternal matter structure involving a motion, that is another motion than the free fall: the Kepler motion. And, leaving aside the facts of faith, Newton made out of this structure, just via a process of *thinking by analogy*, the first one of the physical structures used today as *depositories of memory*. Quoting from a second letter to Bishop Richard Bentley:

> To the last part of your letter, I answer, first, that if the earth (without the moon) were placed any where with its centre in the orbis magnus, and stood still there without any gravitation or projection, and there *at once* were infused into it *both a gravitating energy* towards the sun, and *a transverse impulse* of a just quantity moving it directly in a tangent to the orbis magnus; the compounds of this attraction and projection would, according to my notion, cause a circular revolution of the earth about the sun. But the transverse impulse must be a just quantity; for if it is too big or too little, it will cause the earth to move in some other line. Secondly, *I do not know any power in nature which would cause this transverse motion* without the divine arm. Blondel tells us somewhere in his book of Bombs, that Plato affirms, that the motion of the planets is such, as if they had all of them been created by God in some region very remote from our system, and left fall thence towards the sun, and *so soon as they arrived at their several orbs, their motion of falling turned aside into a transverse one*. And this is true, supposing the gravitating power of the sun was double at the moment of time in which they all arrive at their several orbs; but then the divine power is here required in a double respect, namely, to turn the descending motions of the falling planets into a side motion, and, at the same time, to double the attractive power of the sun. So, then, gravity may put the planets into motion, but, without the divine power, it could never put them into such a circulating motion as they have about the sun; and therefore for this, as well as other reasons, I am compelled to ascribe the frame of this system to an intelligent Agent ([81], pp. 209 – 210, our Italics)

There is, therefore, *in the mind of Newton*, an event *imprinted* in the 'transverse' circular motion, that can be described in the following fashion: the initial motion of the planets is radial towards a center represented by the Sun; this radial motion turned into transverse motion at the moment when 'they arrived at their respective orbs'. This description is only true provided some incidentals occur, in the form of a gravitating power of Sun, which has to be double at the moment when, in their free fall, the planets reached their 'orbs'. All these are rational conjectures, produced by imagination according to the rules of logic; no doubt about that. The only reality connected to them is their present motion, that stands witness to the imaginary past event thus described, thereby representing the material form of its memory. This material form can then be taken even as a depository of such a memory in the very modern sense of such a device. The rest of Newton elaborated opinion is a matter of faith, as we said, for there is no other possible explanation.

3.3. NEWTONIAN FORCES ARE NOT FIELD FORCES

The forces in general are defined by Newton using a method that resembles a measurement procedure. Because of this, they are not field forces in the sense defined above according to Maxwell. However, they possess an important scale invariance like no other forces [79], and it is this scale invariance that puts them in connection with the universal Thomson forces in matter. Let us insist on an analytical rendition of the Newton's definition of the forces.

James Whitbread Lee Glaisher's work on Newton's definition of forces [29], does show, among others, that all of the results in characterizing the forces, can be recovered from Newton's essential propositions, by casting them into analytical form. And the basic proposition Glaisher chose to put in analytical form is the Corollary 3 of the Proposition VII from Book I of Newton's *Principia*. Quoting:

> The force by which the body P *in any orbit* revolves about *the center of force* S, is to the force by which the same body *may revolve in the same orbit*, and *the same periodic time*, about *any other center of force* R, as the solid $SP{\cdot}RP^2$, contained under the distance of the body from the first center of forces, and the square of its distance from the second center of force R, to the cube of the right line SG, drawn from the first center of force S parallel to the distance RP of the body from the second center of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in the circle of the same curvature. ([52], p. 51; our Italics)

As we see it, this is a definition of the force based, indeed, on the *idea of measurement*. The only real and, in fact, physically necessary ingredients here are the orbit of motion and the periodic time. Of the two points, *S* and *R*, helping in defining the force, one is our choice – the point toward which we want to calculate the acting force – while the other one can be arbitrary at random in the plane of orbit, provided we know the force acting toward it. Apparently, as Newton presents the idea, both of these points should be chosen inside the orbit. However, with the analytical development of the theory, even this condition became obsolete, allowing, for instance the Newtonian definition of the force correlated with the light phenomenon.

Using the geometry of triangle in order to handle the concepts in the excerpt above, Glaisher reduces the ratio of those two forces, as it was defined by Newton himself, to the expression

$$\frac{FORCE\ to\ R}{FORCE\ to\ S}=\left(\frac{SN}{RM}\right)^{3}\cdot\frac{RP}{SP} \tag{3.3.1}$$

This expression involves, on one hand, the perpendiculars *SN* and *SM* of the two centers of force onto tangent in *P* to the orbit, and, on the other hand, the distances *RP* and *SP* of the moving point to the two centers of force. This expression is particularly prone to an analytical form and, furthermore, to a differential calculus. Let us reproduce them both here.

Choosing S as the origin of a reference frame – which, by the way, means a reference frame fit to the source of force – and referring the generic coordinates, (ξ,η) say, in the plane of motion to such a frame, the equation of orbit in a Kepler motion, can be written in the form

$$C(\xi,\eta)\equiv a_{11}\xi^{2}+2a_{12}\xi\eta+a_{22}\eta^{2}+2a_{13}\xi\eta+2a_{23}\xi\eta+a_{33}=0 \tag{3.3.2}$$

where *C* here is taken as meaning '*Conic*'. The tangent to this conic is one essential ingredient in Newton's procedure. In the current point *P*, of coordinates (*x*, *y*) say, its equation is

$$(a_{11}x+a_{12}y+a_{13})\xi+(a_{12}x+a_{22}y+a_{23})\eta+a_{13}x+a_{23}y+a_{33}=0 \tag{3.3.3}$$

Thus, one can calculate the distances from the two centers of force to this straight line, by a well-known analytical procedure. One gets expressions dependent on the coordinates of the point of application of force:

$$\overline{SN}=\frac{a_{13}x+a_{23}y+a_{33}}{\sqrt{(a_{11}x+a_{12}y+a_{13})^{2}+(a_{12}x+a_{22}y+a_{23})^{2}}}\equiv\frac{1}{\sqrt{z^{2}}} \tag{3.3.4}$$

where z is the vector of components

$$\frac{a_{11}x+a_{12}y+a_{13}}{a_{13}x+a_{23}y+a_{33}},\quad \frac{a_{12}x+a_{22}y+a_{23}}{a_{13}x+a_{23}y+a_{33}} \tag{3.3.5}$$

By the same token, we have

$$\overline{RM}=\frac{(a_{11}x+a_{12}y+a_{13})\varepsilon_1+(a_{12}x+a_{22}y+a_{23})\varepsilon_2+a_{13}x+a_{23}y+a_{33}}{\sqrt{(a_{11}x+a_{12}y+a_{13})^2+(a_{12}x+a_{22}y+a_{23})^2}}\equiv\frac{1+\boldsymbol{\varepsilon}\cdot\mathbf{z}}{\sqrt{\mathbf{z}^2}} \tag{3.3.6}$$

where ε_1 and ε_2 are the components of the position vector of the center of force *R* in the chosen reference frame: $\boldsymbol{\varepsilon}\equiv\overrightarrow{SR}$. On account of (3.3.4) and (3.3.6) the equation (3.3.1) can then be written in the form

$$FORCE\ to\ S=\frac{(1+\mathbf{z}\cdot\boldsymbol{\varepsilon})^3}{\sqrt{\mathbf{r}^2+\boldsymbol{\varepsilon}^2-2\mathbf{r}\cdot\boldsymbol{\varepsilon}}}\cdot r\cdot FORCE\ to\ R \tag{3.3.7}$$

Therefore, if we know the force toward *R*, we can calculate the force toward *S*. Obviously, the reciprocal is also true, and for such an occasion Newton analyzed a number of particular cases in order to be used appropriately according to necessities. One of these cases shows that, if the center of force *R* is located in the center of the conic, then the force between *P* and *R* is proportional to the distance between them [(Newton, 1974), p. 54; Corollary 1 of Proposition X, Problem V]. Therefore, if we choose *R* as the center of the conic section, then (3.3.7) simplifies to:

$$FORCE\ to\ S=\mu\cdot r\cdot(1+\mathbf{z}\cdot\boldsymbol{\varepsilon})^3 \tag{3.3.8}$$

Here μ is a constant coming from the law of force towards center of the conic. So, if *S* is in an arbitrary position in the plane of orbit, while *R* is located at the center of orbit, we can calculate the parenthesis from equation (3.3.8), using the fact that the coordinates of the center of orbit are solution of the system:

$$a_{11}\varepsilon_1+a_{12}\varepsilon_2+a_{13}=0,\quad a_{12}\varepsilon_1+a_{22}\varepsilon_2+a_{23}=0$$

Then, the equation (3.3.6) gives

$$1+\mathbf{z}\cdot\boldsymbol{\varepsilon}=\frac{a_{33}}{a_{13}x+a_{23}y+a_{33}}$$

so that equation (3.3.8) simplifies to

$$FORCE\ to\ S\ =\mu\cdot r\cdot\frac{a_{33}^3}{(a_{13}x+a_{23}y+a_{33})^3} \tag{3.3.9}$$

which is the main of Glaisher's results. It was established earlier by William Rowan Hamilton, which is why it is also known under name the name of *Hamilton's theorem*

[31]. Based on equation (3.3.3) we can formulate it in words: the force toward the center *S* acting on *P* is proportional to the distance from *P* to *S*, and inversely proportional to the cube of distance from *P* to the straight line conjugated to *S* with respect to the orbit. This straight line is the polar of this center of force with respect to orbit, and such an instance gives us the possibility of a significant choice of the position of point *S* so as to correspond to reality. And this reality, embodied, as it were, in the Kepler's laws, shows that *S* needs to be located in one of the foci of orbit. In this case, by the very definition of the ellipse we have, in our reference frame:

$$r = \pm e\left(a_{13}x + a_{23}y + a_{33}\right) \tag{3.3.10}$$

where e is the eccentricity of the orbit, appropriately gauged. Consequently, (3.3.9) becomes:

$$FORCE\ to\ FOCUS\ = \pm\ \frac{\mu e^{3} a_{33}^{3}}{r^{2}} \tag{3.3.11}$$

which is the regular expression of the magnitude of a Newtonian force responsible for the existence of the Kepler motion.

The equation (3.3.11) does not represent the magnitude of a force *field*: such a force needs to be defined irrespective of the motion, everywhere in the realm, and (3.3.11) is valid only along specific cycles, in the form of orbits. This force can be, therefore, of the a priori type only. To wit: it is only defined with respect to a *reference frame with its origin in the center of force* and, on the other hand, according to Maxwell, it describes *a continuum of zero density*. In other words, the Newtonian force is not a field force, in spite of the fact that it can be derived from a potential. It is this property that explains its connection with the Thomson's universal force within matter.

The explanation in question involves oscillators, however, not physical oscillators, but only mathematical, whereby the part of frequency is assigned to different physical magnitudes. Two such physical magnitudes played an essential part in the *problem of regularization* of the Kepler motion: the *energy* and the *area constant* of the second of the Kepler laws. Inasmuch as the concept of frequency is instrumental for the quantization in matter, seems worth our while learning a few things from the theory of regularization.

3.4. THE PHYSICS AND GEOMETRY OF REGULARIZATION

To begin with, let us consider the mathematical procedure of integration of the classical equation of motion of a generic particle moving in a Newtonian field of force, but from an entirely new perspective. When we confer the qualification 'field' to Newtonian forces here, there is no contradiction with the definition of field above: we automatically consider *vacuum* as the matter of zero density. Therefore, the classical dynamical equation describing the Kepler motion in vacuum can be written in a condensed vector form as:

$$\ddot{\boldsymbol{r}} + \kappa \frac{\boldsymbol{r}}{r^3} = \boldsymbol{0} \tag{3.4.1}$$

Here a dot over means differentiation on time, and κ is a physical constant: a monomial cumulating the physical properties of the center of force and the inertial properties of the revolving body. Dot-multiplying here by the velocity vector ($d\boldsymbol{r}/dt$), and then integrating the result – by this we simply mean constructing what sometimes are designated as anti-differentials, while concurrently using the vector properties of the algebraical expression involved – we get what the astronomers used for a long time to designate as 'the integral of energy', which they usually denote by h [72], [73]:

$$\frac{1}{2}\dot{\boldsymbol{r}}^2 - \frac{\kappa}{r} \overset{def}{=} \frac{1}{2}h \tag{3.4.2}$$

Further multiplying this relation by r^2, we can rearrange it as:

$$\boldsymbol{r}^2\dot{\boldsymbol{r}}^2 = 2\kappa r + hr^2 \tag{3.4.3}$$

On the other hand, cross-multiplying (3.4.1) by r and subsequently integrating produces the *vector of the rate of area* defined as $\dot{\boldsymbol{a}} = (\boldsymbol{r}\times d\boldsymbol{r})/dt$, which is, of course, a constant vector. Thus, we have an important *a priori vector identity*, to be conveniently used in what follows: taking just the magnitude of this vector, we can write:

$$\dot{\boldsymbol{a}} \equiv \boldsymbol{r}\times\dot{\boldsymbol{r}} \quad \therefore \quad \dot{\boldsymbol{a}}^2 = r^2(\,\dot{\boldsymbol{r}}^2 - \dot{r}^2) \tag{3.4.4}$$

The equations (3.4.3) and (3.4.4) can still be rewritten in a somewhat simpler form, if we use as time variable not quite the time given by Kepler's time equation, but a time, τ say, which is simply linear in the eccentric anomaly: a 'fictitious time' in astronomical terms. With respect to this time, the time of dynamics (3.4.1) satisfies the first order differential equation $t' = r$, where the diacritic accent means derivative with respect to τ. So, finally, equations (3.4.3) and (3.4.4) provide the result:

$$\left.\begin{aligned} \boldsymbol{r}'^{\,2} = 2\kappa r + hr^2 \\ \boldsymbol{r}'^{\,2} - r'^{\,2} = \dot{\boldsymbol{a}}^2 \end{aligned}\right\} \quad \therefore \quad r'^{\,2} + \dot{\boldsymbol{a}}^2 = 2\kappa r + hr^2 \tag{3.4.5}$$

Notice that the second equality in equation (3.4.4) can be regarded as an identity allowing us to calculate the magnitude of the area rate. In other words, the area rate is always held as such a rate, but only with respect to the original time of the dynamics, not with the 'fictitious' time: while the conservation of the plane of motion is independent of time, the second of the Kepler laws depends on the 'reality' of the time of motion. One more differentiation with respect to τ in the last relation of equation (3.4.5), and the assumption that the area rate $\dot{a}$ is a constant, results in a second order differential equation for the radial coordinate of the Kepler problem:

$$r'' - h \cdot r = \kappa \tag{3.4.6}$$

provided $r' \neq 0$. This equation represents in fact a first integral of the third-order differential equation

$$r''' - hr' = 0 \tag{3.4.7}$$

whereby the integration constant is taken as κ. Starting afresh from equation (3.4.1) and transforming it directly in the time scale described by the parameter τ, we get

$$r\boldsymbol{r}'' - r'\boldsymbol{r}' + \kappa\boldsymbol{r} = \boldsymbol{0}, \quad t' = r \tag{3.4.8}$$

which looks like a three-dimensional damped oscillator, provided we consider r simply a parameter as any other one, depending on time. Differentiating once more with respect to τ and using in the result thus obtained the equation (3.4.6), we get a three-dimensional replica of the equation (3.4.7). Summarizing, the final result of this regularization procedure is:

$$\boldsymbol{r}''' - h\boldsymbol{r}' = \boldsymbol{0}, \quad r'' - hr = \kappa, \quad t' = r \tag{3.4.9}$$

provided $r' \neq 0$. The first one of these equations, when compared to equation (3.2.14) gives a clue as to the interpretation of these results. Namely, the Kepler motion can be always described in a Thomson realm, it is part of the transversal space: it is an essential example showing what the transversality actually means. Now, elimination of the time parameter from the general solutions of (3.4.9), leads to cone surfaces, as in the equation (3.2.16). This means that Thomson's hypothesis actually sanctions a fact to be taken into consideration in this physics. For the rest, equation (3.4.9) leads to a corresponding three-dimensional replica of the equation (3.4.6) itself:

$$\boldsymbol{r}'' - h\boldsymbol{r} = -\boldsymbol{A} \tag{3.4.10}$$

where $\boldsymbol{A}$ is the Laplace-Runge-Lenz vector of this problem. The transition from equation (3.4.8) to equation (3.4.9) depends on the constant from the right-hand side of equation (3.4.6). Should we start directly from (3.4.7) in order to get (3.4.6) by integration, in the right-hand side of this last equation we would have an arbitrary constant. Not knowing anything about the origin of (3.4.6), the constant could remain arbitrary forever: it is only the fact that we are aware of having originally to deal with the Kepler problem, that allows us to identify our integration constant with κ from this problem, and thus get the previous results. We should therefore assume that the binary representation of the radial coordinate when using the time τ defined by $t' = r$ is the only guarantee of the fact that this procedure might have some physical background and is not just a simple trick of formal validation.

As far as time is concerned in the theory of regularization presented by us up to this point, it is obvious that it was always defined with respect to the first power of the radial distance involved in the Kepler motion by the equation $t' = r$: starting from the equation (3.4.5) on, we have not used but this equation in defining the time with respect to which the derivative is denoted with a prime. The highpoint of this theory is the equation (3.4.10), which depicts a three-dimensional harmonic oscillator representing a conic with center, having the following geometrical characteristics [13]:

$$a = -\frac{1}{h}, \quad \boldsymbol{c} = \frac{\boldsymbol{A}}{h}, \quad e = |\boldsymbol{A}| \tag{3.4.11}$$

Here a is the major semiaxis, $\boldsymbol{c}$ is the position of the center of oscillator with respect to the center of force, and e is the eccentricity of the orbit. However, insofar as the physics is concerned here, we may need a regularization associated with oscillators having their center in the center of force, i.e., at $r = 0$. The reason for this need, although obvious from a natural philosophical point of view, can be presented in more technical terms as follows.

Claude Alain Burdet [13] gets the equation (3.4.10) too, only he is explicit in noticing that this happens as a result of what he calls the *central regularization procedure*. This is the regularization by two oscillators like those from equation (3.4.8), indeed, but when referred to the center of Kepler's orbit. This regularization procedure is, indeed, related to the time definition $t' = r$, as before. However, there is one more possibility to proceed in regularization, to wit, the one involving two oscillators related not to the center of orbit, but to the center of force, whereby the time scale transition is defined by the differential equation $t' = r^2$. This is what Burdet calls the *focal regularization procedure*, and he proceeds in doing it as follows: going back to the original dynamical equation (3.4.1), we rewrite it in the form

$$\ddot{\hat{\boldsymbol{r}}} + 2\dot{r}\dot{\hat{\boldsymbol{r}}} + \left(\ddot{r} + \kappa/r^2\right)\hat{\boldsymbol{r}} = \boldsymbol{0}, \quad \boldsymbol{r} \equiv r\hat{\boldsymbol{r}} \tag{3.4.12}$$

and then go over to the time τ defined by equation $t' = r^2$. The final result of these calculations is:

$$\hat{\boldsymbol{r}}'' + \left\{\frac{r''}{r} - 2\left(\frac{r'}{r}\right)^2 + \kappa r\right\}\hat{\boldsymbol{r}} = \boldsymbol{0}, \quad t' = r^2 \tag{3.4.13}$$

The quantity in the curly brackets of this equation is a known constant for the Kepler motion. Indeed, in view of the fact that the unit vector $\hat{\boldsymbol{r}}$ is always perpendicular to its time derivative, no matter of this time – the relation of orthogonality is actually a comprehensive differential relation, in the sense already mentioned before, namely: $\hat{\boldsymbol{r}}\cdot d\hat{\boldsymbol{r}} = 0$ – when dot-multiplying by $\hat{\boldsymbol{r}}$ in (3.4.13), we get the result:

$$\frac{r''}{r} - 2\left(\frac{r'}{r}\right)^2 + \kappa r = \hat{\boldsymbol{r}}'\cdot\hat{\boldsymbol{r}}' \tag{3.4.14}$$

This quantity is the square of the rate of area swept by the position vector of the revolving body, as given by us before in equation (3.4.4). This can be shown as follows: dot-multiplying (3.4.1) by $\boldsymbol{r}$ results in the equation

$$\boldsymbol{r}\cdot\ddot{\boldsymbol{r}} + \kappa/r = 0 \quad \therefore \quad \boldsymbol{r}\cdot\ddot{\boldsymbol{r}} = -\kappa/r \tag{3.4.15}$$

Now, using this equation into the a priori natural geometrical relation:

$$\boldsymbol{r}\cdot\ddot{\boldsymbol{r}} = r\ddot{r} - \left(\dot{\boldsymbol{r}}^2 - \dot{r}^2\right) \tag{3.4.16}$$

produces the result

$$r\ddot{r} + \frac{\kappa}{r} = \left(\dot{\boldsymbol{r}}^2 - \dot{r}^2\right)$$

Changing here the time over to the 'fictitious' one defined by equation $t' = r^2$ we get on account of (3.4.4):

$$\frac{r''}{r} - 2\left(\frac{r'}{r}\right)^2 + \kappa r = \dot{\boldsymbol{a}}^2, \quad t' = r^2 \tag{3.4.17}$$

which, for once, is to be compared to (3.4.14) in order to justify our present contention. Furthermore, it provides a radial equation by itself, equivalent to the area rate of the Kepler's second law for this definition of time. Let us emphasize again: the vector area law, as defined in equation (3.4.4), equivalent to the physical law of conservation of the momentum, is 'universal' so to speak, in the sense that it is independent on the time used to describe the motion. On the other hand, the genuine second Kepler law involves a measure of the area rate of this motion, and this depends on the measure of time involved in calculations. Anyway, the conclusion is that the focal regularization of Claude Burdet still portrays the mobile position in a Kepler motion by an isotropic harmonic oscillator (3.4.13) on the unit sphere with the center in the center of force, together with one radial harmonic oscillator of equation coming out of (3.4.17). This radial oscillator is constantly forced with a force having the magnitude equal to the constant from the expression of Newtonian force governing the original Kepler dynamics. Notably, this force is due exclusively to the physical characteristics of the bodies involved in the Kepler problem – masses and charges – a fact that may come in handy later on. Both these Burdet oscillators are described by a dynamics in the time τ defined by $t' = r^2$ and have a 'frequency' defined by the area constant specific to the original Kepler problem. In view of the condition of quantization associated with the idea of Newtonian force, this conclusion is of utmost importance: the area constant, or something related to it is quantized too. To wit: this was the very case of the Bohr-Sommerfeld quantization, historically speaking. Summarizing all these results, we can say that the focal regularization procedure provides the following equations:

$$\begin{matrix} \hat{\boldsymbol{r}}'' + \dot{\boldsymbol{a}}^2\hat{\boldsymbol{r}} = \boldsymbol{0} \\ u'' + \dot{\boldsymbol{a}}^2 u = \kappa \end{matrix} \quad u \equiv \frac{1}{r}, \ t' = r^2 \tag{3.4.18}$$

Mention should be made, that Victor Bond obtained the very same results by a more involved independent calculation [10], a fact that can be taken as an alternative verification of the above simpler theory carried out by Claude Burdet. The essential difference between the two approaches resides in an equivalence involved naturally in the Bond's approach – a two-body problem – of which we shall have to discuss later.

Let us close this section by reiterating the main point of regularization: the equivalence of the Kepler dynamical problem with a system of oscillators. In the case of central regularization, it is the energy that plays the part of frequency of the oscillators, while in the case of focal regularization, the area constant plays that part. In this sense there is always a regular dynamics for the Newtonian forces, no matter of the space scale we are using the Kepler model. We just have to adjust the time in order to make the motion cope with the space scale. In the case of central regularization, the 'frequency' of the oscillators, given by the mechanical energy, is still singular for the center of force. As a general conclusion, the central regularization is a *transversal dynamics*, judging by the identity between equations (3.2.14) and (3.4.9). Thus, Thomson sideways motion is a geometrical solution of a transversal dynamics. On the other hand, in the case of focal regularization, such a conclusion is pending on the inversion with respect to a sphere with the center in the center of force. In any case, the theory of regularization points toward the legitimacy of Thomson's dynamics as a fundamental dynamics within matter, showing also that the vacuum is matter of zero density.

4. THE FUNDAMENTAL DYNAMICS IN MATTER

The previous natural philosophy – we like to call it *Thomson's natural philosophy* – like the precursor Newtonian natural philosophy, depends on the validity of a certain dynamics, explaining the basic structure within matter. In the classical Newtonian case, the basic dynamics was the one explaining the Kepler structure, thought to be universal. Here, on the other hand, the basic dynamics is restricted only to the radial part of a classical dynamics, and the legitimacy of Thomson's natural philosophy is pending on the dimension of the domain representing the values of radial coordinate. To wit, such a range should be a binary domain. Indeed, as we have seen in the §3.1, but especially in the §3.4, disregarding a space inversion with respect to a certain sphere, the radial coordinate range in the Thomson realm must be a *binary domain*, for it is the linear span of the ensemble of solutions of a linear second-order differential equation of the type (3.1.13), (3.4.9) or (3.4.18). However, while in the last two cases there is no problem in deciding on this issue, in the first case such a conclusion is pending on the validity of the equation (3.1.12), which may not be in order. Indeed, as we have seen, of the two forces assumed to be issuing from the origin of the reference frame, only one is a *field force* – i.e., a force defined by the local action in any position covered by the reference frame, and that one *is not the Newtonian force*. In other words, a legitimate dynamics in this realm can be based on the equation (3.1.12), however only with $A = 0$, which obviously, can be taken as meaning $r_0 \to \infty$. This condition is in accordance with all of the basic propositions of classical natural philosophy. Suffice it to mention the idea of inertia:

according to Ernst Mach it is controlled by the matter beyond the reach of our senses. In this respect one can say that Thomson's dynamics based on equation (3.1.12) does nothing else than bringing the infinity at a finite distance, as it should be, indeed, the case within matter.

On the other hand, the legitimacy of quantization in matter is pending on the structure of a Planck-type constant for the case of matter, in which case we do not know but of one quantum: the Procopius's quantum. Maintaining the same constant for matter is pending on the existence of the Planck resonators, having the properties indicated for the case of Procopiu's quantization. Up to this point in time the constant deciding the structure of a quantum is considered as being the one discovered by Planck himself for light. This needs to be combined with the frequency as a fundamental variable of the resonator, thus considered as an oscillator. There are, however, reasons to believe that for the case of matter things change quite significantly, and the equivalent of the Planck's constant may acquire a more complex algebraical structure [41]. To wit: the equation (3.4.18) of the focal regularization, indicate the area rate of the second of Kepler laws as playing the part of frequency. If a resonator is such an oscillator, the quantum should be connected with its 'frequency'. As long as the Planck's original constant is maintained in the picture, the closest physically meaningful quantum thus constructed is that correlated with the kinetic momentum, by Niels Bohr [9] and by Arnold Sommerfeld, based on different criteria [68]. However, in general, the case is by far more intricate. The Thomson's dynamics indicates, through equation (3.1.18) the feasibility of Procopiu quantization. The possibility of describing mathematically the matter at the scale where the quantization becomes significant, depends fundamentally on a connection between the time and the radial coordinate. The present chapter addresses this issue.

4.1. The Fundamental Equation of Motion in Matter

According to a Maxwell-type criterion, based on the system of stresses given in the equation (3.2.4), the Newtonian forces cannot be field forces in matter. We can rephrase this apparently harsh conclusion, in order to make it concordant with the historical truth. And this historical truth is that the dynamical description of the Kepler motion was successful with the Newtonian forces considered as field forces, giving the right results. Mention should be made though, that according to the Maxwellian criterion the matter to which these field forces is referring *has zero density*: the Newtonian forces are *vacuum forces*. In view of these facts, we assume that the equation equivalent, *in the matter of non-zero density*, to that generating the dynamics that solves the Kepler problem, should be the one used by Thomson, i.e., the equation (3.1.12), but with the point of equilibrium at infinity:

$$\ddot{r} = \frac{K^2}{r^3} \tag{4.1.1}$$

where K may be, in general, a complex constant. That this should be the basic dynamics in the world of matter, would have been suspected even before Thomson's work cited by us here [37], [42]. One can say that the idea was 'floating in the air' by the beginning of the 20th century. Now, history aside, the general solution of equation (4.1.1) has the form

$$r^2 = at^2 + 2bt + c, \quad K^2 = ac - b^2 \tag{4.1.2}$$

with a, b and c some constants, of which, obviously, only two are independent. Then, with a convenient choice of those two constants, we can define a 'fictitious time' for the Thomson realm, according to the last of the equation (3.4.18):

$$d\tau \overset{def}{=} \frac{dt}{r^2} \quad \therefore \quad \frac{dt}{d\tau} = at^2 + 2bt + c \tag{4.1.3}$$

A first observation: there is an interpretation of the realm within which the force inversely proportional with the third power of the radial coordinate acts universally. This means that there is always an ensemble of free classical particles, of the kind of ideal gas molecules, for which r defined by equation (4.1.2) is the radial coordinate in a Euclidean reference frame. From this point of view, the theory above is akin to the first ever interpretation of the blackbody radiation, once given by Albert Einstein, that led in its time to the concept of photon [22]. There was a continuum in those times, that Einstein *interpreted as a molecular gas ensemble*. Recall, indeed, his conclusion that…

> … radiation of *low density* behaves as though it consisted of a *number of independent energy quanta* ([22], *our Italics*)

We have to notice, though, that at those historical moments the interpretative ensemble was, as a rule, thermodynamically decided, so it had to be thought of as a physical object: an ideal gas in evolution by successive equilibrium states, so that the evolution per se had to be *adiabatic*, in concordance with the character of mechanical theoretical explanation of the *adiabatic invariants*. Here, on the other hand, the interpretation procedure is not quite so handy. One thing still has to be noticed, in agreement with what seems to be one of the points of Sir Michael Berry's general natural philosophy, and this is the fact that *the problem of coordinate space should be free from the adiabatic condition* [79]. One can say that this is a fundamental physical problem, which, in fact, only incidentally flared up as adiabatic invariance in the thermodynamics of light. This fact can be explained by the preexistence of a Thomson realm for the case of light, in the form of a Wien-Lummer *hohlraum* ever-present by the physical necessity of experiments on light [76].

It is at this juncture, that we have to notice that the problem of interpretation takes a significant turn, marked, as it were, by an outstanding solution of an equally outstanding *gauging procedure*. Namely, notice that the equation (4.1.2) offers the *radial* equation of motion for an ensemble of *classical free material points*, which is the undisputable counterpart of the classical ideal gas, staying at the foundation of thermodynamics. Indeed, if we are considering the classical dynamical *vector* equation of motion of a free particle in a Euclidean reference frame, and assume that this free particle is bound *never to reach the origin of the reference frame*, we have the equations of motion:

$$\boldsymbol{r} = \boldsymbol{v}t + \boldsymbol{r}_0, \quad \boldsymbol{r}^2 = \boldsymbol{v}^2 t^2 + 2\boldsymbol{v}\cdot\boldsymbol{r}_0 t + \boldsymbol{r}_0^2, \quad K^2 \equiv (\boldsymbol{r}_0 \times \boldsymbol{v})^2$$

proving the fact. One can say the length gauging the radial motion of a Thomson particle can be considered the free path of a classical material point in an ensemble of classical material points, constrained to never reach the origin of the reference frame. This may constitute, after all, the very definition of the reference frame, but we do not insist on this topic here. Suffice to say that the gauging length in a Thomson dynamics *is the radial coordinate of that motion*, which, in fact, is the analogous of a mean free path in the case of ideal gas. This statement may seem occasionally confusing but, in fact, it sets our thinking on the right track, at least historically speaking.

Whatever has been said up to this point in the present section, regarding the idea of interpretation, is based only on the equation (4.1.2) and can be limited to the interpretation in terms of ensembles of free particles in the classical sense. What then, is the physical interpretation of the very equation (4.1.2)? In order to answer to this question, let us discuss in more detail the equation (4.1.1), which also describes the time evolution of the gauge length in a Berry-Klein gauging theory. That equation is *form-invariant* with respect to the transformation of time and gauge length given by equations

$$T = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad R = \frac{r}{\gamma t + \delta} \tag{4.1.4}$$

that is, invariant in the following sense:

$$\ddot{r} = \frac{K^2}{r^3} \quad \Leftrightarrow \quad R'' = \frac{\left(K/\Delta\right)^2}{R^3} \tag{4.1.5}$$

where $\Delta \equiv \alpha\delta - \beta\gamma$ is the determinant of the time transformation in (4.1.4), and the accent mark means derivative on T. The equation of motion (4.1.1) is, indeed, not quite invariant with respect to the transformation (4.1.4), but only form-invariant as we said. It is plainly invariant only with respect to time transformations of unit determinant, but the quasi-invariance (4.1.5) may prove occasionally salutary, in view of our observation that, at some point of the theory, it may become necessary to take the gauge length as a linear form in the coordinates.

The two equations (4.1.4) give a closed-form $\mathfrak{sl}(2,\mathrm{R})$-type realization of an action represented at the infinitesimal level by three differential vectors forming a linear base of the algebra in question, which we take as follows:

$$\boldsymbol{X}_1 = \frac{\partial}{\partial t}, \quad \boldsymbol{X}_2 = t\frac{\partial}{\partial t} + \frac{1}{2}r\frac{\partial}{\partial r}, \quad \boldsymbol{X}_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} \tag{4.1.6}$$

These operators are vectors of the binary domain (t, r) and satisfy the algebraical structural relations, characteristic for a $\mathfrak{sl}(2,\mathrm{R})$ algebra:

$$[\boldsymbol{X}_1,\boldsymbol{X}_2] = \boldsymbol{X}_1, \ [\boldsymbol{X}_2,\boldsymbol{X}_3] = \boldsymbol{X}_3, \ [\boldsymbol{X}_3,\boldsymbol{X}_1] = -2\boldsymbol{X}_2 \tag{4.1.7}$$

which we take as the standard commutation relations for this algebraic structure. Then the equation (4.1.2) can be viewed as a result of *invariance* with respect to the general action of the realization (4.1.6) of this algebra. Indeed, the most general vector of this binary domain is a linear combination of the base vectors (4.1.6). By definition, the invariant functions with respect to such an action, is a solution of the partial differential equation:

$$(c\boldsymbol{X}_1 + 2b\boldsymbol{X}_2 + a\boldsymbol{X}_3)f(t,r) = 0 \quad \therefore \quad \left(at^2 + 2bt + c\right)\frac{\partial f}{\partial t} + (at + b)r\frac{\partial f}{\partial r} = 0 \tag{4.1.8}$$

which turns out to be an arbitrary continuous function of the ratio

$$\frac{r^2}{at^2+2bt+c} \tag{4.1.9}$$

Therefore, the solution (4.1.2) of the fundamental dynamical equation of motion is actually an *invariant function* under the action (4.1.6) of the $\mathfrak{sl}$(2,R) algebra. Revealing a few more properties of this kind of action can be in order on this occasion.

To this end, let us start by noticing the fact that the equation (4.1.1) can be produced by making stationary the *physical action* corresponding the following Lagrangian [1]:

$$L(r,\dot{r}) = \frac{1}{2}\left(\dot{r}^2 - K^2 r^{-2}\right), \quad i.e. \quad A(t_1,t_2) = \int_{t_1}^{t_2} L(r,\dot{r})dt \tag{4.1.10}$$

where the time t of the classical dynamics is used as independent variable. Here is the point where statistics creeps into the classical dynamics without even being noticed: the *physical* action from equation (4.1.10) can be simply viewed as the *time average* of the difference of the two terms involved in the definition of Lagrangian. The action from equation (4.1.10) represents, indeed, just such an average, with an important specification though, regarding the manner in which this statistic is estimated: it is the time average of the Lagrangian over a sequence of *equally probable* moments of time, arranged in a given order by the free particles used in the interpretation. Such time sequences can be, in fact, decided by arbitrary clocks, but – and this is of an overwhelming importance for what we have to say here – the 'equal probability' in question is statistically conceived in terms of a *uniform distribution* of the time moments, having a constant probability density and described by the elementary measure *dt*. Reformulating, therefore, this conclusion in specific theoretical statistical terms, it sounds like this: the equation (4.1.1) can be mathematically produced by the physical condition of stationarity of the *time average* of the difference between the two energies involved in the definition of Lagrangian, *over a sequence of equally probable moments*. The equal probability in question is decided according to a probability distribution of the time moments, described by a constant probability density.

Once adopting this *statistical meaning* of time average for the physical action – the necessity of which shall be clear soon – we can cite an important result of Morton Lutzky, that settles the position of the equation (4.1.1), usually called the *Ermakov-Pinney equation* in physics (for its origination see [27] and also [53]), which is the name we also adopt here. As we see it, this is one of the most important equations of mathematical physics, and for this statement we have firm reasons soon to become completely obvious. The result of Lutzky we just mentioned [44], can be summarized with reference to the following observation on the previous development.

As we have shown in §3.1, equations (3.1.16) to (3.1.18), the quantization in matter, if conducted along the Planck's idea for light, depends on the equilibrium between the universal force (4.1.1) and the Newtonian force. This last force may not be a field force in this case: it can be a field force in matter only pending on the materiality of vacuum, and on its presence within matter. We then take the presence of the Newtonian force in the equation (3.1.16) as meaning only a description around equilibrium. To wit: *it is a sum* (*algebraically speaking*) *between the universal force and the elastic force*. Then we need to modify the Lagrangian from equation (4.1.10) to the following form, which we call *Lutzky*'*s Lagrangian*, describing the radial motion in the conditions of quantization in matter:

$$L(r,\dot{r}) = \frac{1}{2}\left(\dot{r}^2 - \frac{K^2}{r^2} - \omega^2 r^2\right), \quad i.e. \quad \ddot{r} + \omega^2 r = \frac{K^2}{r^3} \qquad (4.1.11)$$

This definition of the Lagrangian preserves the definition of physical action as given in equation (4.1.10). The resultant differential equation is the complete *Ermakov-Pinney equation* of motion, which, in view of Thomson's observations, proves to be essential in the realm of matter.

The problem is now if the harmonic oscillator is indeed the universal structure that can serve to 'interpretation' of a field. In hindsight, such a question may be taken as only rhetorical, for the harmonic oscillator is the conspicuous structure in all kinds of quantization today. However, in view of the role that it played for the light quantization to Planck – in describing the light structure, as well as a resonator, in which case the role of frequency is played by different physical quantities, as we have seen in the case of regularization procedure – it is legitimate to ask if the equation of motion (4.1.11) is, indeed, universal for this dynamics in the classical way of the dynamical description of the Kepler motion. The answer is affirmative and will be detailed in this very chapter. However, to make the case, we need first to concentrate upon some general properties connected to the Ermakov-Pinney equation, without which we cannot go further along the way.

4.2. PROPERTIES OF THE ERMAKOV-PINNEY EQUATION

Mathematically speaking, the Ermakov-Pinney equation has a few distinctive properties on which we need to insist at some length. Assume indeed, that we have to consider *two* possibilities of fundamental particles moving according to equation (4.1.11), in the Thomson realm, for the same time *t*. One particle is assumed to have the radial distance *r*, the other is assumed to have the radial distance *q*. Both of them are supposed to satisfy a corresponding Ermakov-Pinney equation but for the same *ω*, meaning the same elastic forces, if it is to use a here a classical dynamical view, which turns out to be particularly suggestive. The two equations of motion are then:

$$\ddot{r} + \omega^2 r = \frac{R^2}{r^3}, \quad and \quad \ddot{q} + \omega^2 q = \frac{Q^2}{q^3} \tag{4.2.1}$$

where *R* and *Q* are two real constants. What is the relevance of these equations for what we have to say here, based on dynamics!? First of all, our notations *r* and *q* are intended to suggest a *phase plane* of the two symbols: one of them can be taken, for instance, as a generalized coordinate, the other can be taken as generalized momentum. Now, in case $\omega = 0$, for either *r* or *q*, its Lagrangian reduces to the de Alfaro-Fubini-Furlan Lagrangian (4.1.10), and the corresponding equation of 'motion' for this gauge length reduces to that of *a proper Thomson material particle* (4.1.1). Therefore the (*r*, *q*) phase plane includes the radial motion of a free particle, and so this phase plane is bound to represent – and describe also – those transitions correlated with the properties that led to the very classical quantization of Max Planck. This is, indeed, the case, and we shall describe it here in some detail, but in connection with what we find as quite significant from a physical point of view: the Ermakov-Pinney equation (4.1.11), in order to reveal its meaning in full.

First of all, by extending an observation of Colin Rogers and Usha Ramgulam, we can give the following theorem [63]: the ratio of solutions of two Ermakov-Pinney equations, like the ones from (4.2.1), is also, formally speaking, a solution of an Ermakov-Pinney equation. In order to properly understand such a result, we have to prove it in detail. To start with, by successive direct differentiations on t, and choosing the transform of time as dictated by the gauge length r, according to equation (4.1.3), we get the result:

$$\frac{d}{d\tau}\left(\frac{q}{r}\right)=r\dot{q}-q\dot{r},\quad \frac{d^2}{d\tau^2}\left(\frac{q}{r}\right)=r^2(r\ddot{q}-q\ddot{r}),\quad d\tau\overset{def}{=}\frac{dt}{r^2}$$

Now, by using here the equations from (4.2.1), specifically the second of those equations, we get indeed an Ermakov-Pinney equation for the ratio of q and r, which *does not involve* the pulsation ω anymore, and therefore the elastic forces and inertia properties that come with them:

$$\left(\frac{q}{r}\right)''+R^2\left(\frac{q}{r}\right)=Q^2\left(\frac{r}{q}\right)^3 \tag{4.2.2}$$

Here an accent means differentiation with respect to gauged time τ. Had we have used q instead of r, for the definition of the new time, we would have gotten a similar equation but for the reciprocal ratio of the two radial coordinates, with the constants R and Q switching their places. By direct integration of this equation or, if one likes better, by constructing the Hamiltonian of the corresponding Lagrangian (4.1.11), we get that important result of Morton Lutzky already acknowledged above. It is referring to the expression of a constant integral of the motion of Thomson's particle in the matter:

$$I(r,r';q,q')\equiv l'^2+R^2l^2+Q^2l^{-2}=const \tag{4.2.3}$$

where the prime means differentiation with respect to a newly defined time. Indeed, this result presents the two radial coordinates in their interdependence. Given them as two solutions of Ermakov-Pinney equations *for the same time variable*, either one can be used to 'define the time', as it were, by a transformation of the form (4.1.3):

$$l=\frac{q}{r},\quad d\tau=\frac{dt}{r^2}\quad or\quad l=\frac{r}{q},\quad d\tau=\frac{dt}{q^2} \tag{4.2.4}$$

This allows passing to the time t, in which case we have:

$$I(r,\dot{r};q,\dot{q})\equiv(r\dot{q}-q\dot{r})^2+R^2\left(\frac{q}{r}\right)^2+Q^2\left(\frac{q}{r}\right)^{-2}=const \tag{4.2.5}$$

Either for $R=0$, or for $Q=0$, – i.e., for the cases where the Ermakov-Pinney equation reduces to the equation of motion of the harmonic oscillator – the invariant (4.2.5) is the original *Lewis invariant*, used in the plasma physics, which can be regarded as *a direct generalization of the classical Planck's constant* [41]. This can be made intuitively obvious as a simple fact: for *both* $R=0$ and $Q=0$, therefore for harmonic oscillators in both cases, the invariant I is the square of the elementary area rate of the phase plane of the variables (r, q), which can therefore be connected with the Planck's constant. But the things are by far more intricate, and we need to insist further on this important subject, inasmuch as the Planck's constant is considered today one of cornerstones of the physics.

According to our experience, the harmonic oscillator – which is the main physical structure involved in the dynamics of a Thomson particle – has apparently some sound natural philosophy ingrained in it. To wit, it is a self-contained structure transcending the space scales just about like the light does it: the far universe acts locally to arrange the inertial mass of the particle involved in this physical structure, while the close universe acts locally to arrange the elastic strength necessary to the dynamics embodied in the description of the motion. If the oscillator does not work forever – in practical cases, its motion, like any other motion within matter in fact, is always damped – then the equation of motion changes. In the simplest of the cases, which is the case of the so-called *linear damping*, we have the reputed equation of motion of the *damped harmonic oscillator*:

$$m\ddot{q} + 2\beta\dot{q} + kq = 0 \tag{4.2.6}$$

where β is the *damping coefficient*, describing here a 'fading' of the motion due to a resistance proportional to the instantaneous speed; the factor *2* was chosen here just for convenience. As to the other two parameters involved in (4.2.6), m is the *inertial mass*, controlled, in current views in physics, by the distant matter, while k is the *elastic strength* accordingly controlled by the close environment. The equation (4.2.6) too, admits a constant of motion [20], but this time we can hardly identify it with the energy, for it does not reduce itself to that quantity in obvious instances like the usual Hamiltonian of the undamped harmonic oscillator. As the equation (4.2.6) stands, though, the result of integration is [20]:

$$m^2\left\{\left(\frac{\dot{q}}{m} + \lambda q\right)^2 + \omega^2 q^2\right\} \cdot exp\left(-2\frac{\lambda}{\omega}\tan^{-1}\frac{m^{-1}\dot{q} + \lambda q}{\omega q}\right) = const \tag{4.2.7}$$

where the following notations are used, over the ones already introduced previously:

$$\lambda \overset{def}{=} \frac{\beta}{m}, \quad \omega^2 \overset{def}{=} \omega_0^2 - \lambda^2$$

Now, there is a problem: as well known, the equation of an undamped harmonic oscillator *can be obtained* from a principle of stationary action connected to a Lagrangian whose action has a statistical interpretation:

$$L(q,\dot{q}) = \frac{1}{2}\left(m\dot{q}^2 - k^2q^2\right) \quad with \quad A(t_1,t_2) = \int_{t_1}^{t_2} L(q,\dot{q})\,dt \tag{4.2.8}$$

Indeed, the action in this case is the time average of the difference between the kinetic and the potential energy, just like in the previous case from the equation (4.1.10): this quantity needs to remain stationary along the motion, and the equation of motion warrants the case. There is no such simple algebraic expression for a Lagrangian in the case of a damped harmonic oscillator, and our point is that we need to have such an expression describing the case of a Thomson particle. In fact, Denman's calculations [20] can be associated with more general group methods of the kind presented in §4.1., having the structure given in equation (4.1.7), for which the Lagrangian is only a particular approach [64]. Here we choose to follow these group methods, however only selectively, just revealing the pure connection with the physical reason already noticed before, that applies here too, with outstanding results.

Namely, as Harry Bateman once noticed [4], the equation of motion (4.2.6) can be obtained from a variational principle, exactly like the one formulated based on equation (4.2.8), only with reference to the Lagrangian

$$\frac{1}{2} \cdot e^{2\beta t} \cdot \left(m\dot{q}^2 - k^2q^2\right) \tag{4.2.9}$$

This Lagrangian differs from the one leading to the equation of motion of the undamped harmonic oscillator just by an exponential time factor. However, the corresponding action, can be physically explained just as in the case of the undamped oscillator, by making use of the *statistical properties of time sequences*. Indeed, the physical action corresponding to the Lagrangian (4.2.9) can still be seen as the time average of the difference between the kinetic energy and the potential energy. For, if we construct the action integral with this Lagrangian exactly as in equation (4.2.8), we have:

$$A(t_1,t_2) = \int_{t_1}^{t_2} e^{2\beta t} \cdot L(q,\dot{q})\,dt \tag{4.2.10}$$

where the Lagrangian is given in equation (4.2.8). This physical action is, indeed, a time average of the difference between the very same kinetic and potential energies of an undamped harmonic oscillator, over a time sequence beginning with t_1 and ending with t_2. The only real difference of the action (4.2.10) – which leads to equation of motion

(4.2.6) – from the classical case (4.2.8) – which leads to the equation of an undamped harmonic oscillator – stays in the fact that the *distribution of the time moments is not uniform* anymore. This distribution is governed by an elementary measure of the time range having the exponential form

$$e^{2\beta t}\cdot dt \tag{4.2.11}$$

In other words, it represents a time statistics described by a distribution of an *exponential type* – the type almost exclusively used in physics for statistical purposes [40]. And thus, even though the relationship between energy and motion may be 'lost', as it were, the statistical argument still stands and, moreover, this statistics turns out to be of a *Planck nature*, that led to the idea of quantum, even though applied now to time [48].

And now, for the true point of these conclusions, which turns out to be, in fact, a *kinematical* reading of the Ermakov-Pinney equation. This procedure reveals the fact that the harmonic oscillator can be taken as playing the role of a free particle in the radial motion governed by the equation (4.1.1). The discussion to follow spins around this idea, in order to show a connection between kinematics and dynamics in the case of matter. Indeed, as we have seen above, there is equal room, so to speak, for kinematics as well as for dynamics in this theory and, therefore, there does not seem to exist a place where their domains are delimited with respect to each other. And yet, if one adopts the physical interpretation of the action integral by a time statistics, as discussed above, there seem to be a place neatly described by a gauging theory, where the dynamics goes into kinematics and vice versa, allowing us to distinguish the time we call classical, which is the time of dynamics.

Indeed, the physical action (4.2.8) for the undamped harmonic oscillator is not invariant with respect to groupal action (4.1.6). As we have seen above, even the genuine **sl**(2,R) action, only results in a *relative* invariance, whereby the Lagrangian needs a gauging in order to be properly used. What would be, in these conditions, the place of harmonic oscillator, so clearly delineated in a Hamiltonian dynamics? The answer can be found, in our opinion, in an incidental result due to Heinz-Jürgen Wagner, in the form of following theorem [74]: there is a Lagrangian – *Wagner's Lagrangian* – of the form

$$L(q,\dot{q};r,\dot{r}) = \frac{1}{2}\frac{1}{r^2}\left((r\dot{q}-q\dot{r})^2 - R^2\frac{q^2}{r^2}\right) \tag{4.2.12}$$

producing the Euler-Lagrange equation of radial motion

$$\ddot{q} - \frac{q}{r}\left(\ddot{r} - \frac{R^2}{r^3}\right) = 0 \tag{4.2.13}$$

According to Wagner, this would simply mean that in a world where the equation (4.1.11) is the fundamental equation of motion, the coordinate q entering the Lagrangian describes a harmonic oscillator, which thus is bound to play the part of the *free particle of the Thomson realm*. Thus, the connection between the Thomson force and the elastic force is just as universal in matter as the Thomson force entering the dynamics (4.1.1): no need for Thomson's approximation in order to get the harmonic oscillator, for it is everywhere!

In order to assess the situation, it is best to provide a demonstration for the Wagner's theorem. An observation based on the theorem of Morton Lutzky from equation (4.2.5), provides the key to proof: applying his theorem above for the particular case $Q = 0$, we get the Lewis invariant:

$$I(r,\dot{r};q,\dot{q}) \equiv r^4\left(\frac{d}{dt}\frac{q}{r}\right)^2 + \left(\frac{q}{r}\right)^2 = const \tag{4.2.14}$$

where we chose the constant R as unity, which is, in fact, Wagner's original choice. Now, switching to a new time, τ say, defined by equation (4.1.3): $d\tau = dt/r^2$, the Lewis invariant becomes: $(q/r)^2+[(q/r)']^2$ where the prime means derivative on τ. This is the energy of an undamped harmonic oscillator working *in the new time* according to an equation of motion produced by the Lagrangian $[(q/p)']^2 - (q/p)^2$, a well-known fact. Switching back to the original time t in this new Lagrangian, gives the Wagner's Lagrangian (4.2.12) that produces the equation of motion (4.2.13). The action corresponding to this equation of motion is, physically speaking, the time average of the difference between the area rate in the phase plane of the coordinates (p, q) and their ratio adjusted by an appropriate constant for dimensional reasons. The average is to be calculated with respect to a time statistical measure given by $d\tau = dt/r^2$, which is, in fact, the invariant function (4.1.9) of a Thomson realm.

However, the true meaning of the Wagner's result concerns the idea of Planck's resonator: in a Thomson realm the harmonic oscillator (a virtual resonator) is connected with the particle obeying the equation (4.1.11) of the fundamental dynamics. In this case, we can assign a precise physical meaning to this resonator: namely, *it is a fundamental physical particle carrying a charge q*. In order to see this, we just need to notice that the Lewis invariant of the fundamental motion in this realm is actually an amended Newtonian force. One can even state the following theorem of the Thomson natural philosophy: *in a Thomson realm, the Newtonian forces are Lewis-Lutzky invariants of the fundamental motion*. To wit: such a particular invariant can be written in the form

$$I(r,\dot{r};q,\dot{q}) = K\cdot\left(\frac{q}{r}\right)^2 + \left\{\left(\frac{q}{r}\right)'\right\}^2,\quad t' = r^2 \tag{4.2.15}$$

where K is a constant, and t is the time of the fundamental motion. Obviously the first term of this invariant is the static Coulomb force between two identical charges of magnitude q, located at the distance r with respect to each other. As to the second term of the invariant (4.2.15), it can be considered as an addition to this force due to the electrodynamical effects. In fact, one can say that the electrodynamical force in a Thomson realm is a complete Lutzky invariant of the form

$$I(r,\dot{r};q,\dot{q}) = K_1\cdot\left(\frac{q}{r}\right)^2 + K_2\cdot\left(\frac{r}{q}\right)^2 + \left\{\left(\frac{q}{r}\right)'\right\}^2,\quad t' = r^2 \tag{4.2.16}$$

where K_1 and K_2 are two constants. So, it seems that the quantization was with us ever since the foundation of the modern science by Newton!

However, this is not the whole story: if this invariant is a generalization of the Planck constant for the case of matter, then the *Procopiu quantum* established in equation (2.4.6) by analogy with the Planck's quantization procedure, is a legitimate quantum in the case of matter. Consequently, the *Procopiu*'*s quantization* is a legitimate procedure, and then we have to search for the analogous of the frequency on the algebraic structure of that very quantum. Obviously, the same goes for the Thomson's general quantization procedure leading to equation (3.1.18), except for the presence of the Planck's constant in that equation.

The formula (4.2.16) suggests a certain symmetry between the radial coordinate and the charge: there should be a continuum of charge, having the same fundamental dynamics as the Thomson realm. With due consideration on the structure of a charge continuum, this is, indeed, the case, as we show presently.

4.3. GAUGING BY THE AMPLITUDE OF CHARGE

In the problem of charge, we are compelled to adopt E. Katz's natural philosophical ideas [36]. According to these ideas there are isolated magnetic charges, existing in the Thomson realm just like the isolated electric charges. The analogy is then even reciprocal, for the charge dipoles are to be seen as *pieces of vacuum* having electric charges as ends, just as the magnets, which are *pieces of matter* having magnetic poles, i.e., magnetic charges, at their ends (see [50], Chapter 1, for more details). Pending a necessary further analogy between *a piece of matter* and *a piece of vacuum*, in need to be properly circumstantiated, we are at liberty to assume that the charge of any kind – electric or magnetic – is actually a binary domain: the charge, no matter of its nature, has always two components, so that its range can be coordinated by a complex number. In fact, this is the main trait of the physics after Augustin Fresnel's physical theory of light [83]: the possibility of replacing of the physical structure with the (complex) time signal offered by the measurements. The Katz's natural philosophy of charge replicates that theory of light to details, which is the mathematical reason that made possible the electromagnetic theory of light.

Assume, therefore, that the charge can be perceived as a signal, continuous in a convenient time, but still involving the *phase of charge* in the sense of Katz:

$$Q(t) = q(t)e^{i\theta(t)} \tag{4.3.1}$$

If the amplitude q of such a signal would be a constant, then no doubt, this signal would be a solution of the second-order differential equation, assimilable to a harmonic oscillator as it is, no matter of t; but if only a time dependent amplitude is necessary, for the description of the charge, the things become a little more complicate. Assuming that $Q(t)$ gets *physical meaning* as a periodic signal associated to a harmonic oscillator, we will have the following conditions by identifications:

$$\ddot{Q}(t) + \omega_0^2 Q(t) = 0 \quad \Leftrightarrow \quad \begin{cases} q^{-1} \cdot \ddot{q} + \omega_0^2 = \dot{\theta}^2 \\ q^{-1} \cdot \dot{q} + (2\dot{\theta})^{-1}\ddot{\theta} = 0 \end{cases} \tag{4.3.2}$$

Thus, from the second of these conditions we have right away a connection between the amplitude and the phase of such a representation:

$$q^2(t) \cdot \dot{\theta}(t) = const \tag{4.3.3}$$

This equation is a first incentive of the analogy between charge and the *radial coordinate* in the fundamental dynamics of a Thomson realm: the constant from the right hand side is entirely analogous to the area law of that Newtonian dynamics. Consequently, we suspect a dynamics of the charge itself. Inserting (4.3.3) into the first of the conditions (4.3.2), results into an equation for the amplitude of the complex charge representation:

$$\ddot{q} + \omega_0^2 q = \frac{\dot{a}^2}{q^3} \tag{4.3.4}$$

where $\dot{a}$ is the constant from the right hand side of equation (4.3.3). Obviously, we used this notation here in view of the resemblance of the equation (4.3.3) with the second of Kepler's laws, where $\dot{a}$ denotes the area rate, like in the §4.1. Thus, the amplitude of this representation of the charge must be a solution of the Ermakov-Pinney equation with respect to the time of representation, just like the radial coordinate in the Thomson realm.

Before going any further, let us treat the case of physical representation of the charge by a damped harmonic oscillator, just for completeness, if nothing else. In fact, if we want to assign a physics to a signal like (4.3.1) recorded somehow within matter, this would be the natural way to do it: find the equivalent *damped* harmonic oscillator, for, within matter, there cannot be but only damped oscillators. In this case, for the representation (4.3.1), we shall have instead of (4.3.2) the conditions

$$\ddot{Q}(t) + 2\lambda\dot{Q} + \omega_0^2 Q(t) = 0 \quad \Leftrightarrow \quad \begin{cases} q^{-1}\cdot\ddot{q} + 2\lambda(q^{-1}\cdot\dot{q}) + \omega_0^2 = \dot{\theta}^2 \\ q^{-1}\cdot\dot{q} + (2\dot{\theta})^{-1}\ddot{\theta} + \lambda = 0 \end{cases} \tag{4.3.5}$$

leading to the following equation for the phase as a function of time:

$$\dot{\theta}^2 = \omega_0^2 - \lambda^2 - \frac{1}{2}\{\theta, t\} \tag{4.3.6}$$

Here $\{\cdot,\cdot\}$ means *Schwarzian derivative* of the first symbol in curly brackets with respect to the second. However, the notation is not unique in the mathematical literature, even though the definition of the symbol is always the same. To wit, we sometimes may see:

$$\{\theta, t\} \equiv S(\theta)(t) \stackrel{def}{=} \frac{d}{dt}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right) - \frac{1}{2}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right)^2 \tag{4.3.7}$$

The notation $S(\cdot)$ is used mostly for geometrical purposes, as for instance in the case of Lorentz surfaces of constant curvature [21]. The geometrical theory is guided by the notion that the Schwarzian derivative is a second order differential having the meaning of a curvature [28], so that the geometrical notation mimics the structure of a curvature tensor.

These things aside, when coming back to our line here, the connection between amplitude and phase for the case of a damped harmonic oscillator, is not as simple as before, for it involves the time explicitly in the connection between amplitude and the phase of the signal representing the charge. Thus, instead of (4.3.3), we have here:

$$q^2(t)\cdot e^{2\lambda t}\cdot\dot{\theta}(t)=const \tag{4.3.8}$$

However, this fact does not change the previous conclusions on $q(t)$, but just 'rephrase' them in a way, because the equation (4.3.6) is actually an Ermakov-Pinney equation 'in disguise', as it were. Indeed, we have the identity:

$$-\frac{1}{2}\{\theta,t\}\equiv\frac{\ddot{\xi}}{\xi}\quad for\quad \xi^{-1}\equiv\sqrt{\dot{\theta}} \tag{4.3.9}$$

so that (4.3.6) becomes an Ermakov-Pinney equation for ξ:

$$\ddot{\xi}+\left(\omega_0^2-\lambda^2\right)\xi=\frac{1}{\xi^3} \tag{4.3.10}$$

Using then equation (4.3.8) we can find the amplitude of this representation in the form

$$q(t)=const\cdot\xi(t)\cdot e^{-\lambda t} \tag{4.3.11}$$

where $\xi(t)$ is a solution of the Ermakov-Pinney equation (4.3.10).

Let us discuss, up to a point, the physics involved here: essential in the previous mathematics is the *first time derivative of the phase*, which can be taken as an *instantaneous frequency*, like in optics for instance [46]. The basic equation on which we choose to discuss here in the statistical spirit of time-frequency analysis [18], is the equation (4.3.6): it is the only equation that can offer the instantaneous frequency in some *physical terms* connected to the structure of a damped harmonic oscillator. Notice that the instantaneous frequency would be a well-defined mechanical frequency:

$$\dot{\theta}^2\equiv\omega^2=\omega_0^2-\lambda^2 \tag{4.3.12}$$

only in the special cases where the phase is a linear fractional function of time:

$$\{\theta,t\}=0\quad\therefore\quad\theta(t)=\frac{\alpha t+\beta}{\gamma t+\delta} \tag{4.3.13}$$

Here we have used the result (4.3.9) in order to integrate the homogeneous Schwarzian differential equation. Such a physical definition of the instantaneous frequency is impossible in general terms: the frequency $\dot{\theta}$ given by (4.3.13) as a function of time cannot be a constant identically, as the condition (4.3.12) asks. The things get in order, as, in fact, they did historically speaking, either if we assume the phase a linear function of time or else assume that the time can be taken as a period. However, in this last case is not simply $\dot{\theta}$ but the square root of this expression, and therefore the period in question is $\left(\sqrt{\dot{\theta}}\right)^{-1}$, which is a solution of the Ermakov-Pinney equation. This observation allows one to construct a probability density based upon the idea of period [30] as we shall do it here right away. But before presenting this construction, the necessity of interpretation reveals another side of the issue, on which we need to insist at some length. The equation (4.3.10) allows us to associate an instantaneous frequency to a particle in the charge realm, based on physical reasons, provided

$$\lambda^2 + \frac{1}{2}\{\theta,t\} = \mu^2 \quad \therefore \quad \dot{\theta} = \pm\omega_0 \tag{4.3.14}$$

according to equation (4.3.6), where μ is a new damping coefficient entering the ‘updated’ definition of frequency of the damped harmonic oscillator which serves for the physical definition of parameters. This would be an instantaneous frequency depending on time, of course. In terms of harmonic oscillator, this represents a new state of *transfer* between the far and close environments of the particle carrying charge. In view of (4.3.9) and (4.3.10) the equation (4.3.14) can be written as a second order differential equation for the function ξ:

$$\ddot{\xi} + \omega^2\xi = 0, \quad \omega^2 \equiv \mu^2 - \lambda^2 \tag{4.3.15}$$

having solutions of the general form

$$\xi(t) = B\cos(\omega t + b) \quad \therefore \quad \dot{\theta}(t) \equiv \sqrt{\omega_0^2 - \mu^2} = \frac{1}{B^2\cos^2(\omega t + b)} \tag{4.3.16}$$

where B and b are integration constants, and the condition $\mu^2 > \lambda^2$ was assumed. The last identity here leads by integration to a phase of the form

$$\theta(t) = \theta_0 + \frac{1}{B^2\omega}\tan(\omega t + b) \tag{4.3.17}$$

The equation (4.3.15), and therefore its solution (4.3.17), can be replicated if $\lambda^2 > \mu^2$, in which case

$$\ddot{\xi} - \omega^2 \xi = 0, \quad \omega^2 \equiv \lambda^2 - \mu^2 \tag{4.3.18}$$

for ω real. Therefore, the instantaneous frequency can be defined by a time sequence, according to equation

$$\xi(t) = B\cosh(\omega t + b) \quad \therefore \quad \dot{\theta}(t) \equiv \omega_0 = \frac{1}{B^2\cosh^2(\omega t + b)} \tag{4.3.19}$$

with the phase θ connected to the time sequence by equation

$$\theta(t) = \theta_0 + \frac{1}{B^2\omega}\tanh(\omega t + b) \tag{4.3.20}$$

The equation (4.3.18) has relevance in physics as the equation of an *inverted harmonic oscillator* [2], a physical structure appropriate in closed spaces occupied by matter, for instance in tunneling problems [3]. This whole theory only means definition of the frequency through a time sequence, as one does routinely in the practice of measurements of this quantity. Only, in this special case, the practice is simply relying upon equation (4.3.19), so that equation (4.3.20) gives a succession of phases corresponding to the time sequence in question. However, the tunneling phenomenon cannot be described but by statistical methods, and the occasion has come to approach here one of the most important problems in physics: the definition of *the probability density*, mentioned above.

According to Emil Julius Gumbel, the definition of a probability density of an event, stays in its *repetition* [30]. This definition seems to fit the case in point here, for the tunneling should be successful only after many 'trials', as it were. In the case of the complex form of a signal, as given in equation (4.3.1), the variable $\theta(t)$ is bound to describe recurring events, so that it may serve, indeed, to define a probability density according to Gumbel. Using the equation (4.3.16), for instance, the phase $\theta(t)$ can define a period $T(t)$ of recurrence by the relation

$$\Theta(t) = C + D \cdot \tanh(\omega t + b) \tag{4.3.21}$$

where C and D are two constants. Then, according to Gumbel's prescription, the corresponding probability density should be given by

$$w(t) \overset{def}{=} \frac{\dot{\Theta}(t)}{\Theta^2(t)} = \frac{E}{\cosh^2(Mt + N)} \tag{4.3.22}$$

with *E, M* and *N* are new, properly chosen constant. This is the probability density function, indeed, for *a skew-symmetric logistic distribution* [69], known to belong to the class of exponential distributions having of *quadratic variance functions* [51]. And this is the class of distribution functions which also comprises the Planck's Negative Binomial Distribution, and Procopiu's $Z(\beta)$ from equation (2.4.7). It would appear that this whole class of distributions, and not just a specific one of them – like the Negative Binomial Distribution from the case of light [15] – should be characteristic to the physics of quantization in general.

As to the case of a harmonic oscillator described by the equation of motion (4.3.15), which leads to the solution (4.3.17), one cannot give the same statistical interpretation right away. In fact, we judge it here from the point of view of a resonator, and it is quite doubtful that a resonator can be simply represented by just a harmonic oscillator. That such a simple structure should enter somehow the more complicated physical structure of the resonator seems, nevertheless, out of question. But the manner it does has to be realized starting from some other viewpoints. Let us try one such idea.

4.4. The Structure of a Resonator

In the spirit of the Procopiu's analysis [61], it seems that the general physical structure of a resonator, which is a material structure, needs itself a special quantization, proper to matter. Such a material structure should be capable of emitting or absorbing light, and our experience tells us nothing about these two phenomena. That is, nothing but an indication of instantaneity, in the case of the production of light, for instance in a storm thunderbolt. It is based on this observation, and on further inference that the reciprocal phenomenon of absorption of light is just as instantaneous as its production, that Niels Bohr has built his model of quantization in matter [9]. In hindsight, this model appears to be so successful just because the Newtonian forces laying at its foundation is not unconditionally a field force: it does not act locally everywhere in the manner described by Maxwell stresses (see §3.2). Rather, as Paul Langevin once has shown [37], the Newtonian forces are tied up with the orbit defining them, just as Newton defined them initially. Remarkably, they can be gauge forces though: as Berry and Klein showed [79], they are invariant to a special gauging involving the Thomson fundamental dynamics expressed by Ermakov-Pinney equation (4.4.1). According to this view, the radial

Thomson coordinate appears as a gauge length in a Berry-Klein gauging, whose fundamental group is given by the action (4.1.4).

This state of the case should allow us to define a resonator in the most general way, based on the common idea of instantaneity. First, notice that the case from equation (4.3.13) is a reference one, undoubtedly: for once, we can always find a physical meaning for the instantaneous frequency, which can be defined under that condition in terms of the parameters of a harmonic oscillator. On the other hand, though, the corresponding function ξ defined in the equation (4.3.9) is practically the original time, up to a linear transformation, for we have:

$$\theta(t) = \frac{\alpha t + \beta}{\gamma t + \delta} \quad \therefore \quad \xi(t) \equiv \frac{\sqrt{\Delta}}{\sqrt{\dot{\theta}(t)}} = \gamma t + \delta \tag{4.4.1}$$

However, the most important incidentals come here with the idea of a *locked phase*, that may be used to represent instantaneity, and then to define, based on it, the appropriate *physical structure of a resonator*. Thus, a resonator turns out to be a $\mathfrak{sl}$(2,R) Riemannian manifold of constant curvature, of the kind describing the physical structure of the nucleus of the planetary atom, or of the planetary model in general [49]. In order to show this, we simply need to write the condition of a *locked phase*, in terms of the variations of the parameters α, β, γ, δ. We then have:

$$d\theta(t) \equiv d\frac{\alpha t + \beta}{\gamma t + \delta} = 0 \quad \therefore \quad dt = \omega^1 t^2 + \omega^2 t + \omega^3 \tag{4.4.2}$$

where $\omega^{1,2,3}$ are the differential forms:

$$\omega^1 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\delta - \beta\gamma}, \quad \omega^2 = \frac{\alpha d\delta - \delta d\alpha + \beta d\gamma - \gamma d\beta}{\alpha\delta - \beta\gamma}, \quad \omega^3 = \frac{\beta d\delta - \delta d\beta}{\alpha\delta - \beta\gamma} \tag{4.4.3}$$

The equation in differentials (4.4.2) is integrable in the sense of Cartan, if it is *conditionally exact* as a differential 1-form [16]:

$$d \wedge (\omega^1 t^2 + \omega^2 t + \omega^3) = 0$$

Here the symbol '∧', which usually means exterior multiplication, gives to differential symbol 'd' the meaning of exterior differentiation, which we symbolize by 'd∧'. Performing the differentiation according to the rules of exterior calculus, results in

$$\left(d \wedge \omega^1 - \omega^1 \wedge \omega^2\right)t^2 + \left(d \wedge \omega^2 + 2\omega^3 \wedge \omega^1\right)t + \left(d \wedge \omega^3 - \omega^2 \wedge \omega^3\right) = 0$$

which shows that the differential 1-forms (4.4.3) must satisfy the Maurer-Cartan structure equations

$$d\wedge\omega^1=\omega^1\wedge\omega^2,\quad d\wedge\omega^2=-2\omega^3\wedge\omega^1,\quad d\wedge\omega^3=\omega^2\wedge\omega^3 \tag{4.4.4}$$

These are characteristic to a **sl**(2,R) type algebraic structure. One can verify by a direct calculation that the differential 1-forms (4.4.3) satisfy indeed these structural relations, and therefore they can be taken as a coframe for the space of parameters. Adding to this a *quadratic metric* that we choose in the form:

$$(\omega^2)^2-4\omega^1\omega^3 \tag{4.4.5}$$

defined up to an arbitrary constant factor, this manifold can be organized as a Riemannian space, as we said: a physical structure *defined for an instant* – an *instanton* proper. The metric form (4.4.5) was chosen by us in view of the fact that it represents the discriminant of the quadratic form (4.4.2), so that it comes in handy in case we seek for a physical interpretation of that equation. Such a physical interpretation can be brought about as follows immediately. It is based on the idea that, inasmuch as the whole physical definition of such a resonator is pending on the definition of phase by equation (4.4.1) some general properties of this phase can, no doubt, come in handy.

Thus, if the homography (4.4.1) will be written in the form used by Élie Cartan as a suggestive example, using three essential parameters ([16], see examples to §§102, 108, 112, 161, and 214 in that reference):

$$\theta=\phi+\frac{t}{\alpha t+\beta} \tag{4.4.6}$$

Here ϕ is a phase that makes physical sense in the problem at hand, like the parameter θ from equation (4.3.3), for instance, in the case of charge, or the central angle of the second of the Kepler laws. In the interest of an incidental further analysis of this relation, notice the 'full closed homography', from equation (4.4.1) would be here:

$$\theta=\frac{(1+\alpha\phi)t+\beta\phi}{\alpha t+\beta} \tag{4.4.7}$$

This reduces to an identity in the cases where $t=0$. Therefore, for this acceptance of the time parameter, we have unconditionally an identity between phase of a recorded signal and a physically meaningful phase, as in in the Kepler motion or the harmonic oscillator cases:

$$\theta(0) = \phi \tag{4.4.8}$$

On the other hand, if the time t goes to infinity, we have, again, a physically accepted interpretation of the situation, whereby the physical phase is increased by a *constant period*, as in the case envisioned in the Bertrand's classical theorem:

$$\theta(\infty) = \phi + \alpha^{-1} \tag{4.4.9}$$

The parameter α is therefore itself a period. If this period is zero, we have a linear relationship between the phase and the time:

$$\theta(t)\Big|_{\alpha=0} = \phi + \beta^{-1} t \tag{4.4.10}$$

Therefore β itself must be a period, for its inverse should be a frequency. This period is distinguished though: only in the cases where it goes to infinity, and therefore the corresponding frequency goes to zero, can the phase θ be *constant at any finite time*, as required, for instance, in the case of a harmonic oscillator, be it damped, or not.

Expressing now the previous condition (4.4.2) of a locked phase, constructed based on a physically meaningful phase, we get the equation in differentials

$$dt = \left(-\frac{\alpha^2}{\beta}d\phi + \frac{d\alpha}{\beta}\right)t^2 + \left(-2\alpha d\phi + \frac{d\beta}{\beta}\right)t - \beta d\phi \tag{4.4.11}$$

Here the differential 1-forms $\omega^{1,2,3}$, can be obtained either by using directly the equation (4.4.7), or by identifying from it the parameters from (4.4.1) and then calculating the differential forms according to the formulas from the equation (4.4.3). Either way, we get the obvious result, to be read on (4.4.11):

$$\omega^1 = -\frac{\alpha^2}{\beta}d\phi + \frac{d\alpha}{\beta}, \quad \omega^2 = -2\alpha d\phi + \frac{d\beta}{\beta}, \quad \omega^3 = -\beta d\phi \tag{4.4.12}$$

Therefore, the circumstances in which the phase is locked are given by times that are solutions of equation in differentials given by (4.4.11). These solutions are referring to a three-dimensional ensemble of times, each one of them located by a set of three values of the parameters (ϕ, α, β).

We can say something about this ensemble if we can describe the time moments as the values of some continuous function, which needs to be even differentially continuous, in view of the equation (4.4.11). According to a classical definition of such a function, this description comes down to associating a continuous parameter to time moments, in order to transform the equation (4.4.11) into a system of ordinary differential equations.

The simplest case of such a parameter, can be exhibited if the differential forms (4.4.12) are exact differentials proportional to the differential of the same parameter, *i.e.*, if we can write

$$-\frac{\alpha^2}{\beta}d\phi+\frac{d\alpha}{\beta}=a^1(ds),\quad -2\alpha d\phi+\frac{d\beta}{\beta}=2a^2(ds),\quad -\beta d\phi=a^3(ds) \tag{4.4.13}$$

where $a^{1,2,3}$ are constants, and s is the parameter in question. This can happen along the geodesics of the Riemannian $\mathfrak{sl}(2,R)$ manifold having the metric (4.4.5). Our notation is intended to suggest that the parameter s has to be taken as the arclength of the Riemannian metric of this Thomson realm, which in turn would mean that the time sequences are ordered by the *geodesic motions of that metric*, just as in the classical case, where the time sequences are given by the free particle motions. Here too, such an interpretation is possible, provided we use classical free particles in a Euclidean reference frame with 'inaccessible origin', as have shown before. But let us continue with our case, to see what all is about. The equation in differentials (4.4.11) becomes an ordinary differential equation of Riccati type, offering, by its solutions, the moments t as the values of a function of the continuous parameter s:

$$\frac{dt}{ds}=a^1t^2+2a^2t+a^3 \tag{4.4.14}$$

As we have shown in this work, this equation has a fundamental bearing on the physical description of the Thomson realm: it represents the interpretable case of the motion of a *Thomson particle* in the field of fundamental forces of magnitude going inversely with the cubic distance. That distance is provided by the square root of the quadratic polynomial from the right hand side of (4.4.14), which is, indeed, classically interpretable. We proceed to finding the solution of this equation in the most general terms.

The theory of solution of a Riccati equation of the type (4.4.14) shows that the general solution is completely determined if we know three particular solutions [78]. We do not have, though, but only two such particular solutions: the roots of quadratic polynomial from the right hand side of the (4.4.14). A third one can be taken as an arbitrary constant value of t, according to the following procedure: notice that the equation (4.4.14) is equivalent to the equation in exact differentials:

$$ds=\frac{dt}{a^1t^2+2a^2t+a^3} \tag{4.4.15}$$

Now use the following expression for the denominator here:

$$a^1\left(a^1t^2+2a^2t+a^3\right)\equiv\left(a^1t+a^2\right)^2-\Delta,\quad \Delta\equiv\left(a^2\right)^2-a^1a^3$$

so that, assuming the case of real roots, $\Delta>0$, and the general solution of (4.4.14) can be written as

$$2\sqrt{\Delta}\left(s-s_0\right)=\ln\left|\frac{\xi-1}{\xi+1}\right|,\quad \xi\equiv\frac{a^1t+a^2}{\sqrt{\Delta}} \tag{4.4.16}$$

where s_0 is an arbitrary constant. There are two possible solutions here, according to the values of ξ that make the expression from the right hand side positive:

$$a^1t+a^2=\begin{cases}-\sqrt{\Delta}\cdot\coth\left\{\sqrt{\Delta}\left(s-s_0\right)\right\}\\ or\\ -\sqrt{\Delta}\cdot\tanh\left\{\sqrt{\Delta}\left(s-s_0\right)\right\}\end{cases} \tag{4.4.17}$$

according to the position of the values of ξ with respect to the interval $(-1,1)$: outside or inside.

On the other hand, if the two roots t_1 and t_2 are complex, we have the identity

$$a^1\left(a^1t^2+2a^2t+a^3\right)\equiv\left(\sqrt{\Delta}\right)^2+\left(a^1t+a^2\right)^2,\quad \Delta\equiv a^1a^3-\left(a^2\right)^2$$

and this expression is always positive, regardless of the values of the parameters entering its algebraic structure. Then the general solution of the equation (4.4.15) can be written in the form:

$$a^1t+a^2=\sqrt{\Delta}\cdot\tan\left\{\sqrt{\Delta}\left(s-s_0\right)\right\} \tag{4.4.18}$$

where s_0 is an arbitrary constant.

The integration of the system (4.4.13) replicates any of these cases of solution for the equation (4.4.14). In order to see this, notice first that the differential $d\phi$ can be eliminated from (4.4.13) in favor of ds. Then, once this elimination done, we seek for expressions for $d\alpha$ and $d\beta$ in order to construct the differential of the ratio α/β. The final result of this construction is the Riccati equation:

$$d\left(\frac{\alpha}{\beta}\right)=\left\{a^1-2a^2\left(\frac{\alpha}{\beta}\right)+a^3\left(\frac{\alpha}{\beta}\right)^2\right\}(ds) \tag{4.4.20}$$

having a solution of the form (4.4.17) or (4.4.18), only with the places of a^1 and a^3 interchanged, and the sign of a^2 simply changed. Obviously, Δ remains the same under these operations. Therefore, we can have either the correspondent of (4.4.17) i.e., a general solution of the form:

$$a^3\tau - a^2 = \begin{cases} -\sqrt{\Delta}\cdot\coth\left\{\sqrt{\Delta}\left(s-s_0\right)\right\} \\ or \\ -\sqrt{\Delta}\cdot\tanh\left\{\sqrt{\Delta}\left(s-s_0\right)\right\} \end{cases} \tag{4.4.21}$$

where τ denotes the ratio between α and β, or else the correspondent of (4.4.18):

$$a^3\tau - a^2 = \sqrt{\Delta}\cdot\tan\left\{\sqrt{\Delta}\left(s-s_0\right)\right\} \tag{4.4.22}$$

where s_0 is an arbitrary real value in both cases.

Once we got the solution (4.4.21) or (4.4.22), we can concentrate upon finding solutions of (4.4.13) for the other two dependent variables: β and ϕ. For once, using the third equation (4.4.13) in order eliminate $d\phi$ from the second, we get the following differential equation for β:

$$\frac{d\beta}{\beta} + 2\left(a^3\tau - a^2\right)(ds) = 0 \tag{4.4.23}$$

which can be solved by using for the binomial expression entering as the coefficient of the differential of independent variable, one of the two different expressions provided by equations (4.4.21) and (4.4.22), in turn. According to this procedure, corresponding to (4.4.21) we have:

$$\beta(s) = \begin{cases} \beta_0\sinh^2\left\{\sqrt{\Delta}(s-s_0)\right\} \\ or \\ \beta_0\cosh^2\left\{\sqrt{\Delta}(s-s_0)\right\} \end{cases} \tag{4.4.24}$$

while corresponding to (4.4.22), we have:

$$\beta(s) = \beta_0\cos^2\left\{\sqrt{\Delta}(s-s_0)\right\} \tag{4.4.25}$$

where β_0 is an integration constant in both cases, and Δ from (4.4.25) is the negative of that from equation of (4.4.24).

Now, we can concentrate on finding α, whose expression is only a matter of algebra. Thus, using (4.4.21) and the corresponding (4.4.24) we have:

$$a^3\alpha(s) = \begin{cases} \beta_0\sinh\left\{\sqrt{\Delta}(s-s_0)\right\}\left(a^2\sinh\left\{\sqrt{\Delta}(s-s_0)\right\} - \sqrt{\Delta}\cosh\left\{\sqrt{\Delta}(s-s_0)\right\}\right) \\ or \\ \beta_0\cosh\left\{\sqrt{\Delta}(s-s_0)\right\}\left(a^2\cosh\left\{\sqrt{\Delta}(s-s_0)\right\} - \sqrt{\Delta}\sinh\left\{\sqrt{\Delta}(s-s_0)\right\}\right) \end{cases} \tag{4.4.26}$$

while using (4.4.22) and the corresponding (4.4.25) we have

$$a^3\alpha(s)=\beta_0\cos\left\{\sqrt{\Delta}\left(s-s_0\right)\right\}\cdot\left(a^2\cos\left\{\sqrt{\Delta}\left(s-s_0\right)\right\}+\sqrt{\Delta}\cdot\sin\left\{\sqrt{\Delta}\left(s-s_0\right)\right\}\right) \tag{4.4.27}$$

We cannot close the case of the parameters α and β without noticing one of the most important issues concerning their possible physical connection: taken as a virtual position, the pair (α, β) is *located on a conic*, which can be, in particular, a closed cycle. In order to show this, we consider the three cases of parametrization in turn. Thus, for the case of the first of equations (4.4.21) and the first of (4.4.24) we have the hyperbola:

$$\left(a^3\alpha-a^2\beta\right)^2-\Delta\cdot\beta^2=\Delta\cdot\beta_0\beta \tag{4.4.28}$$

and for the case of the second ones of those equations we have a hyperbola of symmetric center with respect to the α-axis:

$$\left(a^3\alpha-a^2\beta\right)^2-\Delta\cdot\beta^2=-\Delta\cdot\beta_0\beta \tag{4.4.29}$$

In both cases here, we must observe the definition:

$$\Delta\equiv\left(a^2\right)^2-a^1a^3>0$$

It is only for the parameterization (4.4.22) and (4.4.25) that we get a cycle proper, for in that case we have the ellipse

$$\left(a^3\alpha-a^2\beta\right)^2+\Delta\cdot\beta^2=\Delta\cdot\beta_0\beta \tag{4.4.30}$$

where we have to observe the condition of definition:

$$\Delta\equiv\left(a^2\right)^2-a^1a^3<0$$

Now, we got the gist of the method, so that for the case of the parameter ϕ would be no need to transcribe all three cases, that would only lengthen the discourse. Suffice to notice that, by its parameterization, this dependent variable is of the same algebraic nature as the time variable. For instance, in the last case of equation (4.4.30) we have, by integrating the last of the equations (4.4.13), the parameterization:

$$\phi(s)=\phi_0-\frac{a^3}{\beta_0\sqrt{\Delta}}\cdot\tan\left\{\sqrt{\Delta}\left(s-s_0\right)\right\} \tag{4.4.31}$$

where ϕ_0 is an arbitrary constant. Obviously, this parameterization is of the same algebraical nature as the time given by (4.4.18): the initial time must be of the same nature as the time *t* itself. It also can possess periods serving in a definition of the probability densities of Gumbel type, necessary in description of the tunneling processes. But we defer the continuation of this theory to a future work.

5. CONCLUSIONS

The results thus far allow us some conclusions defining a future strategy in describing the structure of matter based on the fundamental idea of quantization. By comparing the Procopiu quantization procedure with the prototypical Planck's quantization procedure, and extracting the due theoretical conclusions, we can report that:

1) The quantization in matter should be different from the quantization in light, even though they are based on the same principles and use the same statistics, supported by the exponential distributions having quadratic variance functions. The main difference is that in the case of light the statistics is of discrete type, while in the case of matter it is of continuous type, staying at the base of the modern magnetism physical theory: the Generalized Hyperbolic Secant distribution.

2) The quantization in matter asks for quanta of Procopiu type, that generalize the classical Newtonian central forces with magnitude going inversely with the square of distance. In this sense the classical Newtonian dynamics can count, implicitly, as a quantum theory.

3) A classical dynamics can be constructed in order to describe the motion under forces in matter, and thus support the statistics involved in the definition of the quanta of any kind. The fundamental dynamical equation in matter appears to be the Ermakov-Pinney equation.

4) The concept of resonator is connected to the concept of the phase of charges: it represents an instanton, i.e., a lump of matter for whose interpretation we need to use particles having the same phase of charges. The Procopiu quantization is based on such a statistics that can be constructed as a convolution of two such instantons structures. One can see in this model of resonator the generalization of the classical Ampère element of electrodynamics.

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