The Barycenter of a 4-Gon

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In the first part we introduce the class of *vertex convexity polygons*, and we show that all 4-gons belong to this class. We define a point in some polygons, and in the second part we prove that in a 4-gon this is the center of gravity. We calculate this barycenter by a subdivision of the polygon into two triangles. Then we compute the barycenters of the triangles. The barycenter of the 4-gon results in the barycenter of the two barycenters of the triangles with taking into account the areas of the triangles.

Keywords and phrases: polygon; barycenter

MSC: 51

1 Introduction

We start with a set of points. Let us assume $k+1$ points called $Points \subset \mathbb{R}^2$, where $Points = \{(x_1, y_1),\ (x_2, y_2),\ \ldots (x_{k-1},\ y_{k-1}),\ (x_k, y_k),\ (x_{k+1}, y_{k+1})\}$. We joint the possible edges. We take the set $Union := \bigcup\ [(x_i, y_i), (x_{i+1}, y_{i+1})]$ for $i \in \{1, 2, \ldots k-1, k\}$. With the expression '$[a, b]$' we mean all points on the line segment between a and b and the boundaries a and b.
We call the interval $[(x_i, y_i), (x_{i+1}, y_{i+1})]$ an *edge* and each point (x_i, y_i) a *vertex* for $i \in \{1, 2, 3, \ldots k-1, k\}$.
We call $Union$ a *polygon* if and only if it holds $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$ where $i, j \leq k$. We call $Union$ a *simple polygon* if and only if it is a polygon and it is homeomorphic to a circle, and there are no three consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$. Also we demand that the points $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ and $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ are not collinear. If we have a simple polygon we include its interior, and it holds $(x_{k+1}, y_{k+1}) = (x_1, y_1)$ and $k > 2$.
We say that a *self-intersecting polygon* is a polygon which self-intersects. This means that there are two edges $[(x_i, y_i), (x_{i+1}, y_{i+1})]$ and $[(x_j, y_j), (x_{j+1}, y_{j+1})]$ where $i \neq j$, and the two edges have a common point. Further we demand that the polygon is not homeomorphic to a circle.
An *r-gon* is a simple polygon with r vertices.

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Proposition 1.1. A self-intersecting polygon is not a simple polygon.

Proof. Trivial. □

We introduce a property. We say that a polygon has the property *vertex convexity* if and only if it has a vertex (r, s) such that the intervals $[(r, s), (p, q)]$ are a subset of the polygon for all vertices (p, q). We call the class of those polygons which have the property of vertex convexity vertex convexity polygons. Note that a convex simple polygon has the property 'vertex convexity'. The 4-gon in Figure 3 is an example of a polygon with the property 'vertex convexity', which is not convex.
We call the class of all convex simple polygons convex simple polygons and the class of simple polygons simple polygons and the class of all polygons polygons.

Proposition 1.2.

$$\text{convex simple polygons} \subset \text{vertex convexity polygons} \subset \text{polygons}$$

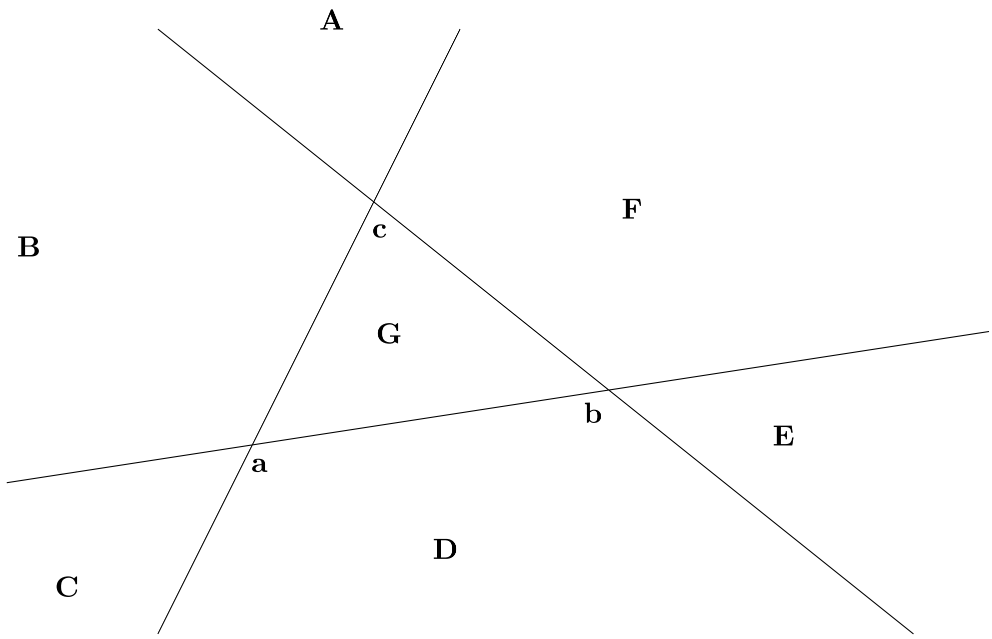
where the inclusions are proper.

Proof. The inclusions are trivial. To prove that they are proper consider the U-shaped 8-gon with the vertex set $\{(-3, -1), (3, -1), (3, 3), (2, 3), (2, 0), (-2, 0), (-2, 3), (-3, 3)\}$. It is a polygon, but no vertex convexity polygon. The 4-gon Q with a vertices $(0, 0), (2, 2), (0, 1), (-2, 2)$ is a vertex convexity polygon since the intervals $[(0, 0), (p, q)]$ are a subset of Q for all vertices (p, q) of Q, which is not convex. Note that the interior of Q is included. □

Proposition 1.3. Every polygon with 4 vertices either is a self-intersecting polygon or it has the property 'vertex convexity'.

Proof. Please see below Figure 1. We assume that the polygon P has vertices **a**, **b**, **c** and **d** (i.e. the edges connect **a**, **b**, **c**, **d** and then **a**). We see the lines that intersect **a**,**b** or **a**,**c** or **b**,**c**, respectively. They form three different lines since some vertices are not collinear. For the same reason **d** is not an element of one of the lines. The plane is subdiveded by the lines in seven sets **A**, **B**, **C**, **D**, **E**, **F**, **G**, where only **G** has a finite area. **G** is the triangle with vertices **a**, **b**, **c**. The fourth vertex **d** of P must be either in **A** or in **B** or in **C** or in **D** or in **E** or in **F** or in **G**. If **d** is in **A** or **B** or **C** or **E** or **G** P is a simple polygon. In these five cases the vertex convexity property is fulfiled, since the intervals $[(r, s), (p, q)]$ are subsets of P where (r, s) is **c** (if **d** is in **A**) or **d** (if **d** is in **B**) or **a** (if **d** is in **C**) or **b** (if **d** is in **E**) or **d** (if **d** is in **G**), and (p, q) is any vertex of P. If **d** is in **D** or in **F** we get a self-intersecting polygon. Note that in a simple polygon its interior is a part of the polygon.
The proposition is proved. □

Figure 1:



2 The Barycenter

We got the following well-known formulas for the *barycenter* $B = (B_x, B_y)$ of a simple polygon from [1] or [2]. Please see also [3] and [4]. Area is the area of a simple polygon. Note that $\mathsf{Area} > 0$, and that in [1] and [3] the barycenter is called a *Centroid.* Note that B is the center of gravity of the simple polygon, if it is realized with homogeneous material of constant thickness. Further note that the order of the vertices in the simple polygon is counterclockwise. We write

$$D_i = x_i \cdot y_{i+1} - x_{i+1} \cdot y_i, \text{ where } 1 \leq i \leq k \qquad \text{and it holds} \tag{2.1}$$

$$\mathsf{Area} = \frac{1}{2} \cdot \sum_{i=1}^{k} D_i \tag{2.2}$$

$$B_x = \frac{1}{6 \cdot \mathsf{Area}} \cdot \sum_{i=1}^{k} (x_i + x_{i+1}) \cdot D_i, \quad B_y = \frac{1}{6 \cdot \mathsf{Area}} \cdot \sum_{i=1}^{k} (y_i + y_{i+1}) \cdot D_i \tag{2.3}$$

3 Theorem

Let us consider either a convex simple polygon or a non-convex 4-gon which we call P. We assume that it has k vertices $(x_1, y_1), (x_2, y_2) \ldots (x_{k-1}, y_{k-1}), (x_k, y_k)$ and $(x_{k+1}, y_{k+1}) = (x_1, y_1)$ where $k > 2$. It has the area Area (see the chapter 'The Barycenter'). By Proposition 1.3 it has the property 'vertex convexity'. Without restrictions of generality let (x_1, y_1) be the vertex such that the intervals $[(x_1, y_1), (p, q)]$ are a subset of P for all vertices (p, q) of P. The polygon can be represented by $k-2$ triangles $T_2 \cup T_3 \cup \ldots \cup T_{k-2} \cup T_{k-1}$, where T_i is the triangle with vertices $(x_1, y_1), (x_i, y_i)$ and (x_{i+1}, y_{i+1}), for $2 \leq i \leq k-1$.

Definition 3.1. We use the abbreviation $A_i := x_1 \cdot y_i - y_1 \cdot x_i + x_i \cdot y_{i+1} - y_i \cdot x_{i+1} + x_{i+1} \cdot y_1 - y_{i+1} \cdot x_1$. A_i is twice the area of the triangle T_i, for $i = 2, 3, \ldots k-1$. We define the point $C = (C_x, C_y) \in \mathbb{R}^2$, where C_x is

$$\frac{1}{6 \cdot \mathsf{Area}} \cdot \sum_{i=2}^{k-1} (x_1 + x_i + x_{i+1}) \cdot A_i \tag{3.1}$$

and C_y has the value

$$\frac{1}{6\cdot\mathsf{Area}}\cdot\sum_{i=2}^{k-1}(y_1+y_i+y_{i+1})\cdot A_i. \tag{3.2}$$

Theorem 3.2. In a triangle or in a 4-gon it holds $C = B$.

Proof. First we show that $C_x = B_x$ holds in a triangle. We assume vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), where $(x_4, y_4) = (x_1, y_1)$. A_2 is twice the area of the triangle.
To show the equation $C_x = B_x$ for a triangle we have to prove

$$(x_1+x_2+x_3)\cdot A_2 \;=\; (x_1+x_2)\cdot D_1+(x_2+x_3)\cdot D_2+(x_3+x_4)\cdot D_3 \tag{3.3}$$

where

$$D_1 \;=\; x_1\cdot y_2-x_2\cdot y_1,\; D_2 \;=\; x_2\cdot y_3-x_3\cdot y_2,\; D_3 \;=\; x_3\cdot y_4-x_4\cdot y_3 \tag{3.4}$$

and

$$A_2 \;=\; D_1 \;+\; D_2 \;+\; D_3 \tag{3.5}$$

By using the commutativity of the multiplication the confirmation of equation (3.3) is straightforward.

Let us presume a 4-gon with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) and (x_5, y_5), where $(x_5, y_5) = (x_1, y_1)$. (Mostly we omit the multiplication point '$\cdot$'.) We calculate

$$\begin{aligned}
C_x &= (x_1+x_2+x_3)\cdot A_2+(x_1+x_3+x_4)\cdot A_3 && (3.6)\\
&= \;x_1x_1y_2-x_1y_1x_2+x_1x_2y_3-x_1y_2x_3+x_1x_3y_1-x_1y_3x_1 && (3.7)\\
&+\; x_2x_1y_2-x_2y_1x_2+x_2x_2y_3-x_2y_2x_3+x_2x_3y_1-x_2y_3x_1 && (3.8)\\
&+\; x_3x_1y_2-x_3y_1x_2+x_3x_2y_3-x_3y_2x_3+x_3x_3y_1-x_3y_3x_1 && (3.9)\\
&+\; x_1x_1y_3-x_1y_1x_3+x_1x_3y_4-x_1y_3x_4+x_1x_4y_1-x_1y_4x_1 && (3.10)\\
&+\; x_3x_1y_3-x_3y_1x_3+x_3x_3y_4-x_3y_3x_4+x_3x_4y_1-x_3y_4x_1 && (3.11)\\
&+\; x_4x_1y_3-x_4y_1x_3+x_4x_3y_4-x_4y_3x_4+x_4x_4y_1-x_4y_4x_1 && (3.12)\\
&= \;x_1x_1y_2-x_1y_1x_2 && (3.13)\\
&+\; x_2x_1y_2-x_2y_1x_2+x_2x_2y_3-x_2y_2x_3 && (3.14)\\
&\qquad\qquad +x_3x_2y_3-x_3y_2x_3 && (3.15)\\
&\qquad\qquad\qquad\qquad +x_1x_4y_1-x_1y_4x_1 && (3.16)\\
&\qquad\qquad +x_3x_3y_4-x_3y_3x_4 && (3.17)\\
&\qquad\qquad +x_4x_3y_4-x_4y_3x_4+x_4x_4y_1-x_4y_4x_1 && (3.18)
\end{aligned}$$

and

$$\begin{aligned}
B_x &= (x_1+x_2)\cdot D_1+(x_2+x_3)\cdot D_2+(x_3+x_4)\cdot D_3+(x_4+x_1)\cdot D_4 && (3.19)\\
&= x_1x_1y_2-x_1y_1x_2+x_2x_1y_2-x_2y_1x_2 && (3.20)\\
&+x_2x_2y_3-x_2y_2x_3+x_3x_2y_3-x_3y_2x_3 && (3.21)\\
&+x_3x_3y_4-x_3y_3x_4+x_4x_3y_4-x_4y_3x_4 && (3.22)\\
&+x_4x_4y_1-x_4y_4x_1+x_1x_4y_1-x_1y_4x_1 && (3.23)
\end{aligned}$$

We leave gaps where pairs erase itself due to different signs. $C_x = B_x$ is shown.
The identity $C_y = B_y$ is demonstrated in the same way, both for triangles and for 4-gons. The theorem is proven. □

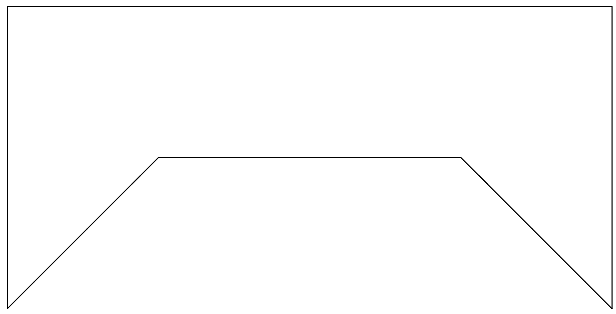
The cases for larger k can be treated in the same manner. We consider the case $k = 5$. We have to prove the equation

$$(x_1 + x_2 + x_3) \cdot A_2 + (x_1 + x_3 + x_4) \cdot A_3 + (x_1 + x_4 + x_5) \cdot A_4 \tag{3.24}$$
$$= (x_1 + x_2) \cdot D_1 + (x_2 + x_3) \cdot D_2 + (x_3 + x_4) \cdot D_3 + (x_4 + x_5) \cdot D_4 + (x_5 + x_1) \cdot D_5 \tag{3.25}$$

for a 5-gon with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) and $(x_6, y_6) = (x_1, y_1)$.
We will not continue the proof.

The following 6-gon shows that simple polygons is not a subclass of vertex convexity polygons.

Figure 2:



Conjecture 3.3. It holds

$$\textsf{vertex convexity polygons} \subset \textsf{simple polygons}$$

Conjecture 3.4. Every 5-gon has the property 'vertex convexity'.

Conjecture 3.5. In a 5-gon it holds $C = B$.

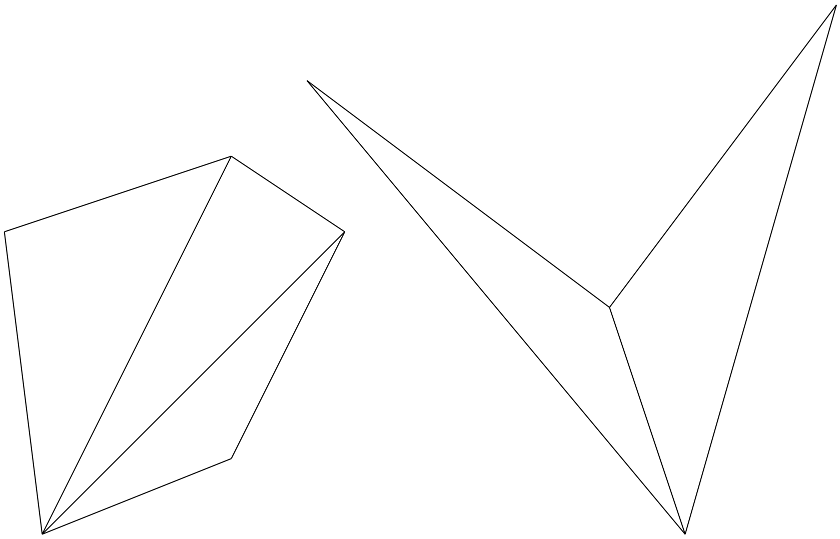
Conjecture 3.6. In all convex simple polygons it holds $C = B$.

Conjecture 3.7. In all simple polygons which have the property 'vertex convexity' where perhaps we have to modify the formulas (3.1) and (3.2) it holds $C = B$.

Conjecture 3.8. In all simple polygons where perhaps we have to modify (3.1) and (3.2) it holds $C = B$.

Conjecture 3.9. Every simple polygon can be formed by a finite number of triangles.

Figure 3:
On the right hand we see a 5-gon and a 4-gon, respectively. They are subdivided in three and two triangles, respectively.



References

[1] https://en.wikipedia.org/wiki/Centroid

[2] https://www.biancahoegel.de/geometrie/schwerpunkt_geometrie.html

[3] https://en.wikipedia.org/wiki/Polygon

[4] https://de.wikipedia.org/wiki/Baryzentrische_Koordinaten