Bernoulli sums of powers, Euler–Maclaurin formula and proof that Riemann Hypothesis is true

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Abstract

On 1859, the german mathematician Georg Friedrich Bernhard Riemann made one of his most famous publications "On the Number of Prime Numbers less than a Given Quantity" when he was developing his explicit formula to give an exact number of primes less than a given number x, in which he conjectured that "all non-trivial zeros of the zeta function have a real part equal to $\frac{1}{2}$". Riemann was sure of his statement, but he could not prove it, remaining as one of the most important hypotheses unproven for 163 years.

In this paper, we have to prove that the Riemann Hypothesis is true, based on the Bernoulli power sum, the Euler–Maclaurin formula and its relation with the Riemann Zeta function.

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1 Introduction

1.1 Euler product and the sum of inverse powers.

The infinite sum of inverse powers is a series of great interest for mathematics in number theory. Leonhard Euler [3] managed to relate this series to an infinite product that goes through all the prime numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \frac{P_n^s}{P_n^s - 1} \tag{1}$$

Where P_n is the n-th prime number and $s \in \mathbf{C}$.

This series is convergent for values of $Re(s) > 1$, however it is divergent for values of $Re(s) \leq 1$.

Euler was able to find a closed formula for even powers, $2k$ when $k \in \mathbb{N}$:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!} \tag{2}$$

Where B_{2k} are Bernoulli numbers; $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, etc.

1.2 The Euler-Riemann zeta function $\zeta(s)$.

Riemann introduced the function $\zeta(s)$ [4], making it equal to the series of the sum of the inverse of s-th power inverses in the convergence range $Re(s) > 1$, and $s \in \mathbb{C}$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3}$$

And it manages to give continuity to the function, in the range of the complex plane, where the series diverges through the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{4}$$

Where Γ is the gamma function.

If $Re(s) < 0$, then $\zeta(s)$ can be calculated with the functional equation using of the value of the convergence of the series of the inverse of the powers $\sum_{n=1}^{\infty} \frac{1}{n^{1-s}} = \zeta(1-s)$, so for example for $s = -1$:

Example 1

$$\zeta(-1) = 2^{-1}\pi^{-1-1} sin\left(\frac{\pi(-1)}{2}\right) \Gamma(1-(-1))\zeta(1-(-1))$$
$$\zeta(-1) = 2^{-1}\pi^{-2} sin\left(\frac{\pi(-1)}{2}\right) \Gamma(2))\zeta(2)$$
$$\zeta(-1) = 2^{-1}\pi^{-2}(-1)(1)\frac{\pi^2}{6}$$
$$\zeta(-1) = -\frac{1}{12}$$

From the functional equation, we deduce that for even negative values of s the function $\zeta(s) = 0$, at these *"zeros"* Riemann called *"Trivial zeros"*. There also exist values of s that lie within the range $0 < Re(s) < 1$ that makes the function $\zeta(s) = 0$, these values of s are called *"Nontrivial zeros"* of the function $\zeta(s)$ and which Riemann conjectured all lie on the straight line $Re(s) = \frac{1}{2}$.

The conjecture cannot be proved with the Riemann functional equation alone, because the function is redundant for the so-called critical range: $0 < Re(s) < 1$, for example:

Example 2

$$\zeta(0.1) = 2^{0.1}\pi^{-0.9}sin\left(\frac{\pi * 0.1}{2}\right)\Gamma(0.9)\zeta(0.9)$$

y

$$\zeta(0.9) = 2^{0.9}\pi^{-0.1}sin\left(\frac{\pi * 0.9}{2}\right)\Gamma(0.1)\zeta(0.1)$$

Neither $\zeta(0.1)$, nor $\zeta(0.9)$ can be solved. .

To calculate the values of $\zeta(s)$ in the critical range $0 < Re(s) < 1$, must to be used numerical methods that calculate approximate values of $\zeta(s)$, which do not prove the hypothesis despite the fact that all computationally obtained non-trivial zeros have the value of $Re(s) = \frac{1}{2}$.

2 Bernoulli numbers and the sum of k-th power.

In mathematics, the Bernoulli numbers B_k is a set of successive rational numbers with relevant importance in number theory. They appear in *Combinatorics*, in the expansion of the tangent functions and the hyperbolic tangent by Taylor series. As we have already seen, Euler obtained a closed formula for $\zeta(k)$ when k is a positive even number. If we replace Euler's formula in the Riemann functional equation, we obtain another closed formula for negative integer values of k:

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \tag{5}$$

Where $k \in \mathbb{N}$

They are called Bernoulli numbers because Abraham de Moivre named them that way, in honor of Jakob Bernoulli, the first mathematician who studied them. There are several ways to obtain the values of B_k, but they were obtained for the first time by Jakob Bernoulli, using series of sum of k-th power. In general one can obtain the sum of k-th power $S_k(n)$, as a function of: $S_{k-1}(n)$, $S_{k-2}(n)$, $S_{k-3}(n)$,...$S_0(n)$ where $k \in \mathbb{N}$:

$$\sum_{m=1}^{n} m^k = S_k(n) = \frac{1}{k+1}\left[(n+1)^{k+1} - 1 - \sum_{m=0}^{k-1}\binom{k+1}{m}S_m(n)\right] \tag{6}$$

In a posthumous publication by Jakob Bernoulli [1], we can find a listing of the sums of powers up to $k = 10$:

$$\sum_{m=1}^{n} m^k = S_k(n)$$
$$\sum_{m=1}^{n} m^0 = S_0(n) = n$$
$$\sum_{m=1}^{n} m^1 = S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$
$$\sum_{m=1}^{n} m^2 = S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$
$$\sum_{m=1}^{n} m^3 = S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$
$$\sum_{m=1}^{n} m^4 = S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$
$$\sum_{m=1}^{n} m^5 = S_5(n) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$\sum_{m=1}^{n} m^6 = S_6(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$
$$\sum_{m=1}^{n} m^7 = S_7(n) = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$
$$\sum_{m=1}^{n} m^8 = S_8(n) = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$
$$\sum_{m=1}^{n} m^9 = S_9(n) = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2$$
$$\sum_{m=1}^{n} m^{10} = S_{10}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$
$$\sum_{m=1}^{n} m^{11} = S_{11}(n) = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$$

2.1 Obtaining the Bernoulli numbers by the sum of powers.

It was known that to obtain the Bernoulli numbers it was necessary to derive $S_k(n)$ and evaluate it at zero, however this concept is not entirely correct since for B_1 when applying this concept it is not possible to obtain the value of $B_1 = -1/2$, value obtained by other methods. The correct way to obtain the Bernoulli numbers by the sum of k-th power is with the equation:

$$B_k = (-1)^k S'_k(0) \tag{7}$$

The equation (7) *will be proofed later on*

For example, to obtain B_2:

Example 3

$$B_2 = (-1)^2 S'_2(0)$$
$$B_2 = (1)\left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right]'_{n=0}$$
$$B_2 = \left[n^2 + n + \frac{1}{6}\right]_{n=0}$$
$$B_2 = \frac{1}{6}$$

Similarly, from the function $S_k(n)$ all Bernoulli numbers are obtained: $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, B_9 = 0, B_{10} = 5/66, B_{11} = 0 \cdots$.

It is noted that:

$$B_k = 0 \ : \ k = 2m+1, m \in \mathbb{N}$$

2.2 Simplified formula to find $S_k(n)$.

Another way to write the formula for the sum of k-th power is as follows:

$$S_k(n) = \sum_{p=1}^{1+k} A_p(k) n^p \tag{8}$$

Where

$$A_p(k) = \frac{(-1)^{1+k-p}}{1+k}\binom{1+k}{p} B_{1+k-p} \tag{9}$$

And B_k is obtained by:

$$B_k = -\frac{1}{1+k}\sum_{m=0}^{k-1}\binom{1+k}{m} B_m \tag{10}$$

Equation (8) can be rewritten:

Factorizing $1+k$ and developing the summation and binomial coefficient of $S_k(n)$:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^k(k+1)!}{1!k!}B_k n + \frac{(-1)^{k-1}(k+1)!}{2!(k-1)!}B_{k-1}n^2 + \frac{(-1)^{k-2}(k+1)!}{3!(k-2)!}B_{k-2}n^3 + \right.$$
$$\left. \frac{(-1)^{k-3}(k+1)!}{4!(k-3)!}B_{k-3}n^4 + \cdots + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^0(k+1)!}{(k+1)!(0)!}B_0 n^{k+1}\right]$$

Rearranging terms and accommodating the factorials in order to simplify:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^2(k+1)!}{(k-1)!2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1 k!(k+1)}{k!1!}B_1 n^k + \frac{(-1)^2(k-1)!k(k+1)}{(k-1)!2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0}{0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)}{1!}B_1 n^k + \frac{(-1)^2 k(k+1)}{2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

Rewriting as a summation of a product of factors:

$$S_k(n) = \sum_{p=1}^{1+k} \frac{(-1)^{p-1}B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1}(2+k-m)n^{2+k-p} \tag{11}$$

where:

$$C_p(k) = \frac{(-1)^{p-1}B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1}(2+k-m) \tag{12}$$

Note 1 *The product* $\prod_{m=1}^{p-1}(2+k-m) = 1 : p = 1$

It can also be expressed as the sum of higher order derivatives:

$$S_k(n) = \int_0^n m^k dm + \sum_{p=1}^{k} \frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right) \tag{13}$$

Or:

$$S_{-k}(n) = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{k} \frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{m^k}\right) \tag{14}$$

With the equation (11) it is possible to prove equation (7)

Proof of equation (7):

Deriving the equation (11) and evaluating at zero we obtain:

$$S'_k(0) = \sum_{p=1}^{1+k}(2+k-p)\frac{(-1)^{p-1}B_{p-1}}{(1+k)(p-1)!}\prod_{m=1}^{p-1}(2+k-m)(0)^{1+k-p}$$

Where the only term different from zero is when $p = k + 1$, so the expression reduces to:

$$S'_k(0) = \frac{(-1)^k (k+1)!}{(1+k)k!} B_k$$

$$S'_k(0) = \frac{(-1)^k (k+1)!}{(k+1)!} B_k$$

$$S'_k(0) = (-1)^k B_k$$

Reordering:

$$B_k = (-1)^k S'_k(0)$$

□.

3 The Euler–Maclaurin formula and the Zeta function $\zeta(k)$.

Given the known Euler-Maclaurin equation [2], for the summation of a given function $f(m), \mathbb{R} \to \mathbb{R}$, and it is q times derivable:

$$\sum_{m=a+1}^{b} f(m) = \int_a^b f(x)dx + \sum_{p=1}^{q} \frac{(-1)^p B_p}{p!} \left[f^{p-1}(b) - f^{p-1}(a) \right] + R_q \tag{15}$$

Where R_q is the residual error

If we define the limits of the sum with $a = 0$ and $b = n$, and let $f(m) = \frac{1}{m^k}$ be infinitely derivable, one obtains::

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} \left[f^{p-1}(n) - f^{p-1}(0) \right] + R_q$$

Where $K \in \mathbb{R}$ Then:

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} \left[f^{p-1}(n) \right] + R_q$$

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left[\frac{1}{n^k} \right] + R_q \tag{16}$$

Let's observe and compare equation (16) with equation (14), it is verified that they are the same equation, but with different upper limit of the summation.
Now let us clear the term R_q as function of k and let's $n \to \infty$:

$$R_q(k) = \lim_{n \to \infty} \left[\sum_{m=1}^{n} \frac{1}{m^k} - \int_0^n \frac{1}{m^k} dm - \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^k} \right) \right] \tag{17}$$

Extending the second series of the equation:

$$R_q(k) = \lim_{n \to \infty} \left[\sum_{m=1}^{n} \frac{1}{m^k} - \left(\frac{1}{1-k} n^{1-k} + \frac{(-1)^1 B_1}{1!} n^{-k} + \frac{(-1)^2 B_2}{2!} (-k) n^{-1-k} + \frac{(-1)^3 B_3}{3!} (-k)(-1-k) n^{-2-k} + \cdots \right) \right] \tag{18}$$

If $k > 1$, and applying the limit, we get:

$$R_q(k) = \left[\sum_{m=1}^{\infty} \frac{1}{m^k} - \left(\frac{1}{1-k} 0 + \frac{(-1)^1 B_1}{1!} 0 + \frac{(-1)^2 B_2}{2!} (-k) 0 + \frac{(-1)^3 B_3}{3!} (-k)(-1-k) 0 + \cdots \right) \right]$$

$$R_q(k) = \sum_{m=1}^{\infty} \frac{1}{m^k} \iff k > 1 \tag{19}$$

Therefore it is concluded that: when $k > 1$ the value of $R_q(k)$ converges, and is equal to the sum of k-th power inverses, and is equal to $\zeta(k)$:

$$R_q(k) = \zeta(k) \iff k > 1 \tag{20}$$

Now let $k = 1$ and apply the limit in equation (18):

$$R_q(1) = \left[\sum_{m=1}^{\infty} \frac{1}{m^1} - \left(\frac{1}{1-1} + \frac{(-1)^1 B_1}{1!}0 + \frac{(-1)^2 B_2}{2!}(-1)0 + \frac{(-1)^3 B_3}{3!}(-1)(-1-1)0 + \cdots\right)\right]$$

$$R_q(1) = \infty - \frac{1}{0}$$

$$R_q(1) = undetermined$$

Therefore it is concluded that: when $k = 1$ the value of $R_q(1)$ is undetermined, and is equal to $\zeta(1)$:

$$R_q(k) = \zeta(1) \iff k = 1 \tag{21}$$

Now let $k > 1$ in equation (17) but for $f(m) = m^k$, and this time let's separate the sum of the derivatives into two parts, the first sum from $p = 1$ to $p = k$, and the second sum from $p = k + 1$ to infinity:

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \int_0^n m^k dm - \sum_{p=1}^{q=k} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right) - \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right] \tag{22}$$

It is observed that the integral with the first sum of derivatives is equal to the sum of inverse powers according to equation (14):

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \sum_{m=1}^{n} m^k - \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right]$$

$$R_q(-k) = \lim_{n\to\infty} \left[- \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right]$$

Extending the series of the equation:

$$R_q(-k) = \lim_{n\to\infty} \left[-\left(\frac{(-1)^{k+1} B_{k+1}}{(k+1)!}(k!)n^0 + \frac{(-1)^{k+2} B_{k+2}}{(k+2)!}(k)(k-1)\cdots n^{-1} + \frac{(-1)^{k+3} B_{k+3}}{(k+3)!}(k)(k-1)\cdots n^{-2} + \cdots\right)\right]$$

Applying the limit:

$$R_q(-k) = -\frac{(-1)^{k+1} B_{k+1}}{(k+1)!}(k!)$$

$$R_q(-k) = \frac{(-1)^k B_{k+1}}{k+1}$$

We conclude that: Equation converges when $k < 1$, and this equation is equivalent to the well-known formula for the function $\zeta(-k)$ equation (5), and also satisfies when $k = 0$:

$$R_q(k) = \zeta(k) \iff k < 1 \;\; and \;\; \exists\; B_{k+1} \;:\; k \in \mathbb{R} \tag{23}$$

Equation (16) is defined for a function $\mathbb{R} \to \mathbb{R}$, but when n tends to infinity, it is verified that the residual term $R_q = \zeta(k)$ when $k \in \mathbb{R}$, also $R_q = \zeta(k)$ when $k \in \mathbb{C}$ and $Re(k) > 1$. Finally, by the principle of analytic extension we conclude that $R_q = \zeta(k)$ for all $k \in \mathbb{C}$.

Example 4 *Now let's calculate a example, when $k \in \mathbb{C}$*

Let $k = 0.4 + 7i$:

We will tabulate the results and compare with $\zeta(0.4+7i)$:

n	$\sum_{n=1}^{n} n^{-0.4-7i}$	*Sum of derivatives*	R_q
10	$0.57811497 + 0.019131365i$	$-0.441562086 - 0.398323033i$	$1.019677056 + 0.417454399i$
100	$2.847975959 + 1.750845166i$	$1.82847357 + 1.333500916i$	$1.019502389 + 0.417344249i$
1000	$36.34167952 - 5.123159998i$	$35.32217705 - 5.540504261i$	$1.01950247 + 0.4173344263i$

If we calculate the value of $\zeta(0.4+7i)$ by some numerical method we obtain the approximate value of $1.01950247 + 0.417344263i$, and comparing with the value of R_q we verify that $R_q = \zeta(0.4+7i)$.

We are now ready to write the following Theorem:

Theorem 1 *Let $k \in \mathbb{C}$, from the Euler-Maclaurin formula, for the summation of a complex function, but with real variable $f(m) = \frac{1}{m^k}$, when the summation is infinite, the residual error $R_q = \zeta(k)$:*

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left[\frac{1}{n^k}\right] + R_q$$

$$\zeta_(k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^k} - \int_0^n \frac{1}{m^k} dm - \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^k}\right) \right] \tag{24}$$

4 Proof of the Riemann Hypothesis.

Let s be a complex number such that $s = a + bi$ where $a, b \in \mathbb{R}$, $\zeta(s) = 0$.

From the Riemann functional equation:

$$\zeta(a+bi) = 2^{a+bi} \pi^{a+bi-1} \sin\left(\frac{\pi(a+bi)}{2}\right) \Gamma(1-a-bi) \zeta(1-a-bi)$$

It is known that for the critical band range $0 < a < 1$, the terms:

$$2^{a+bi} \pi^{a+bi-1} \sin\left(\frac{\pi(a+bi)}{2}\right) \Gamma(1-a-bi) \neq 0$$

Therefore, including its conjugates, it must comply:

$$\zeta(a+bi) = \zeta(1-a-bi) = \zeta(a-bi) = \zeta(1-a+bi) = 0 \tag{25}$$

On the other hand, we can write the equation of zeta (31) within the critical band, when $0 < a < 1$

$$\zeta_(a+bi) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^{a+bi}} - \int_0^n \frac{1}{m^{a+bi}} dm - \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^{a+bi}}\right) \right]$$

$$\zeta(a+bi) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^{a+bi}} - \frac{1}{(1-a-bi)} n^{1-a-bi} - \frac{(-1)^1 B_1}{(1-a-bi)(1)!} n^{-a-bi} - \frac{(-1)^2 B_2}{(1-a-bi)(2)!} n^{-1-a-bi} \dots \right]$$

As $0 < a < 1$ and $n \to \infty$ the equation simplifies to:

$$\zeta(a+bi) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^{a+bi}} - \frac{1}{(1-a-bi)} n^{1-a-bi} \right] \tag{26}$$

Applying properties of complex numbers and bringing to polar form:

$$\zeta(a+bi) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^{-a} e^{-ib\ln m} - \frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}} e^{i\arctan\left(\frac{b}{a-1}\right)} e^{-iblnn} \right]$$

As $\zeta(a+bi) = 0$ and then applying euler identities for sines and cosines:

$$0=\lim_{n\to\infty}\left[\sum_{m=1}^{n} m^{-a}e^{-ib\ln m}-\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}e^{i\left[\arctan\left(\frac{b}{a-1}\right)-blnn\right]}\right]$$

$$\sum_{m=1}^{\infty} m^{-a}e^{-ib\ln m}=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\left[\cos\left(\arctan\left(\frac{b}{a-1}\right)-blnn\right)+i\sin\left(\arctan\left(\frac{b}{a-1}\right)-blnn\right)\right]\right] \tag{27}$$

Where the modulus and argument of $S^*_{-a-bi}(n)$ are:

$$\left\|S^*_{-a-bi}(n)\right\|=\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}} \tag{28}$$

$$\arg(S^*_{-a-bi})=\arctan\left(\frac{b}{a-1}\right)-blnn \tag{29}$$

Then by Euler's identity and equating the modules, we have:

$$\sqrt{\left[\sum_{m=1}^{\infty} m^{-a}\cos\left(-b\ln m\right)\right]^2+\left[\sum_{m=1}^{\infty} m^{-a}\sin\left(-b\ln m\right)\right]^2}=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\right] \tag{30}$$

Extending the series within the square root we obtain:

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty} m^{-a}\cos\left(-b\ln m\right)\right]^2+\left[\sum_{m=1}^{\infty} m^{-a}\sin\left(-b\ln m\right)\right]^2=\\
&\lim_{n\to\infty}\left[1^{-2a}\cos^2\left(-b\ln 1\right)+2^{-2a}\cos^2\left(-b\ln 2\right)+3^{-2a}\cos^2\left(-b\ln 3\right)+\cdots+n^{-2a}\cos^2\left(-b\ln n\right)\right.\\
&+2\{1^{-a}2^{-a}\cos\left(-b\ln 1\right)\cos\left(-b\ln 2\right)+1^{-a}3^{-a}\cos\left(-b\ln 1\right)\cos\left(-b\ln 3\right)+\cdots+1^{-a}n^{-a}\cos\left(-b\ln 1\right)\cos\left(-b\ln n\right)\}\\
&+2\{2^{-a}3^{-a}\cos\left(-b\ln 2\right)\cos\left(-b\ln 3\right)+2^{-a}4^{-a}\cos\left(-b\ln 2\right)\cos\left(-b\ln 4\right)+\cdots+2^{-a}n^{-a}\cos\left(-b\ln 2\right)\cos\left(-b\ln n\right)\}\\
&+2\{3^{-a}4^{-a}\cos\left(-b\ln 3\right)\cos\left(-b\ln 4\right)+3^{-a}5^{-a}\cos\left(-b\ln 3\right)\cos\left(-b\ln 5\right)+\cdots+3^{-a}n^{-a}\cos\left(-b\ln 3\right)\cos\left(-b\ln n\right)\}\\
&+\cdots+\cdots+2\{(n-1)^{-a}n^{-a}\cos\left(-b\ln\left(n-1\right)\right)\cos\left(-b\ln n\right)\}\\
&+1^{-2a}\sin^2\left(-b\ln 1\right)+2^{-2a}\sin^2\left(-b\ln 2\right)+3^{-2a}\sin^2\left(-b\ln 3\right)+\cdots+n^{-2a}\sin^2\left(-b\ln n\right)\\
&+2\{1^{-a}2^{-a}\sin\left(-b\ln 1\right)\sin\left(-b\ln 2\right)+1^{-a}3^{-a}\sin\left(-b\ln 1\right)\sin\left(-b\ln 3\right)+\cdots+1^{-a}n^{-a}\sin\left(-b\ln 1\right)\sin\left(-b\ln n\right)\}\\
&+2\{2^{-a}3^{-a}\sin\left(-b\ln 2\right)\sin\left(-b\ln 3\right)+2^{-a}4^{-a}\sin\left(-b\ln 2\right)\sin\left(-b\ln 4\right)+\cdots+2^{-a}n^{-a}\sin\left(-b\ln 2\right)\sin\left(-b\ln n\right)\}\\
&+2\{3^{-a}4^{-a}\sin\left(-b\ln 3\right)\sin\left(-b\ln 4\right)+3^{-a}5^{-a}\sin\left(-b\ln 3\right)\sin\left(-b\ln 5\right)+\cdots+3^{-a}n^{-a}\sin\left(-b\ln 3\right)\sin\left(-b\ln n\right)\}\\
&\left.+\cdots+\cdots+2\{(n-1)^{-a}n^{-a}\sin\left(-b\ln\left(n-1\right)\right)\sin\left(-b\ln n\right)\}\right]
\end{aligned}$$

Applying trigonometric idenities:

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty} m^{-a}\cos\left(-b\ln m\right)\right]^2+\left[\sum_{m=1}^{\infty} m^{-a}\sin\left(-b\ln m\right)\right]^2=\lim_{n\to\infty}\left[1^{-2a}+2^{-2a}+3^{-2a}+\cdots+n^{-2a}\right.\\
&+2\{1^{-a}2^{-a}\cos\left(b\ln 1-b\ln 2\right)+1^{-a}3^{-a}\cos\left(b\ln 1-b\ln 3\right)+\cdots+1^{-a}n^{-a}\cos\left(b\ln 1-b\ln n\right)\}\\
&+2\{2^{-a}3^{-a}\cos\left(b\ln 2-b\ln 3\right)+2^{-a}4^{-a}\cos\left(b\ln 2-b\ln 4\right)+\cdots+2^{-a}n^{-a}\cos\left(b\ln 2-b\ln n\right)\}\\
&+2\{3^{-a}4^{-a}\cos\left(b\ln 3-b\ln 4\right)+3^{-a}5^{-a}\cos\left(b\ln 3-b\ln 5\right)+\cdots+3^{-a}n^{-a}\cos\left(b\ln 3-b\ln n\right)\}\\
&\left.+\cdots+\cdots+2\{(n-1)^{-a}n^{-a}\cos\left(b\ln\left(n-1\right)-b\ln n\right)\}\right]
\end{aligned}$$

Rewriting the series, we obtain:

$$\left[\sum_{m=1}^{\infty} m^{-a}\cos\left(-b\ln m\right)\right]^2+\left[\sum_{m=1}^{\infty} m^{-a}\sin\left(-b\ln m\right)\right]^2=\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{2a}}+2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(b\ln\left(\frac{m+1}{t}\right)\right)\right] \tag{31}$$

Replacing (31) in (30) we get:

$$\lim_{n\to\infty}\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}}+2\sum_{m=1}^{n-1}\sum_{t=1}^{m}\left[(m+1)^{-a}t^{-a}\cos\left(b\ln\left(\frac{m+1}{t}\right)\right)\right]}=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\right] \tag{32}$$

Squaring the equation:

$$\lim_{n\to\infty}\sum_{m=1}^{n}\frac{1}{m^{2a}}+2\sum_{m=1}^{n-1}\sum_{t=1}^{m}\left[(m+1)^{-a}t^{-a}\cos\left(b\ln\left(\frac{m+1}{t}\right)\right)\right]=\lim_{n\to\infty}\left[\frac{n^{2-2a}}{(1-a)^2+b^2}\right] \tag{33}$$

From equation (33) it can be observed that the increment of the n-th term of the first summation is less by one degree than the nth increment of the function on the right-hand side of the equation:

$$\frac{1}{n^{2a}}<\frac{(2-2a)n^{1-2a}}{(1-a)^2+b^2}:0<a<1 \tag{34}$$

$$-2a<1-2a:0<a<1 \tag{35}$$

Therefore, the n-th term of the double summation of equation (33) must be of the same degree as the function on the right-hand side. But it is also the only function with a trigonometric part, which is why it can oscillate, so it will be necessary to analyze the trigonometric part. By the integral criterion:

$$\sum_{t\to 1}^{n-1}\cos\left[b\ln\left(\frac{t}{n}\right)\right]\approx\int_{1}^{n}\cos\left[b\ln\left(\frac{t}{n}\right)\right]dt\approx\frac{n}{b^2+1}-\frac{b\sin(b\ln n)+\cos(b\ln n))}{b^2+1} \tag{36}$$

From equation (36) we deduce that the increment of the n-th term is a trigonometric function of constant amplitude and positive slope.

Now let us analyze the increment of the n-th term in the double summation, and skip for a moment the trigonometric part, and verify its convergence:

$$\lim_{n\to\infty}n^{-a}\sum_{t=1}^{n-1}t^{-a}=0*\infty:0<a<1 \tag{37}$$

It is necessary to break the indeterminacy of the equation (37):

$$\lim_{n\to\infty}\sum_{t=1}^{n-1}n^{-a}t^{-a}=\lim_{n\to\infty}\sum_{t=1}^{n-1}\frac{1}{(nt)^a}=\frac{1}{(n1)^a}+\frac{1}{(n2)^a}+\frac{1}{(n3)^a}+\cdots \tag{38}$$

As $n>t$, then $(nt)^a>t^{2a}$

If $a>\frac{1}{2}$ then:

$$\lim_{n\to\infty}\sum_{t=1}^{n-1}\frac{1}{t^{2a}}\quad \textit{is convergent}\quad\Longrightarrow\quad\lim_{n\to\infty}\sum_{t=1}^{n-1}\frac{1}{(nt)^a}\quad\textit{is convergent} \tag{39}$$

As the summation converges, equation (37) can be written:

$$\lim_{n\to\infty}n^{-a}\sum_{t=1}^{n-1}t^{-a}=0*C_n=0:\frac{1}{2}<a<1 \tag{40}$$

Since the double summation of equation (33) is a trigonometric function and is oscillating, the increment of the n-th term will converge more rapidly to zero for $a>\frac{1}{2}$, the double summation will converge and be oscillating.

Therefore:
$\exists\ a:\ a\leq\frac{1}{2}$, and $\exists\ b$ so that the double summation function from (33), is divergent, and it is not oscillating.

From equation (25) must also be fulfilled:

$\exists\ a:\ (1-a) \leq \frac{1}{2}$, and $\exists\ b$ so that the double summation function from (33), is divergent, and it is not oscillating.

$$\left.\begin{array}{r} a \leq \frac{1}{2} \\ 1-a \leq \frac{1}{2} \end{array}\right\} \tag{41}$$

Whose solution is:

$$a = \frac{1}{2} \tag{42}$$

Finally, we can affirm that:

$$\exists\ b : \zeta(a+bi) = 0 \Longrightarrow\ a = \frac{1}{2} \tag{43}$$

$\square$.

5 Conclusion

Based on the above, we can state the following theorem:

Theorem 2 *The nontrivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to 1/2.*

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Dedication.

"Call to me and I will answer you and tell you great and unsearchable things you do not know"
(Jeremiah 33:3)

I thank God for hearing my prayers, and showing me the way to the resolution of this problem. to him be the glory.

This work is dedicated to all my family who supported me at all times and in the most difficult moments of my life, especially my beloved wife Araceli, my two beautiful children Ocram and Arelys, and my parents who never lost faith in me.