The Grothendieck - Krivine number

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Abstract

We give some formulas for the Grothendieck – krivine number

Introduction

The Grothendieck – krivine number is defined by

$$K_G = \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} = 1.7822... \tag{1}$$

where

$$\pi = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\right) = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{2}$$

The Grothendieck inequality and Grothendieck constants are named after Alexander Grothendieck, who proved the existence of the constants in a paper published in 1953. For details see [2],[5],[6],[7],[8],[9].

Brief Explanation:

The real Grothendieck constant K_G is the infimum over those $K \in (0,\infty)$ such that for every $m,n \in \mathbb{N}$ and every $m \times n$ real matrix $\left(a_{ij}\right)$ we have

$$\max_{\{x_i\}_{i=1}^m,\{y_j\}_{j=1}^n \subseteq S^{n+m-1}} \sum_{i=1}^m\sum_{j=1}^n a_{ij}\left(x_i, y_j\right) \le K \max_{\{\varepsilon_i\}_{i=1}^m,\{\delta_j\}_{j=1}^n \subseteq \{-1,1\}} \sum_{i=1}^m\sum_{j=1}^n a_{ij}\,\varepsilon_i\,\delta_j \tag{3}$$

The classical Grothendieck inequality asserts the non-obvious fact that the above inequality does hold true for some $K \in (0,\infty)$ that is independent of m,n and $\left(a_{ij}\right)$. Since Grotendieck's 1953 discovery of this powerful theorem, it has found numerous applications in a variety of areas, but despite attracting a lot of attention, the exact value of the Grothendieck constant K_G remains a mystery. In 1977 Krivine proved that $K_G \le \pi / \left(2\ln\left(1+\sqrt{2}\right)\right)$ and conjectured that his bound is optimal.

The conjecture was refuted in 2011 by Braverman, Makarychev and Naor , who showed that K_G is strictly less than Krivine's bound.

Theorem (Braverman et al.): There exists $\varepsilon_0 > 0$ such that

$$K_G < \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} - \varepsilon_0$$

For details see [7], [8].

In this note we give some formulas for the Grothendieck-Krivine number .

Representations for K_G (Grotendieck-krivine number)

Entry 1.

$$\frac{1}{K_G} = \frac{2\ln\left(1+\sqrt{2}\right)}{\pi} = \sqrt{2}\sum_{k=0}^{\infty} 2^{-k-4} \sum_{n=0}^{k} \frac{42n+5}{2k-2n+1}\binom{2n}{n}^3 2^{-11n} \tag{4}$$

$$\frac{1}{K_G} = \frac{2\ln\left(1+\sqrt{2}\right)}{\pi} = \sqrt{2}\sum_{k=0}^{\infty} 2^{-k-2} \sum_{n=0}^{k} \frac{6n+1}{2k-2n+1}\binom{2n}{n}^3 2^{-7n} \tag{5}$$

$$\frac{1}{K_G} = \frac{2\ln\left(1+\sqrt{2}\right)}{\pi} = \frac{1}{2}\sum_{k=0}^{\infty}\binom{4k}{2k}\binom{2k}{k}\frac{2^{-6k}}{2k+1} \tag{6}$$

$$\frac{1}{K_G} = \frac{2\ln\left(1+\sqrt{2}\right)}{\pi} = \frac{1}{2}\,{}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};1,\frac{3}{2};1\right) \tag{7}$$

Remark: ${}_3F_2$ is the generalized hypergeometric function.

Entry 2.

$$\frac{1}{K_G} = \frac{2\ln\left(1+\sqrt{2}\right)}{\pi} = \sqrt{2}\sum_{k=0}^{\infty}\frac{2^{-12k-4}}{(2k+1)!}A_k \tag{8}$$

where

$$A_k = \sum_{n=0}^{k}\binom{2k-2n}{k-n}^3\binom{2k+1}{2n+1}(42k-42n+5)(2k-2n)!(2n)!2^{11n} \tag{9}$$

$$\frac{1}{K_G} = \frac{2\ln\left(1+\sqrt{2}\right)}{\pi} = \sqrt{2}\sum_{k=0}^{\infty}\frac{2^{-8k-2}}{(2k+1)!}B_k \tag{10}$$

where

$$B_k = \sum_{n=0}^{k}\binom{2k-2n}{k-n}^3\binom{2k+1}{2n+1}(6k-6n+1)(2k-2n)!(2n)!2^{7n} \tag{11}$$

Entry 3.

$$\frac{1}{K_G}=\frac{2\ln\left(1+\sqrt{2}\right)}{\pi}=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\left(1+\sqrt{2}\right)^{1/\pi}-1\right)^{n} \tag{12}$$

$$\frac{1}{K_G}=\frac{2\ln\left(1+\sqrt{2}\right)}{\pi}=\ln 2-2\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\frac{1}{\sqrt{2}}\left(1+\sqrt{2}\right)^{1/\pi}\right)^{n} \tag{13}$$

Entry 4.

$$\frac{1}{K_G}=\frac{2\ln\left(1+\sqrt{2}\right)}{\pi}=\lim_{n\to\infty}2^{n}\left(\sqrt[2^n]{\left(1+\sqrt{2}\right)^{2/\pi}}-1\right) \tag{14}$$

Entry 5.

$$\frac{1}{K_G}=\frac{2\ln\left(1+\sqrt{2}\right)}{\pi}=\lim_{n\to\infty}\left(\frac{2\left(1+\sqrt{2}\right)^{2^{-n}}-2}{\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n-radicals}}\right) \tag{15}$$

Entry 6. If $n=0,1,2,3,\ldots$, we have

$$x_0=2\,,\; x_{n+1}=x_n\sinh\left(\frac{\pi}{2x_n}\right)\Rightarrow x_n\to K_G=\frac{\pi}{2\ln\left(1+\sqrt{2}\right)} \tag{16}$$

$$x_0=2\,,\; x_{n+1}=\frac{x_n}{\sqrt{2}}\cosh\left(\frac{\pi}{2x_n}\right)\Rightarrow x_n\to K_G=\frac{\pi}{2\ln\left(1+\sqrt{2}\right)} \tag{17}$$

$$x_0=2\,,\; x_{n+1}=x_n\left(\sqrt{2}-1\right)\exp\left(\frac{\pi}{2x_n}\right)\Rightarrow x_n\to K_G=\frac{\pi}{2\ln\left(1+\sqrt{2}\right)} \tag{18}$$

$$x_0=2\,,\; x_{n+1}=x_n+\cos\left(x_n\ln\left(1+\sqrt{2}\right)\right)\Rightarrow x_n\to K_G=\frac{\pi}{2\ln\left(1+\sqrt{2}\right)} \tag{19}$$

Entry 7.

$$K_G=\frac{\pi}{2\ln\left(1+\sqrt{2}\right)}=2\sum_{n=0}^{\infty}c_n\left(\sqrt{2}-1\right)^{2n} \tag{20}$$

where

$$c_0=1\,,\; c_n=\frac{(-1)^n}{2n+1}-\sum_{k=1}^{n}\frac{c_{n-k}}{2k+1}\quad ,n=1,2,3,\ldots \tag{21}$$

$$c_n=\left\{1,-\frac{2}{3},\frac{2}{9},-\frac{214}{945},\frac{358}{2835},-\frac{63754}{467775},\ldots\right\} \tag{22}$$

Entry 8.

$$K_G = \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} = \frac{3}{\sqrt{2}}\sum_{n=0}^{\infty} c_n \, 2^{-n} \tag{23}$$

where

$$c_0 = 1 \, , \, c_n = \frac{2^{-3n}}{2n+1}\binom{2n}{n} - \sum_{k=1}^{n} \frac{c_{n-k}}{2k+1} \quad , n = 1, 2, 3, \ldots \tag{24}$$

$$c_n = \left\{1, -\frac{1}{4}, -\frac{47}{480}, -\frac{2203}{40320}, -\frac{345481}{9676800}, -\frac{2974553}{116121600}, \ldots\right\} \tag{25}$$

Entry 9.

$$K_G = \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} = 2\left(\int_0^\infty \left(1+\sqrt{2}\right)^{-x^2} dx\right)^2 = 2\left(\int_0^\infty \left(\sqrt{2}-1\right)^{x^2} dx\right)^2 \tag{26}$$

Entry 10.

$$\sqrt{K_G} = \sqrt{\frac{\pi}{2\ln\left(1+\sqrt{2}\right)}} = \sqrt{2}\sum_{n=0}^{\infty}\left(\sqrt{2}-1\right)^{n^2} f(n) \tag{27}$$

where

$$f(n) = \int_0^1 \left(\sqrt{2}-1\right)^{2nx+x^2} dx \tag{28}$$

Entry 11.

$$K_G = \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} = \int_0^\infty \frac{1}{\left(\sqrt{1+\sqrt{2}}\right)^{x} + \left(\sqrt{1+\sqrt{2}}\right)^{-x}} dx \tag{29}$$

$$K_G = \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} = \int_0^\infty \frac{1}{\cosh\left(x\ln\left(1+\sqrt{2}\right)\right)} dx \tag{30}$$

$$K_G = \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} = \int_0^\infty \frac{1}{x^2 + \left(\ln\left(1+\sqrt{2}\right)\right)^2} dx \tag{31}$$

$$K_G = \frac{\pi}{2\ln\left(1+\sqrt{2}\right)} = \int_0^\infty \int_0^\infty \left(\sqrt{\sqrt{2}-1}\right)^{x^2+y^2} dx dy \tag{32}$$

Entry 12.

$$\frac{1}{K_G} = \frac{2\ln\left(1+\sqrt{2}\right)}{\pi} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{\left(3\cdot 2^{4n} - 5\cdot 2^{2n} + 2\right) B_n}{n\left(2n\right)!}\left(\frac{\pi}{4}\right)^{2n-1} \tag{33}$$

where B_n are the Bernoulli numbers, $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, B_5 = \frac{5}{66}, \ldots$.

Entry 13.

$$\left(\sqrt[2\sqrt{2}]{1+\sqrt{2}}\right)^{K_G}=\sqrt[2\sqrt{2}]{\sqrt{2}-1}\cdot\frac{e}{\sqrt[5]{e}}\cdot\frac{\sqrt[9]{e}}{\sqrt[13]{e}}\cdot\frac{\sqrt[17]{e}}{\sqrt[21]{e}}\cdot\frac{\sqrt[25]{e}}{\sqrt[29]{e}}\cdots \tag{34}$$

where $e=\sum_{n=0}^{\infty}\frac{1}{n!}$.

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