Proof of 16 Formulas Barnes function

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Abstract

I have already published several months ago in the papers "Values of Barnes Function" and "Another Values of Barnes Function and Formulas" in total 16 conjectural formulas that I find with unusual methods.So, in this article, I write the proof of 16 formulas.

1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G\left(1+z\right) = \left(2\,\pi\right)^{\frac{z}{2}}\,\mathrm{e}^{-\frac{z\,(1+z)}{2}-\frac{\gamma\,z^2}{2}}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{k}\mathrm{e}^{-z+\frac{z^2}{2\,k}} \qquad (1)$$

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

$$G(1) = 1 \qquad (2)$$

$$G\left(1+z\right) = G\left(z\right)\Gamma\left(z\right) \qquad (3)$$

$$\log\left(G\left(1+z\right)\right) = \frac{z\log\left(2\,\pi\right)}{2} - \frac{z\left(1+z\right)}{2} + z\log\left(\Gamma\left(1+z\right)\right) - \int_0^z \log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t \qquad (4)$$

$$\int_0^z \log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t = \frac{z\log\left(2\,\pi\right)}{2} - \frac{z\left(1+z\right)}{2} + z\log\left(\Gamma\left(1+z\right)\right) - \log\left(G\left(z\right)\right) - \log\left(\Gamma\left(z\right)\right) \qquad (5)$$

3 Introduction

I need 5 relations:

$\zeta^{(1)}(-1,z) = \zeta^{(1)}(-1) - \log(G(z)) + (z-1)\log(\Gamma(z))$ (6) where

$\zeta^{(1)}(-1,z)$ is the first derivative of Hurwitz Zeta at z. (z is a positiv real)

The Adamchik-Miller's relation (7) for $\zeta^{(1)}\left(1-2n,\frac{h}{k}\right)$:

$$\frac{(\psi(2n)-\log(2\pi k))\,\mathrm{B}_{2n}(h/k)}{2n} - \frac{(\psi(2n)-\log(2\pi))\,\mathrm{B}_{2n}}{2n\,k^{2n}} + \frac{(-1)^{n+1}\,\pi}{(2\pi k)^{2n}}\sum_{r=1}^{k-1}\sin\left(\frac{2\pi rh}{k}\right)\Psi^{(2n-1)}\left(\frac{r}{k}\right)$$

$$+\frac{2\,(-1)^{n+1}\,(2n-1)!}{(2\pi k)^{2n}}\sum_{r=1}^{k-1}\cos\left(\frac{2\pi rh}{k}\right)\zeta^{(1)}\left(2n,\frac{r}{k}\right) + \frac{\zeta^{(1)}(1-2n)}{k^{2n}}$$

where

$\mathrm{B}_{2n}(h/k)$ is Bernoulli polynomial at h/k (8) .Here h and k both positiv integer.

B_{2n} is Bernoulli numbers. (9)

$\Psi^{(2n-1)}\left(\frac{r}{k}\right)$ is the polygamma function order 2n-1 at r/k.But here, in this study, n=1 and just we have the trigamma function. (10)

The Adamchik-Miller's relation is very complicated but I remark with this formula, I can make connection between two Barnes G-Function or if I use the relation (6) a connection between two expressions of first derivative of Hurwitz Zeta.Of course, we must choose two parameters a and b correctly.

So the principle is simple: just I evaluate closed form of $\zeta^{(1)}(-1,a)+\zeta^{(1)}(-1,b)$ or $\zeta^{(1)}(-1,a)-\zeta^{(1)}(-1,b)$ and in particular I can evaluate the complicated second sum.

The relation

$$\sum_{r=1}^{k-1}\zeta^{(1)}\left(s,\frac{r}{k}\right)$$

And we know that this sum $= \zeta^{(1)}(s)\,(k^s-1)+k^s\zeta(s)\log(k)$ (11)

Here, in this study, s=2.

Remember the value: $\zeta^{(1)}(2)$=

$$\frac{\pi^2\gamma}{6}+\frac{\pi^2\log(2)}{6}-\frac{\pi^2}{6}+2\pi^2\zeta^{(1)}(-1)+\frac{\pi^2\log(\pi)}{6}$$

The integral

$$\int_0^z \pi t cot(\pi t)\, dt$$

And we know that this integral $= z\log(2\pi)+\log\left(\frac{G(1-z)}{G(1+z)}\right)$ (12) integral originally due to Kinkelin.

4 About the log(G(1/5)), log(G(2/5)), log(G(3/5)) and log(G(4/5))

First case

I find log(G(4/5) easily: just I use relation (4) with z=-1/5 and we obtain log(G(4/5) in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\, dt\right)$$

Second case

Now I find log(G(1/5)) with the Kinkelin's integral: the software MAPLE give me a closed form in terms of complex dilogarithm and I can simplify the expression in terms of trigamma function and we have

$$\int_0^{\frac{1}{5}} \pi t cot(\pi t)\, dt$$

equals to

$$\frac{\log(2)}{10}+\frac{\log(5)}{20}-\frac{\log\left(\sqrt{5}+1\right)}{10}+\frac{\left(5+\sqrt{5}\right)\Psi^{(1)}\left(\frac{1}{5}\right)+2\sqrt{5}\,\Psi^{(1)}\left(\frac{2}{5}\right)-4\pi^2\left(\sqrt{5}+1\right)}{50\pi\sqrt{10+2\sqrt{5}}}$$

And I use the relation (12) with z=1/5 and I obtain log(G(1/5)) in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\, dt\right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

Third case

Now I search log(G(2/5)), I evaluate the closed form of $\zeta^{(1)}\left(-1, \frac{1}{5}\right)+$ $\zeta^{(1)}\left(-1, \frac{2}{5}\right)$

And I use 2 times the Adamchik-Miller's relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{5-1}\cos\left(\frac{2\pi r*1}{5}\right)\zeta^{(1)}\left(2n, \frac{r}{5}\right)+\sum_{r=1}^{5-1}\cos\left(\frac{2\pi r*2}{5}\right)\zeta^{(1)}\left(2n, \frac{r}{5}\right)$

We have

$\cos\left(\frac{2\pi}{5}\right)\zeta^{(1)}\left(2, \frac{1}{5}\right)-\cos\left(\frac{\pi}{5}\right)\zeta^{(1)}\left(2, \frac{2}{5}\right)-\cos\left(\frac{\pi}{5}\right)\zeta^{(1)}\left(2, \frac{3}{5}\right)+\cos\left(\frac{2\pi}{5}\right)\zeta^{(1)}\left(2, \frac{4}{5}\right)-$ $\cos\left(\frac{\pi}{5}\right)\zeta^{(1)}\left(2, \frac{1}{5}\right)+\cos\left(\frac{2\pi}{5}\right)\zeta^{(1)}\left(2, \frac{2}{5}\right)+\cos\left(\frac{2\pi}{5}\right)\zeta^{(1)}\left(2, \frac{3}{5}\right)-\cos\left(\frac{\pi}{5}\right)\zeta^{(1)}\left(2, \frac{4}{5}\right)$

I can simplify

$-\frac{\zeta^{(1)}\left(2, \frac{1}{5}\right)}{2}-\frac{\zeta^{(1)}\left(2, \frac{2}{5}\right)}{2}-\frac{\zeta^{(1)}\left(2, \frac{3}{5}\right)}{2}-\frac{\zeta^{(1)}\left(2, \frac{4}{5}\right)}{2}$

And now I use the relation (11) and finally

$-12\zeta^{(1)}(2)-\frac{25\pi^2\log(5)}{12}$

So I can finish the calcul with the trigamma function's rules and I have the closed form of $\zeta^{(1)}\left(-1, \frac{1}{5}\right)+\zeta^{(1)}\left(-1, \frac{2}{5}\right)$

We obtain

$$-\frac{2\zeta^{(1)}(-1)}{5}-\frac{\log(5)}{120}+\frac{\left(5+3\sqrt{5}\right)\Psi^{(1)}\left(\frac{1}{5}\right)+\left(-5+\sqrt{5}\right)\Psi^{(1)}\left(\frac{2}{5}\right)-\pi^2\left(6\sqrt{5}+2\right)}{100\sqrt{10+2\sqrt{5}}\,\pi}$$

Hence, with the relation (6), I find log(G(2/5)) in terms of

$$\left(\int_0^{-\frac{1}{5}}\log(\Gamma(t+1))\,dt\right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

Fourth case

Now I search log(G(3/5)): I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}\left(-1, \frac{2}{5}\right)-\zeta^{(1)}\left(-1, \frac{3}{5}\right)$

Finally I have

$$\frac{\left(-\sqrt{5}-5\right)\Psi^{(1)}\left(\frac{2}{5}\right)+2\sqrt{5}\,\Psi^{(1)}\left(\frac{1}{5}\right)-2\pi^2\left(\sqrt{5}-1\right)}{50\sqrt{10+2\sqrt{5}}\,\pi}$$

So I have log(G(3/5)) in terms of

$$\left(\int_0^{-\frac{1}{5}}\log(\Gamma(t+1))\,dt\right),\Psi^{(1)}\left(\frac{1}{5}\right),\Psi^{(1)}\left(\frac{2}{5}\right)$$

5 About the log(G(1/8)), log(G(3/8)), log(G(5/8)) and log(G(7/8))

First case

I find log(G(7/8) easily: just I use relation (4) with z=-1/8 and we obtain log(G(7/8) in terms of

$$\left(\int_0^{-\frac{1}{8}}\log(\Gamma(t+1))\,dt\right)$$

Second case

Now I find log(G(1/8)) with the Kinkelin's integral: the software MAPLE give me a closed form in terms of complex dilogarithm and I can simplify the expression in terms of trigamma function and we have

$$\int_0^{\frac{1}{8}}\pi t cot(\pi t)\,dt$$

equals to

$$-\frac{\sqrt{2}\,K}{4\pi}+\frac{K}{8\pi}-\frac{\pi}{32}-\frac{\pi\sqrt{2}}{32}+\frac{\sqrt{2}\,\Psi^{(1)}\left(\frac{1}{8}\right)}{64\pi}+\frac{\log(2)}{32}-\frac{\log\left(1+\sqrt{2}\right)}{16}$$

where K is the Catalan's constant. (13)

And I use the relation (12) with z=1/8 and I obtain log(G(1/8)) in terms of

$$\left(\int_0^{-\frac{1}{8}}\log(\Gamma(t+1))\,dt\right),\Psi^{(1)}\left(\frac{1}{8}\right)$$

Third case

Now I search log(G(3/8)), I evaluate the closed form of $\zeta^{(1)}\left(-1,\frac{1}{8}\right)+\zeta^{(1)}\left(-1,\frac{3}{8}\right)$

And I use 2 times the Adamchik-Miller's relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{8-1}\cos\left(\frac{2\pi r*1}{8}\right)\zeta^{(1)}\left(2n,\frac{r}{8}\right)+\sum_{r=1}^{8-1}\cos\left(\frac{2\pi r*3}{8}\right)\zeta^{(1)}\left(2n,\frac{r}{8}\right)$

We have

$-2\zeta^{(1)}\left(2,\frac{1}{2}\right)$

I find the value of $\zeta^{(1)}\left(2,\frac{1}{2}\right)$ with the relation (11) with k=2 and I have $3\zeta^{(1)}(2)+\frac{2\pi^2\log(2)}{3}$

Finally I have $-6\zeta^{(1)}(2)-\frac{4\pi^2\log(2)}{3}$

So I can finish the calcul with the trigamma function's rules and I have the closed form of $\zeta^{(1)}\left(-1,\frac{1}{8}\right)+\zeta^{(1)}\left(-1,\frac{3}{8}\right)$

We obtain

$$-\frac{\zeta^{(1)}(-1)}{16}+\frac{\log(2)}{192}+\frac{\sqrt{2}\,\Psi^{(1)}\left(\frac{1}{8}\right)}{64\pi}-\frac{\pi\sqrt{2}}{32}-\frac{\sqrt{2}\,K}{4\pi}-\frac{\pi}{32}$$

Hence, with the relation (6), I find log(G(3/8)) in terms of

$$\left(\int_0^{-\frac{1}{8}}\log(\Gamma(t+1))\,dt\right)$$

Fourth case

Now I search log(G(5/8)): I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}\left(-1,\frac{3}{8}\right)-\zeta^{(1)}\left(-1,\frac{5}{8}\right)$

Finally I have

$$-\frac{K}{8\pi}-\frac{\pi\sqrt{2}}{32}-\frac{\pi}{32}-\frac{\sqrt{2}\,K}{4\pi}+\frac{\sqrt{2}\,\Psi^{(1)}\left(\frac{1}{8}\right)}{64\pi}$$

So I have log(G(5/8)) in terms of

$$\left(\int_0^{-\frac{1}{8}}\log(\Gamma(t+1))\,dt\right),\Psi^{(1)}\left(\frac{1}{8}\right)$$

6 About the log(G(1/10)), log(G(3/10)), log(G(7/10)) and log(G(9/10))

It's easy: the duplication formula (14) is well-known:

$$G(2z)$$

is equals to

$$e^{-\frac{1}{4}} A^3 2^{2z^2-3z+\frac{11}{12}} \pi^{\frac{1}{2}-z} G(z)\, G\left(z+\frac{1}{2}\right)^2 G(1+z)$$

where A is the Glaisher-Kinkelin constant's (15)

If z=3/5, I have directly log(G(1/10)) in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\, dt\right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

If z=4/5, I have directly log(G(3/10)) in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\, dt\right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

If z=1/5, I have directly log(G(7/10)) in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\, dt\right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

If z=2/5, I have directly log(G(9/10)) in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\, dt\right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

7 About the log(G(1/12)), log(G(5/12)), log(G(7/12)) and log(G(11/12))

First case

I find log(G(11/12) easily: just I use relation (4) with z=-1/12 and we obtain log(G(11/12) in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1))\, dt\right)$$

Second case

Now I find log(G(1/12)) with the Kinkelin's integral: the software MAPLE give me a closed form in terms of complex dilogarithm and I can simplify the expression in terms of trigamma function and we have

$$\int_0^{\frac{1}{12}} \pi t cot(\pi t)\, dt$$

equals to

$$\frac{\sqrt{3}\,\Psi^{(1)}\left(\frac{1}{3}\right)}{48\pi} - \frac{\pi\sqrt{3}}{72} + \frac{K}{3\pi} - \frac{\log\left(1+\sqrt{3}\right)}{12} + \frac{\log(2)}{24}$$

And I use the relation (12) with z=1/12 and I obtain log(G(1/12)) in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1))\, dt\right), \Psi^{(1)}\left(\frac{1}{12}\right)$$

If you prefer, we can use the trigamma identity $\Psi^{(1)}\left(\frac{1}{12}\right)$=$10\Psi^{(1)}\left(\frac{1}{3}\right)+ 2\pi^2\sqrt{3} - \frac{8\pi^2}{3} + 40K$

Third case

Now I search log(G(5/12)), I evaluate the closed form of $\zeta^{(1)}\left(-1, \frac{1}{12}\right)+ \zeta^{(1)}\left(-1, \frac{5}{12}\right)$

And I use 2 times the Adamchik-Miller's relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{12-1} \cos\left(\frac{2\pi r*1}{12}\right) \zeta^{(1)}\left(2n, \frac{r}{12}\right)$+$\sum_{r=1}^{12-1} \cos\left(\frac{2\pi r*5}{12}\right) \zeta^{(1)}\left(2n, \frac{r}{12}\right)$

We have

$\zeta^{(1)}\left(2, \frac{1}{6}\right) - \zeta^{(1)}\left(2, \frac{1}{3}\right) - 2\zeta^{(1)}\left(2, \frac{1}{2}\right) - \zeta^{(1)}\left(2, \frac{2}{3}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right)$

I have successively $\sum_{r=1}^{6-1} \zeta^{(1)}\left(2n, \frac{r}{6}\right)$=$6\pi^2\log(2)$+$6\pi^2\log(3)$+$35\zeta^{(1)}(2)$

is equals to $\zeta^{(1)}\left(2, \frac{1}{6}\right)$+$\zeta^{(1)}\left(2, \frac{1}{3}\right)$+$\zeta^{(1)}\left(2, \frac{1}{2}\right)$+$\zeta^{(1)}\left(2, \frac{2}{3}\right)$+$\zeta^{(1)}\left(2, \frac{5}{6}\right)$

And $\zeta^{(1)}\left(2, \frac{1}{6}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right)$=$35\zeta^{(1)}(2) + 6\pi^2\log(6) - \zeta^{(1)}\left(2, \frac{1}{3}\right) - \zeta^{(1)}\left(2, \frac{1}{2}\right) - \zeta^{(1)}\left(2, \frac{2}{3}\right)$

And $\sum_{r=1}^{3-1} \zeta^{(1)}\left(2n, \frac{r}{3}\right)$=$8\zeta^{(1)}(2) + \frac{3\pi^2\log(3)}{2}$

is equals to $\zeta^{(1)}\left(2,\frac{1}{3}\right)+\zeta^{(1)}\left(2,\frac{2}{3}\right)$

I have $\zeta^{(1)}\left(2,\frac{1}{6}\right)+\zeta^{(1)}\left(2,\frac{5}{6}\right)=\frac{16\pi^2\log(2)}{3}+\frac{9\pi^2\log(3)}{2}+24\zeta^{(1)}(2)$

Finally I have $4\pi^2\log(2)+3\pi^2\log(3)+10\zeta^{(1)}(2)$

So I can finish the calcul with the trigamma function's rules and I have the closed form of $\zeta^{(1)}\left(-1,\frac{1}{12}\right)+\zeta^{(1)}\left(-1,\frac{5}{12}\right)$

We obtain

$$\frac{\zeta^{(1)}(-1)}{12}+\frac{K}{3\pi}+\frac{\log(3)}{288}$$

Hence, with the relation (6), I find log(G(5/12)) in terms of

$$\left(\int_0^{-\frac{1}{12}}\log(\Gamma(t+1))\,dt\right),\Psi^{(1)}\left(\frac{1}{12}\right)$$

Fourth case

Now I search log(G(7/12)): I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}\left(-1,\frac{5}{12}\right)-\zeta^{(1)}\left(-1,\frac{7}{12}\right)$

Finally I have

$$-\frac{\sqrt{3}\,\Psi^{(1)}\left(\frac{1}{3}\right)}{48\pi}+\frac{\pi\sqrt{3}}{72}+\frac{K}{3\pi}$$

So I have log(G(7/12)) in terms of

$$\left(\int_0^{-\frac{1}{12}}\log(\Gamma(t+1))\,dt\right)$$

Conclusion:

The 16 formulas Barnes G-function are proved.

I remark that this paper is complementary with my 2 papers "Values of Barnes function" and "Another values of Barnes function and formulas":

In this article, I prove the 16 formulas but I have no information about the closed form of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1))\, dt\right)$$

or

$$\left(\int_0^{-\frac{1}{8}} \log(\Gamma(t+1))\, dt\right)$$

or

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1))\, dt\right)$$

In the 2 papers "Values of Barnes function" and "Another values of Barnes function and formulas", I don't prove the formulas but in the same time, I have more information about the integral log gamma but just I can evaluate some terms, hence it isn't sufficient to obtain a final closed form of the integrals.

The priority is to find closed form of three integrals and the trigamma identity $\Psi^{(1)}\left(\frac{2}{5}\right)$ in terms of $\Psi^{(1)}\left(\frac{1}{5}\right)$.

8 References

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