# Critical Analysis of Trigonometry
# Based on the Unity of Formal Logic and Rational Dialectics

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**Abstract**. The critical analysis of the foundations of standard trigonometry is proposed. The unity of formal logic and rational dialectics is methodological basis of the analysis. The analysis leads to the following main results: (1) trigonometry does not treat a right triangle as a material system. Therefore, trigonometry does not satisfy the system principle; (2) trigonometric functions do not satisfy the mathematical definition of a function. The terms "sine", "cosine", "tangent", "cotangent" and others are not identical to the concept of function. Symbols "cos", "sin", "tg", "ctg", etc. indicate only that there is a correspondence (connection) between the values of the quantities of the angle and the lengths of the sides in a right-angled triangle. Therefore, the standard definitions of trigonometric functions do not represent mathematical (quantitative) relationships between the quantities of the angle and the lengths of the sides in a right-angled triangle. Trigonometric functions are neither explicit nor implicit functions; (3) the range of definition of trigonometric functions does not satisfy the condition for the existence of a right-angled triangle because the definitions of trigonometric functions contradict to the system principle. These facts prove the assertion that the trigonometric functions, the trigonometric identities, the trigonometric form of the Pythagorean theorem and the inverse trigonometric functions are blunders; (4) the values of mathematical quantities are always neutral numbers. Therefore, logical contradictions arise if the quantity of the angle and the symbols "cos", "sin", "tg", "ctg" take on negative values. (5) it is proved that the standard theorems of addition (difference) of two arguments for cosine and sine are blunders. This means that the addition (difference) theorems for all trigonometric functions, the reduction formula, the formula for double and half argument are blunders; (6) in the point of view of the Cartesian coordinate system, the abscissa and ordinate scales are identical and have the dimension "meter". Therefore, the quantity of the angle (which has the dimension "degree") does not exist in the Cartesian coordinate system; (7) the graphs of trigonometric functions are built in an inadmissible coordinate system because the scales are not identical: the abscissa scale has the dimension "degree", and the ordinate scale has the dimension "meter". The non-identity of the dimensions leads to absurdity: "meter" is "degree". Therefore, the graphs of trigonometric functions have no geometric meaning; (8) if the material point is the end point of the moving radius in the material system "circle + mobile radius + Cartesian coordinate system", then the graph of the dependence of the ordinate of the material point on the length of the path traveled (i.e., on the circumference of a given radius) has the form of a sinusoid, but the graph is not a trigonometric sinusoid.

Consequently, standard trigonometry is a pseudoscientific theory.

**Keywords**: general mathematics, trigonometry, geometry, methodology of mathematics, mathematical physics, physics, engineering, formal logic, dialectics, philosophy of mathematics, philosophy of science.

**MSC**: 00A05, 00A30, 00A30g, 00A35, 00A69, 00A79, 03A05, 03A10, 03B42, 03B44, 03B80, 33B10, 03F50, 97E20, 97E30, 97G60, 97G70, 97M50, 51M15, 51N35, 51P05.

## Introduction

As is well known, trigonometry is a branch of mathematics [1-7] and an important part of the mathematical formalism of theoretical physics [8]. Trigonometry as an analytical science was created by I. Newton, L. Euler, J. Fourier, N. Lobachevsky (Lobachevski) and other classics of science. The works of eminent scientists have generated faith in the firmness (indestructibility, irrefutability, constancy) of the foundations of standard trigonometry. But faith is not a proof of the truth of theorems and theories. Faith is not the criterion of truth. Faith rejects doubt about the validity (truth) of the standard theorems and theories. Faith prevents the search and cognition of truth within the framework of the correct methodological basis: the unity of formal logic and rational dialectics.

The critical analysis of the works of the classics of mathematics and physics shows [9-43] that the classics relied on their intuition, but not on the methodological basis. The classics could not find the correct methodological basis and criterion of truth. Therefore, their works do not satisfy the correct criterion of truth.

For the first time methodological errors in trigonometry were revealed (detected) and analyzed in [19-23]. The purpose of the present work is to propose the critical analysis of the foundations of standard trigonometry within the framework of the correct methodological basis: the unity of formal logic and rational dialectics. This way of analysis gives an opportunity to understand the erroneous essence (erroneous concepts) of standard trigonometry.

**1. On the essence of the right-angled triangle**

As is well known, the right-angled triangle is one of the most important figures in geometry, trigonometry, and engineering. This figure as a material system can be constructed and studied as follows [19-23].

1) The right-angled triangle is constructed as follows. If the sides of the angle are bound up with the rectilinear segment, then the synthesized system (the constructed geometrical figure) $\Delta\, AOB$ is called right-angled triangle (Figure 1).

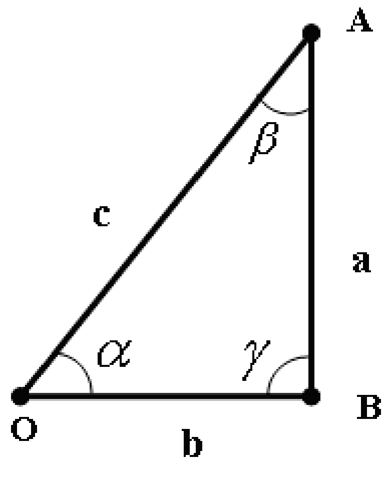


**Figure 1**. Geometrical figure "right-angled triangle $\Delta\, AOB$" as a given material system. Points $O$, $A$, $B$ are universal joints, $\gamma = 90^\circ$.

Three points $O$, $A$, $B$ are called vertexes of triangle. The points $O$, $A$, $B$ are universal joints. The rectilinear segments $a$, $b$, $c$ bounded by vertexes are called legs of triangle $\Delta\, AOB$; the interior angle (concluded angle) $\gamma$ is equal to $90^\circ$. Triangle as a material system does not exist, if length of any leg is equal to zero. Existence of the interior angles (concluded angles) $\alpha$, $\beta$, $\gamma$

of triangle leads to rise of the essential feature of system: the sum $S = \alpha + \beta + \gamma$. Value of $S = 180^\circ$ can be determined only by means of experimental (practical) investigation of properties of triangle as a material system. Therefore, the relationship $(\alpha + \beta) = 90^\circ$ is always true. This implies that the relationship $90^\circ - (\alpha + \beta) \neq 0^\circ$ is incorrect. The right-angled triangle $\Delta\, AOB$ is a basis for experimental and system analysis of the relationships between angles and lengths of triangle sides. The values of quantities are neutral numbers.

2) "Material circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$" is the following geometrical (material) system (Figure 2):

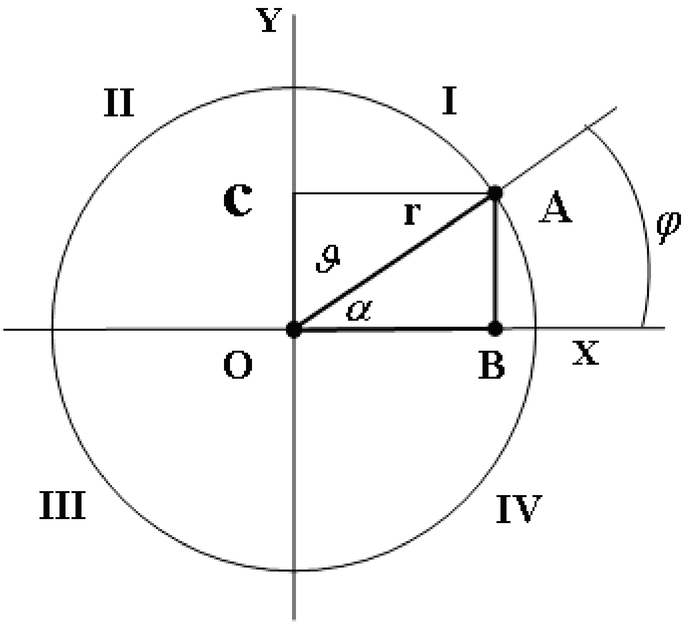


**Figure 2**. Geometrical figure "material circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$" as a given material system. Segment $\overline{OA}$ is mobile radius; hypotenuse $\overline{OA}$, legs $\overline{OB}$ and $\overline{AB}$ are material elements of the $\Delta\, AOB$; lengths of legs $\overline{OB}$ and $\overline{AB}$ are measured using rulers $OX$ and $OY$; $\alpha$ is quantity of the angle between the segments $\overline{OB}$ and $\overline{OA}$; $0^\circ < \alpha < 90^\circ$; points $O$, $A$, $B$, $C$ are universal joints; the angle $\varphi$ is the quantity angle of rotation of the mobile radius $\overline{OA}$; $0^\circ \leq \varphi \leq 360^\circ$. The values of quantities are neutral numbers.

To analyze correctly a geometric system, one must take into consideration the essence of mathematics and geometry. As is known [9-43], the essence of mathematics, geometry and trigonometry is based on the following statements:

(a) the concept "negative number" is an erroneous concept. Any number is a neutral number, i.e. the neutral number does not have sign "+" or "–" . The symbols "+" and "–" are symbols of mathematical (quantitative) operations;

(b) the coordinate system $XOY$ represents four connected material rulers on the material plane: two horizontal rulers $OX$ and two vertical rulers $OY$ . Ruler scales are marked by neutral numbers. The rulers have a common origin: the neutral number "zero". These neutral numbers have the dimension "*meter*" . Therefore, rulers are tools (means) for measuring the lengths of material segments. The results of measurements are expressed by variables $x$ and $y$, which take on numerical values; the values of the quantities are neutral numbers with the dimension "*meter*" ;

(c) the projection (image) of some material point in the coordinate system $XOY$ is a material point (without dimension) on the coordinate scale. Coordinates $x$, $y$ of point in the coordinate system $XOY$ are material segments of coordinate scales $X$, $Y$ and therefore coordinates have the dimension "*meter*" ;

(d) projections (images) of any segment of a material line in the coordinate system $XOY$ are segments of rectilinear material lines (having dimension "*meter*" ) on coordinate scales $X$, $Y$;

(e) the concepts "direction", "direction of change" and "direction of rotation" are not mathematical concepts. Therefore the direction of rotation of the mobile radius cannot be described mathematically. The direction of rotation of the mobile radius is neither positive nor negative characteristic of rotation. The direction of rotation has no sign;

(f) an angle as a geometric figure is a material system that consists of two intersecting straight line segments (elements). The material segments are called the sides of the angle. This is the genetic geometric (qualitative, practical) definition of the system. The intersection point of the straight line segments can be the end points of the line segments. If the end points of the straight segments are connected by a joint, then the segments can be rotated relative to each other. The mutual position of the sides of the angle represents the quantitative determinacy of the angle. The mutual position of the sides of the angle is a variable quantity. If the mutual position of the sides of the angle takes 360 elementary positions (states) under the condition of full turn of the side of the angle, then each elementary position (state) of the sides of the angle is $1/360$ th part of a complete turn. The number $1/360$ is a neutral number (because "part of the whole" can be neither positive nor negative characteristic) and is called "degree". "Degree" (the designation is "$^\circ$") is the unit of measurement for the quantity of the angle. A variable quantity that takes on numerical values from $0^\circ$ to $360^\circ$ is called the quantity of angle. The values of the quantity "degree" are neutral numbers. The quantity of the angle is independent of the lengths of the sides, the positive or negative properties of the sides, and conditions of formation (generation) of the angle (in particular, the quantity of the angle is independent of the direction of rotation of the mobile radius). The value of angle has no dimension "*meter*" and therefore does not exist in the coordinate system $XOY$. In the practical point of view, an angle is a useless geometric figure if it is not an element of a complex geometric figure. There is no correct mathematical (quantitative) definition of the quantity of angle.

(g) Rotation of the mobile radius $\overline{OA}$ is not a periodic motion if the rotation from the value $0^\circ$ of the quantity $\varphi$ to the value $360^\circ$ of the quantity $\varphi$ occurs once. The quantity $\varphi$ does not take on a value greater than $360^\circ$ because one divided the circle into 360 parts. The beginning of a quantitative change in the quantity $\varphi$ (i.e. the value $0^\circ$) and the end of a quantitative change (i.e. the value $360^\circ$) are dialectically connected: the beginning of a change in values of $\varphi$ has an end; the end of change in values of $\varphi$ has a beginning; the end turns to the beginning if the change in values of $\varphi$ is continued. If the rotations from the value $0^\circ$ of the quantity $\varphi$ to the value $360^\circ$ of the quantity $\varphi$ occur several times, then the rotations are a periodic motion. In this case, rotation number (number of revolutions, number of cycles) is abstract (absolute) number $n$. Speed of revolution (rotation) is $v = n/t$ where $t\,(\text{sec})$ is time. If $n = 1$, $t = T$, then rotation frequency is $\nu = 1/T$ where $T\,(\text{sec})$ is period of revolution (rotation) (in other words, $T\,(\text{sec})$ is a time of one revolution (rotation)). The quantity $\nu t = t/T = n$ is number of revolution (rotation) during the $0 \leq t < \infty$. The expression $(\nu t + \varphi)$ is absurd because the quantities $\nu t$ and $\varphi$ have different dimensions and meanings.

(h) Can rotation of the mobile radius $\overline{OA}$ lead to displacement of the right-angled triangle $\Delta\, AOB$ from the first quadrant of circle to the fourth quadrant of circle? The given triangle $\Delta\, AOB$ does not exist under $\alpha = 0^\circ$ and $\alpha = 90^\circ$ (i.e., under the values $y = 0$ and $x = 0$ of the coordinates of the point $A$ of the radius $\overline{OA}$): the material figure $\Delta\, AOB$ degenerates under $\alpha = 0^\circ$ and $\alpha = 90^\circ$. The values $x = 0$ and $y = 0$ are inadmissible values. But the given triangle $\Delta\, AOB$ can be moved from the first quadrant of circle to the fourth quadrant of circle under rotation of the mobile radius $\overline{OA}$ (Figure 2).

Explanation is that material cathetus $\overline{OB}$ moves on the material scale $X$ due to universal joints $O$, $A$, $B$, $C$. The angle $\alpha$ is the angle between the hypotenuse $\overline{OA}$ and the material scale $X$. The angle $\alpha$ is a cyclic quantity under rotation of the mobile radius $\overline{OA}$: the angle $\alpha$ increases in the quadrant **I** if the angle $\varphi$ increases from value $0^\circ$ to value $90^\circ$; the angle $\alpha$ decreases in the quadrant **II** if the angle $\varphi$ increases from value $90^\circ$ to value $180^\circ$; the angle $\alpha$ increases in the quadrant **III** if the angle $\varphi$ increases from value $180^\circ$ to value $270^\circ$; the angle $\alpha$ decreases in the quadrant **IV** if the angle $\varphi$ increases from value $270^\circ$ to value $360^\circ$. Thus, the right-angled triangle $\Delta\, AOC$ moves from the quadrant **I** to the quadrant **IV** under rotation of the mobile radius $\overline{OA}$. The quadrants **II**, **III**, and **IV** are mirror images of the quadrant **I**.

(i) in the case of the system "material circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$" (Figure 2), the correct experimental relationship between the dimensional variable quantities $\alpha$ and $y$ in the linear approximation has the following form:

$$\frac{\alpha - \alpha_i}{\alpha_i} = \frac{y - y_i}{y_i}, \quad \alpha = \left(\frac{\alpha_i}{y_i}\right) y, \; 0^\circ < \alpha < 90^\circ, \; 0\, meter < y\, meter$$

where the variable $y$ is the length of the leg $\overline{AB}$ which is measured with the ruler $OY$; the length of the hypotenuse $\overline{OA}$ is $r = const$; $y_i$ and $\alpha_i$ are experimental values; $i = 1, 2, 3, \ldots$ . This linear relationship represents the proportion of the relative increments of the variable quantities $\alpha$ and $y$ describing the different elements of the right-angled triangle $\Delta\, AOB$;

(j) in the case of the system "circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$", the correct experimental relationship between the dimensional variable quantities $\alpha$ and $x$ in the linear approximation has the following form:

$$\frac{\alpha - \alpha_i}{\alpha_i} = \frac{1/x - 1/x_i}{1/x_i}, \; \alpha = (\alpha_i x_i)\frac{1}{x}, \quad 0^\circ < \alpha < 90^\circ, \quad 0\, meter < x\, meter$$

where the variable $x$ is the length of the leg $\overline{OB}$ which is measured with the ruler $OX$; the length of the hypotenuse $\overline{OA}$ is $r = const$; $x_i$ and $\alpha_i$ are experimental values; $i = 1, 2, 3, \ldots$ . This linear relationship represents the proportion of relative increments of the variable quantities $\alpha$ and $x$ describing the different elements of the right-angled triangle $\Delta\, AOB$;

(k) in the case of the system "material circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$", the experimental relationship between the dimensional variable quantities $y$ and $x$ in the linear approximation has the following form:

$$\frac{y - y_1}{y_1} = \frac{1/x - 1/x_1}{1/x_1}, \quad x \neq 0, \quad y \neq 0$$

where the length of the hypotenuse $\overline{OA}$ is $r = const$; $y_i$ and $x_i$ are experimental values; $i = 1, 2, 3, \ldots$ . This relationship represents the proportion of relative increments of the variable quantities $y$ and $x$ describing the different elements of the right-angled triangle $\Delta\, AOB$;

(l) in the case of the system "material circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$" (Figure 2), the correct experimental relationship between the dimensional variable quantities $\alpha$ and $\beta$ in the linear approximation has the following form:

$$\frac{\alpha - \alpha_i}{\alpha_i} = \frac{1/\beta - 1/\beta_i}{1/\beta_i}$$

where the length of the hypotenuse $\overline{OA}$ is $r = const$; $y_i$ and $x_i$ are experimental values; $i = 1, 2, 3, \ldots$ .;

(m) in the case of the system "material circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$", the quantitative relationships between $\alpha$ and $\varphi$ have the following form:

$$\text{under } 0^\circ \leq \varphi \leq 90^\circ,$$
$$\alpha \neq \varphi \text{ under } 90^\circ \leq \varphi \leq 180^\circ,$$
$$\alpha \neq \varphi \text{ under } 180^\circ \leq \varphi \leq 270^\circ,$$
$$\alpha \neq \varphi \text{ under } 270^\circ \leq \varphi \leq 360^\circ.$$

The qualitative relationship between $\alpha$ and $\varphi$ have the form of law of lack of contradiction:

*"internal (concluded) angle $\alpha$ of right-angled triangle $\Delta\, AOB$*
*is not angle $\varphi$ between the mobile radius $\overline{OA}$ and the scale $OX$"*;

(n) in the case of the system "material circle + right-angled triangle $\Delta\, AOB$ + coordinate system $XOY$", the quantitative relationships between the circumference $l^{(circle)}$ and the radius $r$ has the following form of proportion:

$$\left(\frac{l^{(circle)} - l_1^{(cirkle)}}{l_1^{(cirkle)}}\right) = \left(\frac{r - r_1}{r_1}\right),$$
$$l^{(circle)} = \left(\frac{l_1^{(circle)}}{r_1}\right) r .$$

Therefore, the dependence of the ordinate of the material point $A$ of the moving radius $\overline{OA}$ on the length of the traversed path length $l^{(circle)}$ (i.e., the circumference $l^{(circle)}$) has the following form (Figure 3):

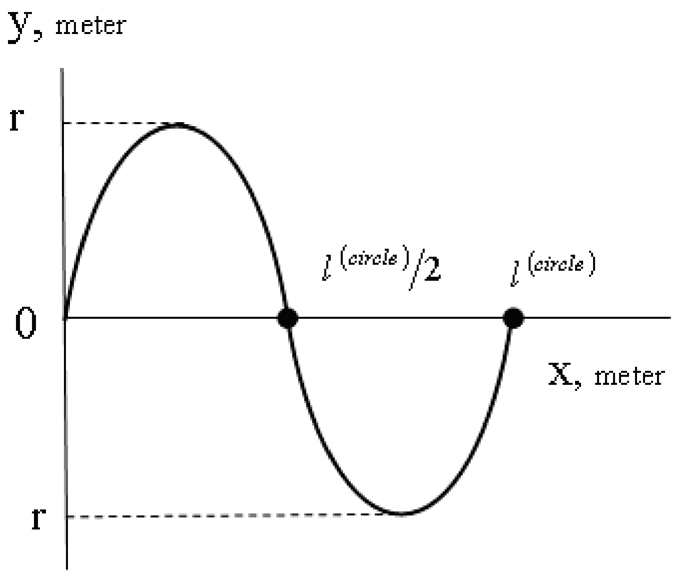


**Figure 3.** Dependence of the ordinate of the material point $A$ on the length of the traversed path length $l^{(circle)}$. The material point $A$ represents the end point of the rotating radius $\overline{OA}$ of the material system "circle + mobile radius $\overline{OA}$ + coordinate system $XOY$". $l^{(circle)}$ is the circumference; $r$ is the radius of the circle.

Obviously, the graph (diagram) is not a sinusoid.

**2. On the foundations of standard trigonometry**

As is known (Russian Wikipedia), standard trigonometry is not based on consideration of the right-angled triangle $\Delta\, AOB$. Standard trigonometry is based on consideration of the system "circle + mobile radius $\overline{OA}$ + connected right-angled triangles $\Delta\, AOB$ and $\Delta\, AOC$ + coordinate system $XOY$" (Figure 2). The essence of the foundations of standard trigonometry is the set of the following unfounded assertions.

(a) Definitions of trigonometric functions are:

$$, \quad \sin\varphi = \frac{y}{r}, \quad tg\,\varphi = \frac{\sin\varphi}{\cos\varphi}, \quad ctg\,\varphi = \frac{\cos\varphi}{\sin\varphi},$$

where $r = const$ is the length of the mobile radius $\overline{OA}$; $x$ and $y$ are the coordinates of the point $A$ of the mobile radius $\overline{OA}$ (in other words, $x$ and $y$ are the segments of $OX$ and $OY$); $\varphi$ is an angle between the mobile radius $\overline{OA}$ and the scale $OX$ (Figure 2). The mobile radius $\overline{OA}$ and the coordinates $x$ and $y$ (the segments of $OX$ and $OY$) form a right triangle. The range of definition of the functions $\cos\varphi$ and $\sin\varphi$ is $0^\circ \leq \varphi < \infty$.

In the case of the right triangle $\Delta\, AOB$, the designations (notations) is as follows:

$$\cos\alpha^{(\Delta AOB)} = \frac{x^{(\Delta AOB)}}{r^{(\Delta AOB)}}, \quad \sin\alpha^{(\Delta AOB)} = \frac{y^{(\Delta AOB)}}{r^{(\Delta AOB)}},$$

where $r^{(\Delta AOB)} = const$ is the length of the hypotenuse.

(b) The relationships

$$\cos^2 \alpha^{(\Delta AOB)} + \sin^2 \alpha^{(\Delta AOB)} = 1,$$
$$\cos^2 \frac{\alpha^{(\Delta AOB)}}{2} + \sin^2 \frac{\alpha^{(\Delta AOB)}}{2} = 1$$

represent the trigonometric form of the Pythagorean theorem.

(c) The addition theorem for cosine is formulated as follows. The cosine of the sum (difference) of two angles (arguments) has the following form:

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta,$$
$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta.$$

The properties of evenness of cosine and oddness of sine are used in the formula for $\cos(\alpha + \beta)$.

(d) The addition theorem for the sine is formulated as follows. The sine of the sum (difference) of two angles (arguments) has the following form:

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta,$$
$$\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta.$$

The addition theorem for the sine is a consequence of the relationship $90^\circ - (\alpha + \beta) \neq 0^\circ$, the properties of evenness of the cosine and the oddness of the sine, and the addition (difference) theorem for the cosine.

(e) The addition theorems for tangent and cotangent are consequences of the addition theorems for cosine and sine.

(f) The reduction formulae express the trigonometric functions of the arguments $-\alpha$, $90^\circ \pm \alpha$, $180^\circ \pm \alpha$, $270^\circ \pm \alpha$, $360^\circ \pm \alpha$ in terms of the functions of the argument $\alpha$.

(g) The duplication formulae for the argument are as follows:

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha,$$
$$\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha.$$

These formulae are a consequence of the addition formulae for cosine and sine under $\alpha = \beta$.

(h) The formulae for division of the argument in half express the trigonometric functions of the half argument $\alpha/2$ in terms of the trigonometric functions of the argument $\alpha$:

$$2\cos^2 \frac{\alpha}{2} = 1 + \cos\alpha,$$
$$2\sin^2 \frac{\alpha}{2} = 1 - \cos\alpha.$$

These formulae are a consequence of the formula for the cosine of the double argument and the trigonometric form of the Pythagorean theorem for the half argument.

**3. Objections to the foundations of the standard trigonometry**

1) The standard definitions of trigonometric functions do not satisfy the system principle [19-23], because one does not treat a right triangle as a material system.

2) The relationships

$$\cos\alpha^{(\Delta AOB)} = \frac{x^{(\Delta AOB)}}{r^{(\Delta AOB)}}, \quad \sin\alpha^{(\Delta AOB)} = \frac{y^{(\Delta AOB)}}{r^{(\Delta AOB)}}, \quad r^{(\Delta AOB)} = const$$

represent the following expressions:

$$f_c\left(\alpha^{(\Delta AOB)}\right) = \frac{x^{(\Delta AOB)}}{r^{(\Delta AOB)}}, \quad f_s\left(\alpha^{(\Delta AOB)}\right) = \frac{y^{(\Delta AOB)}}{r^{(\Delta AOB)}},$$

$$\text{i.e.,} \quad f_c(\alpha) = \frac{x}{r}, \quad f_s(\alpha) = \frac{y}{r}, \quad r = const.$$

These relationships express the experimental fact that the experimental values $\alpha_i$ correspond to the experimental values $x_i$ and $y_i$ ($i = 1, 2, 3, ...$ ). By definition, the symbol $f$ is a designation of the law of functional dependence (i.e., a designation of the law of connection between variables). The law of functional dependence represents a set of mathematical (quantitative) operations that must be performed on an argument in order to obtain a value of function.

But the symbols $f_c \equiv \cos$ and $f_s \equiv \sin$ do not indicate (determine, define) the set of mathematical operations that one must perform on the quantity $\alpha$ in order to obtain the quantities $x$ and $y$. There are no mathematical operations that would convert the quantity of the angle into the length of straight line segments. Consequently, the symbols $f_c \equiv \cos$ and $f_s \equiv \sin$ do not represent analytic definitions (representations) of functions. If these symbols were an analytic definitions (representations) of functions, then these functions could be classified (i.e., polynomial functions, rational functions, explicit algebraic functions, implicit algebraic functions, transcendental functions, etc.). But the symbols $f_c \equiv \cos$ and $f_s \equiv \sin$ are just signs that denote an experimental fact: the existence of a correspondence between the experimental values $\alpha_i$ and $x_i$, $y_i$ ($i = 1, 2, 3, ...$ ). The words "$\cos$" and "$\sin$" can be replaced by the symbol: "$\alpha \leftrightarrow x/r$".

3) As is known, the concept of function is introduced as follows:

$$x = x, \quad ax = ax, \quad ax + b = ax + b, \quad y = ax + b,$$
$$y = f(x), \quad f(x) = ax + b;$$
$$(ax + b)^2 = (ax + b)^2, \quad z = (ax + b)^2,$$
$$z = y^2, \quad z = F(y, x), \quad F(y, x) = (ax + b)^2.$$

From this point of view, $f_c \equiv \cos$, $f_s \equiv \sin$ and expressions $f_c^2(\alpha) \equiv \cos^2\alpha$, $f_s^2(\alpha) \equiv \sin^2\alpha$ are meaningless expressions because the symbols (characters) $f_c \equiv \cos$ and $f_s \equiv \sin$ are just icons. Consequently, the trigonometric form of the Pythagorean theorem $\cos^2\alpha + \sin^2\alpha = 1$ is a meaningless expression.

4) The values $y = 0$ and $x = 0$ are inadmissible values because the right-angled triangle does not exist under the values $y = 0$ and $x = 0$. Therefore, in the cases of $y = 0$ and

$x = 0$, the Pythagorean theorem loses its meaning: $y^2 = r^2$ (under $x = 0$) and $x^2 = r^2$ (under $y = 0$).

5) Standard definitions of trigonometric functions contain the following uncertainty. Which right-angled triangle – $\Delta AOB$ or $\Delta AOC$ in the Figure 2 – do the following standard definitions correspond to

$$\frac{x}{r} = \frac{l^{(\overline{OB})}}{l^{(\overline{OA})}} = \frac{l^{(\overline{CA})}}{l^{(\overline{OA})}} \quad ?$$

(where $l$ is the length of the segment, $x$ is coordinate of the point $A$ of the mobile radius $\overline{OA}$).

6) The direction of rotation of the mobile radius $\overline{OA}$ does not determine the sign "+" or "−" of the quantity $\varphi$. Indeed, if the mobile radius $\overline{OA}$ were rotated in a negative direction (i.e., in a clockwise direction), $\varphi$ would be a negative quantity. Then the following contradiction would arise: $90^\circ = -90^\circ$ under coincidence (superposition) of the mobile radius $\overline{OA}$ with the coordinate system ruler "$Y$" (where $90^\circ$ is the value of the angle belonging to the coordinate system $XOY$; $\varphi = -90^\circ$ is the value of the angle formed by the mobile radius $\overline{OA}$ in the quadrant **IV**. Consequently, the values of the quantities $\varphi$, $\alpha$, $\beta$, $\gamma$, etc. are neutral numbers. Trigonometric functions are neither even nor odd functions.

7) The fallacy (falsity) of standard trigonometric functions is expressed by the following relationship:

$$\frac{x}{r} = \cos\varphi = \cos\left(90^\circ - \vartheta\right) = \sin\vartheta, \quad 0^\circ \le \varphi \le 90^\circ, \; \varphi = 90^\circ - \vartheta.$$

In this relationship, $\varphi$ is the angle between the scale $OX$ and the mobile radius $\overline{OA}$; the angle $\vartheta$ belongs to the right-angled triangle $\Delta AOC$; the angle $90^\circ = const$ belongs to the coordinate system $XOY$ (Figures 2 and 3). In other words, the variables $\varphi$ and $\vartheta$ in this relationship belong to different subsystems. This is a violation of the system principle. A correct relationship must not contain the value $90^\circ = const$ belonging to the coordinate system $XOY$.

8) The standard definitions of trigonometric functions in quadrants **I**, **II**, **III**, and **IV** are not based on consideration of the given right-angled triangle $\Delta AOB$ (Figure 1, 2, 3). The standard definitions of trigonometric functions in quadrants **I, II, III,** and **IV** are based on consideration of the positions of the mobile radius $\overline{OA}$ in the system "the mobile radius $\overline{OA}$ + connected right-angled triangles $\Delta AOB$ and $\Delta AOC$" under change in the values of the angle $\varphi$ (Figure 2). In this case, the standard definitions take the following form:

$$|\cos\varphi| = \left|\frac{x}{r}\right|, \; |\sin\varphi| = \left|\frac{y}{r}\right|, \; |tg\,\varphi| = \left|\frac{\sin\varphi}{\cos\varphi}\right|, \; |ctg\,\varphi| = \left|\frac{\cos\varphi}{\sin\varphi}\right|, \; 0^\circ \le \varphi \le 360^\circ.$$

The relationships between the trigonometric functions of the arguments $\varphi$ and $\vartheta$ represent the following expressions:

in the first quadrant: $0^\circ \le \varphi \le 90^\circ$, $\cos\varphi = \sin\vartheta$;
in the second quadrant: $90^\circ \le \varphi \le 180^\circ$, $\varphi = 180^\circ - \vartheta$, $|\cos\varphi| = |\cos\vartheta|$;
in the third quadrant: $180^\circ \le \varphi \le 270^\circ$, $\varphi = 270^\circ - \vartheta$, $|\cos\varphi| = |\sin\vartheta|$;

in the fourth quadrant: $270^{\circ} \leq \varphi \leq 360^{\circ}$, $\varphi = 360^{\circ} - \vartheta$, $\left|\cos\varphi\right| = \left|\cos\vartheta\right|$
where the angle $\vartheta$ belongs to the right-angled triangle $\Delta\, AOC$ .

The relationships between the trigonometric functions of the arguments $\varphi$ and $\vartheta$ do not satisfy the formal-logical law of the lack of qualitative contradiction. The law of the lack of qualitative contradiction states the following:

*"the geometric figure representing the angle between the scale $OX$ and the mobile radius $\overline{OA}$ is not identical with the geometric figure representing the internal (concluded) angle of right-angled triangle $\Delta\, AOC$ ".*

9) The standard statements about the evenness and the oddness of the trigonometric functions are erroneous because the values of the quantities $x$, $y$, $\alpha$ and $\beta$ for a right-angled triangle are neutral numbers.

10) As is known, the standard theorem of sum (difference) of two angles (arguments) for cosine reads as follows: the cosine of the sum (difference) of two angles (arguments) is expressed by the following formulae:

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta,$$
$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta.$$

The standard proof of the addition (difference) theorem for cosine is based on Figures 4, 5.

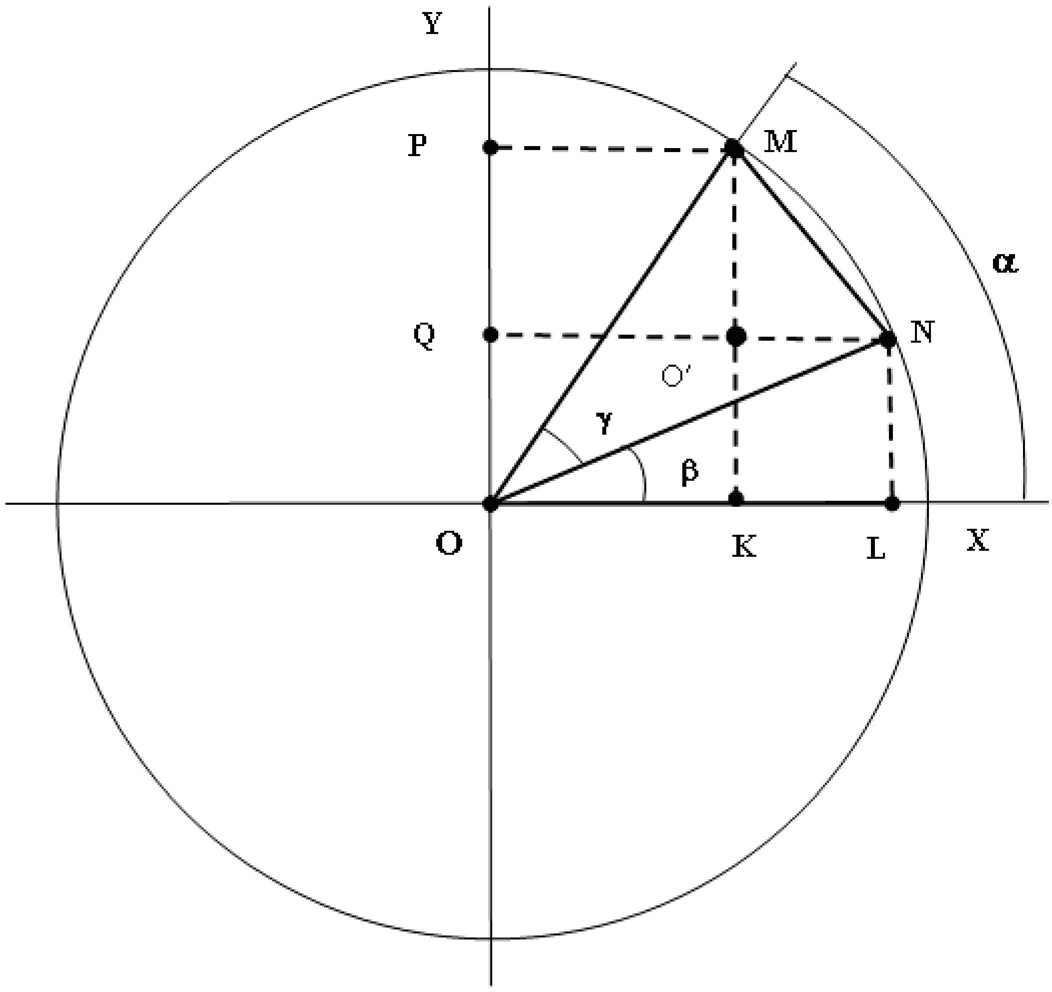


**Figure 4.** The initial position of the material triangle $\Delta MON$ in the coordinate system $XOY$. The mobile radii $\overline{OM}$, $\overline{ON}$ and segment $\overline{MN}$ are the sides of the triangle $\Delta MON$. Length of the segment $\overline{MN}$ is constant. The relationship between the variable quantities $\alpha$ and $\beta$ has the following form: $\gamma = \alpha - \beta$ where $\gamma = const$. Values of quantities are neutral numbers not equal to zero.

(a) In accordance with Figure 4 and the Pythagorean theorem, the geometric relationship

$$\left(d_1^{(\overline{MN})}\right)^2 = \left(d_1^{(\overline{O'M})}\right)^2 + \left(d_1^{(\overline{O'N})}\right)^2 =$$
$$= \left(x_1^{(\overline{OL})} - x_1^{(\overline{OK})}\right)^2 + \left(y_1^{(\overline{MK})} - y_1^{(\overline{O'K})}\right)^2$$

is correct under the conditions

$$\gamma = \gamma_1 = \alpha_1 - \beta_1, \quad \alpha_1 \neq 0, \ \beta_1 \neq 0, \ \gamma_1 \neq 0$$

($\alpha_1$, $\beta_1$, $\gamma_1$ are the values of the variables).

If

$$\frac{x_1^{(\overline{OL})}}{r} = \cos\beta_1, \quad \frac{x_1^{(\overline{OK})}}{r} = \cos\alpha_1, \quad \frac{y_1^{(\overline{MK})}}{r} = \sin\alpha_1, \quad \frac{y_1^{(\overline{NL})}}{r} = \sin\beta_1,$$

then the relationship

$$\left(d_1^{(\overline{MN})}\right)^2 = \left[2 - 2\left(\cos\alpha_1 \cdot \cos\beta_1 + \sin\alpha_1 \cdot \sin\beta_1\right)\right] r^2$$

represents a trigonometric expression for the square of the length of the hypotenuse $\overline{MN}$ of the right-angled triangle $\Delta MO'N$ (Figure 4).

(b) In accordance with Figure 4, clockwise rotation of the right-angled triangle $\Delta MO'N$ around the point $O$ and superposition of the side $\overline{ON}$ with the ruler $\overline{OX}$ results in Figure 5.

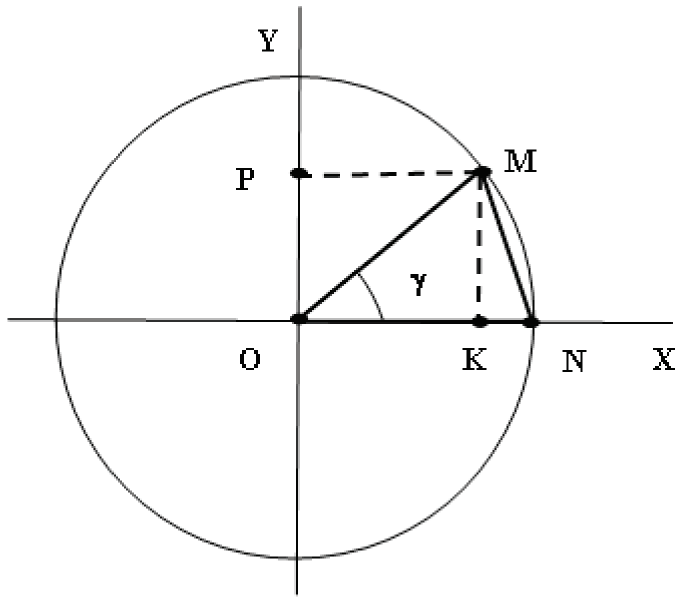


**Figure 5.** The final position of the material triangle $\Delta MON$ in the coordinate system $XOY$. The mobile radii $\overline{OM}$, $\overline{ON}$ and segment $\overline{MN}$ are the sides of the triangle $\Delta MON$. Length of the segment $\overline{MN}$ is constant. The relationship between the quantities of angles has the following form: $\gamma = \alpha - \beta$ where $\gamma = const$, $\beta = 0$. The values of quantities are neutral numbers.

In this case, the geometric relationship

$$\left(d_2^{(\overline{MN})}\right)^2 = \left(d_2^{(\overline{KN})}\right)^2 + \left(d_2^{(\overline{KM})}\right)^2 =$$
$$= \left(x_2^{(\overline{ON})} - x_2^{(\overline{OK})}\right)^2 + \left(y_2^{(\overline{MK})}\right)^2$$

is correct under the conditions

$$\gamma = \gamma_2 = \alpha_2 - \beta_2, \quad \alpha_2 \neq 0,$$
$$\beta_2 = 0, \; \gamma_2 \neq 0, \quad d_2^{(\overline{NL})} = 0$$
($\alpha_2$, $\beta_2$, $\gamma_2$ are values of variables).

If

$$\frac{x_2^{(\overline{ON})}}{r} = 1, \quad \frac{x_2^{(\overline{OK})}}{r} = \cos\alpha_2, \quad \frac{y_2^{(\overline{MK})}}{r} = \sin\alpha_2$$

(where $r$ is the length of the mobile radius), then the relationship

$$\left(d_2^{(\overline{MN})}\right)^2 = \left(2 - 2\cos\alpha_2\right)r^2$$

represents trigonometric expression for the square of the length of the hypotenuse $\overline{MN}$ of the right-angled triangle $\Delta MKN$ (Figure 5).

(c) Therefore, the geometric relationship

$$\left(d_1^{(\overline{MN})}\right)^2 = \left(d_2^{(\overline{MN})}\right)^2$$

has the following trigonometric form:

$$\left[2 - 2\left(\cos\alpha_1 \cdot \cos\beta_1 + \sin\alpha_1 \cdot \sin\beta_1\right)\right]r^2 = \left(2 - 2\cos\alpha_2\right)r^2,$$

This expression leads to the standard formula for the cosine of the difference of the arguments:

$$\cos\left(\alpha_1 - \beta_1\right) = \cos\alpha_1 \cdot \cos\beta_1 + \sin\alpha_1 \cdot \sin\beta_1$$
$$\text{if } \alpha_2 = \alpha_1 - \beta_1.$$

But $\alpha_2 \neq \alpha_1 - \beta_1$.

(d) If correct detailed designation is introduced, then one can detect formal-logical errors in the formula for the cosine of the difference of the arguments. Correct detailed designations have the following form:

$$\frac{x_1^{(\overline{OL})}}{r^{(\Delta LON)}} = \cos\beta_1^{(\Delta LON)}, \quad \frac{x_1^{(\overline{OK})}}{r^{(\Delta MOK)}} = \cos\alpha_1^{(\Delta MOK)},$$
$$\frac{y_1^{(\overline{MK})}}{r^{(\Delta MOK)}} = \sin\alpha_1^{(\Delta MOK)}, \quad \frac{y_1^{(\overline{LN})}}{r^{(\Delta LON)}} = \sin\beta_1^{(\Delta LON)},$$

$$\frac{x_2^{(\overline{ON})}}{r} = 1, \quad \frac{x_2^{(\overline{OK})}}{r^{(\Delta MOK)}} = \cos\alpha_2^{(\Delta MOK)}, \quad \frac{y_2^{(\overline{MK})}}{r^{(\Delta MOK)}} = \sin\alpha_2^{(\Delta MOK)},$$

$$\cos\alpha_2^{(\Delta MOK)} \neq \cos\left(\alpha_1 - \beta_1\right)$$

where the hypotenuses of the triangles $\Delta LON$ and $\Delta MOK$ are equal to the length $r$ of the movable radius.

The first formal-logical error in the standard formula is that $\cos\left(\alpha_1 - \beta_1\right)$ does not exist because the quantity $\left(\alpha_1 - \beta_1\right)$ does not belong to any right-angled triangle. The second formal-logical error in the standard formula is that $\cos\beta_1^{(\Delta LON)}$, $\cos\alpha_1^{(\Delta MOK)}$, $\sin\alpha_1^{(\Delta MOK)}$, $\sin\beta_1^{(\Delta LON)}$, $\cos\alpha_2^{(\Delta MOK)}$, $\sin\alpha_2^{(\Delta MOK)}$ do not belong to the same right-angled triangle.

The detailed expression

$$\left[2 - 2\left(\cos\alpha_1^{(\Delta MOK)} \cdot \cos\beta_1^{(\Delta LON)} + \sin\alpha_1^{(\Delta MOK)} \cdot \sin\beta_1^{(\Delta LON)}\right)\right]r^2 = \\ = \left(2 - 2\cos\alpha_2^{(\Delta MOK)}\right)r^2$$

shows that the standard formula for the cosine of the difference of arguments represents the following formal-logical error: violation of the law of lack of contradiction. The law of lack of contradiction read as follows:

*"the left and right sides of the mathematical (quantitative) relationship should not belong to different triangles (qualitative determinacy)".*

(e) The standard formula for the cosine of the sum of the arguments $\cos\left(\alpha_1 + \beta_1\right) = \cos\alpha_1 \cdot \cos\beta_1 - \sin\alpha_1 \cdot \sin\beta_1$ is a consequence of the following expression:

$$\left[2 + 2\left(\cos\alpha_1 \cdot \cos\beta_1 - \sin\alpha_1 \cdot \sin\beta_1\right)\right] = \\ = \left(2 + 2\cos\alpha_2\right), \quad \alpha_2 = \alpha_1 + \beta_1$$

where

$$\left[2 + 2\left(\cos\alpha_1 \cdot \cos\beta_1 - \sin\alpha_1 \cdot \sin\beta_1\right)\right] = \\ = \left(\cos\beta_1 + \cos\alpha_1\right)^2 + \left(\sin\alpha_1 - \sin\beta_1\right)^2 \neq \left(d_1^{(\overline{MN})}/r\right)^2,$$

$$\left(2 + 2\cos(\alpha_1 + \beta_1)\right) = \\ = \left(1 + \cos(\alpha_1 + \beta_1)\right)^2 + \sin^2(\alpha_1 + \beta_1) \neq \left(d_2^{(\overline{MN})}/r\right)^2.$$

Consequently, $d_1^{(\overline{NM})} \neq d_2^{(\overline{NM})}$. This implies that the standard formula for the cosine of the sum of the arguments contradict to the Pythagorean theorem.

These expressions are proof of the fact that the standard formula for the cosine of the sum of arguments is a gross geometrical error.

11) As is known, the standard theorem of addition (difference) of two angles for the sine reads as follows: the sine of the sum (difference) of two arguments is expressed by the following formulae:

$$\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta,$$
$$\sin(\alpha-\beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta.$$

The standard proof of the theorem of addition (difference) of two arguments for sine is based on the following statement: the sine of the sum $(\alpha + \beta)$ is equal to the cosine of the additional argument $\left(90^\circ - (\alpha + \beta)\right)$:

$$\sin(\alpha+\beta) = \cos\left(90^\circ - (\alpha + \beta)\right).$$

But the relationship $\left(90^\circ - (\alpha + \beta)\right) = 0$ is a reliable fact for a right-angled triangle. Therefore, the first error in the theorem of addition (difference) of two arguments for the sine is that $\left(90^\circ - (\alpha + \beta)\right) \neq 0$. The second error is that the addition theorem for sine relies on the erroneous theorem of the difference of the arguments for cosine. The third error is that the formula for the sine of the difference of the arguments is based on an impermissible (inadmissible) substitution $\beta \to -\beta$ in the formula for the sine of the sum of the arguments.

12) The standard reduction formulae express the trigonometric functions of the arguments $-\alpha$, $90^\circ \pm \alpha$, $180^\circ \pm \alpha$, $270^\circ \pm \alpha$, $360^\circ \pm \alpha$ via (in terms of) functions of the argument $\alpha$. But they are incorrect, because they contradict to the existence condition for right-angled triangle.

13) The standard formulae for the double argument are as follows:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$
$$\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha.$$

But these formulae are a consequence of the addition formulae for cosine and sine under $\alpha = \beta$. Therefore, these formulae are incorrect.

14) The standard bisection formulae express the trigonometric functions of the half argument $\alpha/2$ via (in terms of) the trigonometric functions of the argument $\alpha$. But these formulae are incorrect because they are based on the double argument formulae and the following substitution: $2\alpha \to \alpha/2$.

Thus, all definitions and relationships of standard trigonometry (including Inverse trigonometric functions) represent blunders.

15) The graphs of trigonometric functions are built in the coordinate system $\varphi OY$. But, in the point of view of the coordinate system $XOY$, the coordinate system $\varphi OY$ has no geometric meaning, because: (a) the dimensions of the quantities $\varphi$ ("degree") and $x$, $y$ ("meter") are different; (b) the quantity $\varphi$ ("degree") does not exist in the system $XOY$. The difference in dimensions leads to the following absurd (Figure 6):

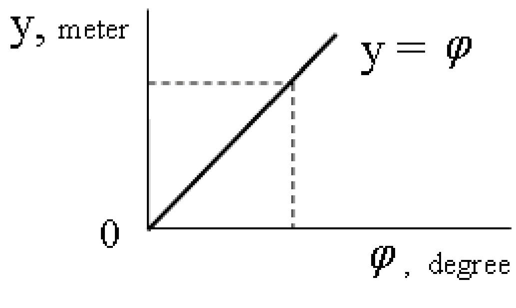


**Figure 6**. Graph of a straight line

segment in the coordinate system $\varphi OY$ .

The absurd is that $y\ meter = \varphi\ degree$, $0\ meter = 0\ degree$. Therefore, the standard sinusoid is absurd (Figure 7).

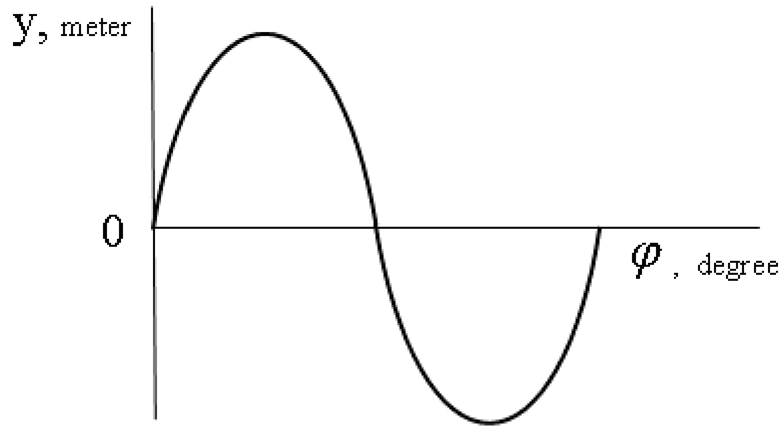


**Figure 7**. Standard sinusoid as absurd

Also, if the arguments and the standard trigonometric functions represent a set of abstract numbers, then the graphs of trigonometric functions do not exist in the metric coordinate system $XOY$ and have no theoretical and practical importance. For example, the sinusoid $y = A\sin(\nu t + \varphi_0)$ (where $A$ is a coefficient, $(\nu t + \varphi_0)$ is a phase) represents a meaningless expression because: (a) the quantities $\nu t$ and $\varphi_0$ have different meanings; (b) the formula for the sine of the sum of the arguments is an error.

## 4. Discussion

Thus, standard trigonometry contains blunders. If standard trigonometry is a pseudoscientific theory, then the following questions arise: Why did the classics of science make scientific mistakes in their work? Why did subsequent generations of scientists not discover errors in science? Why do the errors are not removed from science today? In my opinion, the answers to these questions could be as follows.

(a) The sciences arise from the needs of practice and inductively progress according to the following scheme: “practice $\to$ theory $\to$ practice”.

(b) The creation of a theory does not lead to the creation of a criterion of truth. Practice is not a complete criterion of truth for a theory. Special sciences - mathematics and physics - do not contain the criterion of truth.

(c) The classics of mathematics and physics could not find the criterion of truth. The starting point of their creative works was simple practice and intuition. Unfounded (i.e., doubtful and unclear) points in the created theories were overcome by them with the help of intuition. This means that the inductive method of cognition inevitably leads to boundless accumulation of errors. The inductive method of cognition does not lead to complete truth.

(d) The criterion of truth can only be formulated within the framework of the general sciences: formal logic and rational dialectics. The unity of formal logic and rational dialectics is a correct methodological basis and, consequently, a correct criterion of truth.

(e) Modern scientists are unwilling or unable to critically analyze unfounded (i.e., doubtful and unclear) points of theories because they do not work within a correct methodological basis. Therefore, methodological errors exist in the scientific literature. (For example, the standard definition $y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ of the derivative function is a consequence of the following logical contradiction: $0 \neq \Delta x = 0$). This means that the inductive method of cognition does not eliminate errors from science. The inductive method of cognition does not lead to complete truth. In accordance with system principle, part of truth does not exist without complete, absolute truth.

(f) The elimination of methodological errors leads to the abolition of standard theories. Critical analysis of theories within the framework of the correct methodological basis and the abolition of standard theories open the way to the search for truth. But, as the history of science shows, the development of erroneous theories is probably preferable to the search for truth in a competitive environment. “False hypotheses often rendered more services than the true ones” (H. Poincare). Is a lie better than the truth?

**Conclusion**

Thus, the critical analysis of the foundations of standard trigonometry within the framework of the unity of formal logic and rational dialectics leads to the following main results:

(1) trigonometry does not treat a right triangle as a material system. Therefore, trigonometry does not satisfy the system principle;

(2) trigonometric functions do not satisfy the mathematical definition of a function. The terms “sine”, “cosine”, “tangent”, “cotangent” and others are not identical to the concept of function. Symbols “cos”, “sin”, “tg”, “ctg”, etc. indicate only that there is a correspondence (connection) between the values of the quantities of the angle and the lengths of the sides in a right-angled triangle. Therefore, the standard definitions of trigonometric functions do not represent mathematical (quantitative) relationships between the quantities of the angle and the lengths of the sides in a right-angled triangle. Trigonometric functions are neither explicit nor implicit functions;

(3) the range of definition of trigonometric functions does not satisfy the condition for the existence of a right-angled triangle because the definitions of trigonometric functions contradict to the system principle. These facts prove the assertion that the trigonometric functions, the trigonometric identities, the trigonometric form of the Pythagorean theorem and the inverse trigonometric functions are blunders;

(4) the values of mathematical quantities are always neutral numbers. Therefore, logical contradictions arise if the quantity of the angle and the symbols “cos”, “sin”, “tg”, “ctg” take on negative values.

(5) it is proved that the standard theorems of addition (difference) of two arguments for cosine and sine are blunders. This means that the addition (difference) theorems for all trigonometric functions, the reduction formula, the formula for double and half argument are blunders;

(6) in the point of view of the Cartesian coordinate system, the abscissa and ordinate scales are identical and have the dimension “meter”. Therefore, the quantity of the angle (which has the dimension “degree”) does not exist in the Cartesian coordinate system;

(7) the graphs of trigonometric functions are built in an inadmissible coordinate system because the scales are not identical: the abscissa scale has the dimension “degree”, and the ordinate scale has the dimension “meter”. The non-identity of the dimensions leads to absurdity: “meter” is “degree”. Therefore, the graphs of trigonometric functions have no geometric meaning;

(8) if the material point is the end point of the moving radius in the material system “circle + mobile radius + Cartesian coordinate system”, then the graph of the dependence of the ordinate

of the material point on the length of the path traveled (i.e., on the circumference of a given radius) has the form of a sinusoid, but the graph is not a trigonometric sinusoid.

Consequently, standard trigonometry is a pseudoscientific theory.

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