On Theoretical Contingency of Quantum Mechanics

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Abstract

According to our current understanding of quantum mechanics, a 'measurement' violates unitarity. In other words as the act of measurement 'resets' the probabilities described by the Schrödinger equation, in the very 'moment' of the collapse of the wavefunction, conservation of probability does not hold. To make progress in our understanding of the measurement problem it is thus expected that one must encompass in a single equation both otherwise mutually-exclusive processes of measurement and unitary evolution. In this light, turning to the continuity equation, we realize the possibility that if we allow for existence of sources and sinks in the 'fluid of probability' we get closer to such a theory by arriving at nonlinear generalizations of Schrödinger and Klein-Gordon equations. The generalized equations derived are then shown to satisfy all conditions that are expected from a proper generalization: simplification to their linear counterparts by a well-defined dynamical condition, and Galilean and Lorentz invariance.

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1 Introduction

1.1 A brief history

In the foundations of modern physics there is nothing more controversial than the *measurement problem* of quantum mechanics[1]: How should one reconcile the fact that on one hand, the wave function evolves deterministically according to the Schrödinger equation as a linear superposition of different states, while on the other hand, the outcome of measurements are always a single definite state?
This problem is a mere reappearance of the old issue of *wave-particle duality*, for as long as the wavefunction evolves one is dealing with a wave, while as soon as a measurement is performed one observes a single particle, a single dot on the screen for example. The reconciliation of the wave and particle pictures therefore is a crucial problem whose resolution is necessary for making any progress regarding the measurement problem. The pursuit of such reconciliation is at least as old as the quantum theory itself[1]. It began with Mie's ideas[2] and was pursued seriously by Einstein after his successful explanation of the photoelectric effect[3]. Although in a letter to Besso[4] Einstein admitted that '*[...] All the fifty years of conscious ruminations have not gotten me closer to an anwser for the question: "What are light quanta?" These days any rascal may believe that he knows, but he deludes himself.*', his idea of *particles being concentrations (lumps) of continuous fields*[3, 5] is still the best concrete idea we have for our attempts along such lines. This idea was a persistent theme of Einstein's attempts and was pursued through his grand programme of *Unified Field Theory*[5].
In fact Schrödinger himself initially thought in terms of *wavepackets*: a particle was assumed to be a 'parcel' of matterwaves (an *envelope*) that moved together as a whole by a group velocity equal to the velocity of the particle meanwhile the inner waves (in the parcel) oscillated by a frequency equal to the phase velocity of de Broglie's matterwaves. But this view soon faced a serious problem[6]: wavepackets which were supposed to be particles did not maintain their integrity due to linearity and *dispersion*[2]. This objection and the consequent 'victory' of the idealist founders like Bohr was serious enough to discourage even de Broglie

[1]We use 'quantum *theory*' to include also the so-called *old quantum theory*.
[2]*Non*linearity and dispersion however solves this problem by allowing the existence of *soliton* solutions. This is one of our achievements in this paper.

for some twenty years [7]. de Broglie's interest in the problem was revitalised by Bohm's theory[8]; the theory that is now called *de Broglie-Bohm* or *Pilot wave* theory, according to which the particle and wave pictures are both maintained simultaneously and the wave 'guides' the particle[9] by the *guiding equation*

$$\mathbf{p} = \nabla S$$

where S is the phase of the wavefunction (classically *Hamilton's principal function*)[10].
de Broglie himself however considered pilot wave theory a 'degenerate' form of his early attempts[7]; such attempts evolved to what de Broglie in his late years dubbed as *the theory of the double solution*, according to which, particles were to be described by a *nonlinear equation* which has the Schrödinger equation as its approximation, hence the return of the Einsteinian theme. But neither Einstein nor de Broglie were able to derive any such equation satisfactorily. In this paper we derive an equation that

- Is a generalization of Schrödinger equation: it simplifies to Schrödinger equation by a certain mathematical condition, hence fulfilling Einstein-de Broglie's maxim.
- Unlike the so-called Nonlinear Schrödinger equation[11] and the Ghirardi-Rimini-Weber (GRW) theory[12], it contains no new arbitrary parameters.
- Is *nonlinear and dispersive*, hence allows for soliton solutions[13].
- Satisfies Weinberg's *homogeneity property*[14] according to which if ψ is a solution then so must be $Z\psi$ for arbitrary complex number Z; in other words pure states must be represented by *rays* in the Hilbert space, not vectors.

Our focus in this paper is on the physical and conceptual aspects and we do not attempt at solving the proposed equation, neither do we enter techincal discussions regarding the existence and uniqueness of solutions. As the purpose of this paper is to scrutinise and revise the very foundations of quantum theory we hope that the mathematically-inclined reader indulges us patiently whenever a heuristic approach is pursued in lieu of abiding by the established formalism.

1.2 Motivation

According to the current understanding, Schrödinger equation governs the evolution of the wavefunction *between* measurements and measurements 'reset' the probabilities. If we compare this statement with the issue of wave-particle duality, we realise that observation of particles violates unitarity and conservation of probability

$$\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left(|\psi|^2 \frac{\nabla S}{m} \right) = 0, \tag{1}$$

in other words we maintain that it is the conservation of probability that obstructs the application of Schrödinger equation to the collapse process. Hence any reconciliation of wave and particle pictures must go beyond the law of conservation of probability. It now temporarily serves us to recall Penrose's description of quantum mechanics[1] in terms of **U**-process[3] and **R**-process[4]. According to Penrose the *complete* evolution of the quantum state must look like[5]

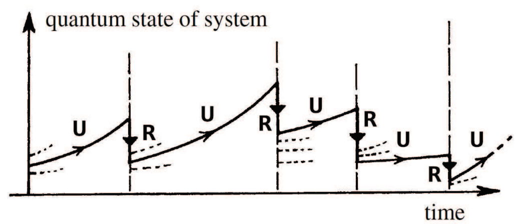

Figure 1: *'The way that the quantum-theoretic world appears to behave, with stretches of deterministic **U**-evolution, punctuated by moments of probabilistic **R**-action, each of which restores some element of classicality'*[1].

If we recall the fluid picture of Madelung[15] and hence think of the above graph as the motion of a quantum fluid of probability (hereafter *Madelung fluid*), we can presume that *the **R**-processes act as the sinks or sources of the Madelung fluid.* But the current (orthodox) quantum mechanics crucially misses this point because the fluid that it describes is one *without* sources and sinks[16, 17]. This condition in non-relativistic quantum mechanics is mathematically expressed by

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{p} = \nabla \cdot \mathbf{k} = 0. \tag{2}$$

The conclusion is that a complete picture of reality in which a serious attempt is made for resolution of the wave-particle duality problem must *allow for the possibility of a 'fluid' of probability in which sources and sinks do exist.* It is an interesting fact that to the best of our knowledge relaxation of this significant restriction has not yet been considered.
Methodologically we know it is difficult to begin with the condition (2) and arrive at a nonlinear equation. We must instead reverse the process and by starting from a firm physical motivation, find a nonlinear equation which simplifies to the Schrödinger equation by imposing (2).
Notwithstanding a key observation can be made immediately from (2), that the current scheme of quantum mechanics cannot handle condition (2) and in fact yields a contradiction: as

$$\hat{\mathbf{p}}(\psi) = -i\hbar\nabla\psi$$

condition (2) for a test wavefunction ψ reads

$$\nabla \cdot \hat{\mathbf{p}}(\psi) = -i\hbar\nabla^2\psi = 0 \tag{3}$$

which is an absurdity since the derivation of (1) assumes $\nabla^2\psi \neq 0$ beforehand.

[3]Unitary evolution.

[4]Reduction (collapse of the wavefunction).

[5]The graph has a mere explanatory role and we do not consider it to accurately describe reality, for in our view there is not any such *discontinuous* **R**-process.

2 Harmonisation

It is the received wisdom that Planck-Einstein-de Broglie law[6] $p^\nu = \hbar k^\nu$ belongs to the era of 'old quantum mechanics' and that in the realm of quantum mechanics the right (and more fundamental) perspective is to solve the Schrödinger equation for any case at hand. Although from an instrumentalist point of view this perspective has been quite successful, in this paper we advocate another perspective which will prove to be more fruitful with regard to the foundational questions of quantum mechanics. Our perspective is that *quantum mechanics is basically 'all' about* $p^\nu = \hbar k^\nu$. To adopt such perspective we need to first scrutinise our understanding of its essential ingredient k^ν. By any rigorous mathematical definition[7] it is required that a wave be defined as a field[8] on spacetime which satisfies a certain equation, *without any **explicit** reference to its four-wavevector.* On the other hand, according to our perspective $p^\nu = \hbar k^\nu$ is a fundamental law of nature and appearance of k^ν in such a law suggests that we must enforce all waves to acquire a mathematically well-defined four-wavevector. Therefore we must find a definition for the four-wavevector of a wave ψ in terms of the ψ *itself*; we shall call this technique *harmonisation.* Considering the simplest case of a (complex) scalar harmonic wave[9,10],

$$\psi = Ae^{-ik_\mu x^\mu},$$

if we apply the gradient operator to both sides we have,

$$\partial_\mu \psi = -ik_\mu \psi = -i\psi k_\mu \tag{4}$$

we realise that there are *three* possibilites for harmonisation:

1. Linear-operatorial approach, in which $\hat{k}_\mu \in End\big(L^2(\mathbb{R})\big)$[11] i.e. four-wavevector is a linear operator (endomorphism) on the space of square-integrable functions, such that

$$\psi \xmapsto{\hat{k}_\mu} i\partial_\mu \psi$$

[6] The metric signature $(+,-,-,-)$ is used everywhere in this paper. Greek indices run over 0 to 3 (four dimensions).

[7] For example, an entity ϕ which satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \nabla^2 \phi,$$

where v is the speed of propagation of the wave. Or, an entity ϕ which satisfies the Schrödinger equation.

[8] We only consider scalar fields in this paper. Sufficient conditions of smoothness are also assumed implicitly.

[9] Notice that in these definitions only forward-in-time waves are considered. It is not clear whether this preference of time direction affects the theory in a decisive manner. Woit has elaborated on this issue in the context of QFT; see *Appendix A* to [18].

[10] Technically plane waves are not physically legitimate as they are not normalisable (square-integrable), but this issue can be rigorously avoided; see[19]. We do not involve in such technical details hereafter in this paper.

[11] Again, technically quantum-mechanical operators are not defined on the *entire* $L^2(\mathbb{R})$ but only on a dense subspace. As we have already said, we do not involve in such technical details in this paper.

i.e.

$$\hat{k}_\mu(\psi) := i\partial_\mu\psi$$

plus the *eigenvalue hypothesis*

$$\hat{k}_\mu(\psi) = k_\mu\psi = \psi k_\mu.$$

An immediate elaboration is needed here. Equation (4) is all that we have to begin with. It is valid only as it stands, and as it stands k_μ is a vector, *not an operator* on the L^2 space of wavefunctions. It is not impermissible to promote k_μ to an operator but to be logically consistent we cannot change the fundamental equation (4) that we have started with. Therefore if we choose to promote k_μ to an operator –as we do in this approach– we must keep (4) as it stands, and that is why we need the additional eigenvalue hypothesis. What we just stated is not anything controversial; it is one of the *axioms*[19] of quantum mechanics, only in light of a different perspective.

2. *Nonlinear-operatorial*[12] approach, in which $\check{k}_\mu : L^2(\mathbb{R})\backslash\{0\} \to L^2(\mathbb{R})$ such that

$$\psi \xmapsto{\check{k}_\mu} i\frac{\partial_\mu\psi}{\psi}$$

i.e.

$$\check{k}_\mu(\psi) := i\frac{\partial_\mu\psi}{\psi} \neq \big(k_\mu\psi = \psi k_\mu\big)$$

without the need to add the eigenvalue hypothesis; because with the eigenvalue hypothesis in this case we would have

$$i\frac{\partial_\mu\psi}{\psi} = k_\mu\psi$$

$$\Rightarrow \partial_\mu\psi = -k_\mu\psi^2$$

which is incompatible with (4).

3. *Neoclassical* nonlinear approach, in which $\psi \in L^2(\mathbb{R})\backslash\{0\}$ and $k_\mu : \mathbb{R}^4 \to \mathbb{C}$ such that

$$x^\nu \xmapsto{k_\mu} i\frac{\partial_\mu\psi(x^\nu)}{\psi(x^\nu)}$$

or

$$k_\mu(x^\nu) := i\frac{\partial_\mu\psi(x^\nu)}{\psi(x^\nu)} \neq \big(k_\mu\psi = \psi k_\mu\big)$$

again **without** the need to add the eigenvalue hypothesis, for the same reason stated above. As we shall see later the eigenvalue hypothesis is automatically included as a special case in this neoclassical approach in the sense that it leads to a generalization of Schrödinger and Klein-Gordon eigenvalue problems (equations). The assumption $\psi \in L^2(\mathbb{R})\backslash\{0\}$ is made only to make connection with the Born Principle, as our

[12]We do not restrict our notion of 'operator' here to one which has the same domain and codomain.

starting motivation requires. But after this (neoclassical) theory is developed it will become conceivable that the structure (Hilbert Space) that comes with this assumption might not be necessary in the neoclassical appraoch; especially the necessity of the inner product structure is hard to see given that in this approach –among other things– we do not need to define the notion of *self-adjointness* (which requires an inner product to be defined).

The second and third approach might seem identical but there are crucial differences. Notably in the nonlinear-operatorial approach –like linear-operatorial– ψ and $\partial_\mu \psi$ are not considered 'independent'; once ψ is given one just puts it into the input of $\check{k}_\mu$ to get momentum. This is basically the reason that in the derivation of Schrödinger equation from the variational principle[20], one varies with respect to ψ (and ψ^*) *but not* $\partial_\mu \psi$. In the neoclassical approach however, ψ and $\partial_\mu \psi$ are considered 'independent'. Indeed this is the reason for calling it *neoclassical* because it reminds one of the way position and momentum are treated in Hamiltonian mechanics.
The linear-operatorial approach is familiar and well-studied, being the foundation for orthodox quantum mechanics. The nonlinear-operatorial approach can handle the condition (2), but *in its current form* is too special to yield a wave equation as it is now defined for a scalar field only. Hence the nonlinear-operatorial approach requires further consideration and a generalization of the definition to be applicable to a vector field; we postpone full investigation of the nonlinear-operatorial approach however to another paper and in this paper focus on the neoclassical approach as it can both handle the condition (2) and yield a straightforward generalization of Schrödinger and Klein-Gordon equations without the need to generalise the harmonisation technique to vector fields.
To the best of our knowledge our approach[13] is new: it is the first time that

$$p_\mu = i\hbar \frac{\partial_\mu \psi}{\psi} \tag{5}$$

is proposed as *the definition* of quantum-mechanical momentum. Although de Broglie-Bohm theory comes quite close to our approach, it misses the point by dispensing with the imaginary part of the momentum in order to make the same predictions as the Schrödinger picture. To be precise, if we write the wavefunction in polar form

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t) e^{iS(\mathbf{x},t)/\hbar}$$

where $R, S : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, we see that according to our proposed definition of quantum-mechanical momentum (5) we have

$$\mathbf{v} = \frac{\nabla S}{m} - \frac{i\hbar}{m} \frac{\nabla R}{R}, \tag{6}$$

while in de Broglie-Bohm theory only the first term is considered[8, 9, 21]; in our view there is no a priori theoretical reason why one should do so, on the contrary we will soon argue

[13] When we use 'our approach' we mean both the neoclassical and nonlinear-operatorial approaches. When a result or discussion is limited to the neoclassical approach and not true for the nonlinear-operatorial approach we specify it.

that neglecting the second term comes from a metaphysical[14] position. *By neglecting this second term one is losing some of the 'information' encoded in the wavefunction.* This can be better seen if we think of S as the Hamilton's principal function in classical mechanics (Hamilton-Jacobi theory[10]). We know that quantum mechanics must tell us more about reality than what classical mechanics does, however by leaving something unchanged (the guiding equation) that *already* exists in classical mechanics we cannot hope to fully achieve this expectation. Inclusion of the imaginary part is an important point of departure for our theory compared to de Broglie-Bohm theory, whose guiding equation is a special case of our definition when the second term

$$\frac{i\hbar}{m}\frac{\nabla R}{R}$$

is neglected. Orthodox quantum mechanics is not different with regard to this limitation because by its conservation of probability (1) it implicitly assumes $\mathbf{v} = \nabla S/m$. The basic reason that de Broglie-Bohm and orthodox quantum theory neglect the second term is that being bound by the eigenvalue hypothesis, they think of eigenvalues as what is actually observed in measurements, hence they require the eigenvalues to be real numbers, which is mathematically expressed by self-adjointness. We think that this is the last remnant of 'classical thinking' in quantum mechanics that presupposes only real numbers 'exist'. It is true that in measurements one only observes real numbers but that can well be a limitation of our understanding: that we cannot observe imaginary numbers is not a reason they cannot exist. This is why we stated earlier that the neglection of the imaginary part of the guiding equation stems from a metaphysical position[15]. By relaxing this restriction, our approach yields novel insights which are obscured by narrowing physical entities to linear self-adjoint operators.

Even if we ignore this important point, on the conceptual side de Broglie-Bohm theory never promotes (5) to *the definition* of momentum in quantum mechanics as the theory only *augments* Schrödinger's theory with the so-claimed *additional* guiding equation.

Two aspects of imperfection of mathematics of quantum theory as it stands currently are therefore shown here. These imperfections provide a reason to think that definition (5) by utilising all the information encoded in the wavefunction, and by successfully handling condition (2) as we shall see, is the 'right' definition[16] for quantum-mechanical momentum by virtue of its generality.

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[14]By *meta*physics we mean *beyond* physics.

[15]In fact our proposal of including the second term in (6) is quite aligned with the work of Renou *et al.*[22] who are discussing the possibility of empirically testing the 'reality' of complex numbers.

[16]Logically there is no right-wrong (true-false) binary system for definitions: a definition is only as good and general as the theory that is built upon it. Therefore a 'right definition' must be defined to be one which is more theoretically *beneficial*.

3 Neoclassical Nonlinear theory

3.1 Generalization of Schrödinger equation

We apply the Planck-Einstein-de Broglie law $p^\mu = \hbar k^\mu$ to our proposed definition (5) to get

$$\mathbf{p} = -i\hbar\frac{\nabla\psi}{\psi} \quad \text{and} \quad E = \frac{i\hbar}{\psi}\frac{\partial\psi}{\partial t}. \tag{7}$$

Similar to the familiar derivation of the Schrödinger equation from the law of conservation of energy, we expect to get the nonlinear equation for evolution of ψ by substituting (7) in the law of conservation of energy

$$E = \frac{p^2}{2m} + V;$$

we must be cautious however, because just as in the case with derivation of Schrödinger equation, the square in the law of conservation of energy can be the source of a subtle confusion: if we think of quantum-mechanical $\mathbf{p}$ as a complex vector in $\mathbb{C}^3$,

$$p^2\psi \stackrel{?}{=} \langle\mathbf{p},\mathbf{p}\rangle\psi = (\mathbf{p}\cdot\mathbf{p}^*)\psi = \hbar^2\nabla^2\psi$$

where dot is the Euclidean inner product[17], we fail to achieve the result we want ($p^2\psi = -\hbar^2\nabla^2\psi$) hence the Schrödinger equation. Therefore this is not the right way to think about quantum-mechanical $\mathbf{p}$, as we can get the intended minus only if we define

$$p^2\psi := \langle\mathbf{p},\mathbf{p}\rangle\psi := (\mathbf{p}\cdot\mathbf{p})\psi = -\hbar^2\nabla^2\psi.$$

As the latter definition is the correct definition by yielding Schrödinger equation, any generalization must conform to it. Consequently for our definition (7) we have

$$p^2 = \langle\mathbf{p},\mathbf{p}\rangle = \mathbf{p}\cdot\mathbf{p} = -\frac{\hbar^2}{\psi^2}\nabla\psi\cdot\nabla\psi =: -\frac{\hbar^2}{\psi^2}\|\nabla\psi\|^2.$$

Substitution in the law of conservation of energy now yields

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\|\nabla\psi\|^2}{\psi} + V\psi} \tag{8}$$

which can also be written as

$$\frac{i\hbar}{2}\frac{\partial\psi^2}{\partial t} = -\frac{\hbar^2}{2m}\|\nabla\psi\|^2 + V\psi^2. \tag{9}$$

Our equation (8) differs from the Schrödinger equation only by the term

$$\frac{\|\nabla\psi\|^2}{\psi},$$

[17] There is a third possibility that

$$p^2\psi \stackrel{?}{=} \langle\mathbf{p},\mathbf{p}\psi\rangle = \mathbf{p}\cdot(\mathbf{p}\psi)^* = \hbar^2\nabla^2\psi^*$$

which is of little interest.

which we now show only in the *special case* that $\mathbf{k}$ is a solenoidal field, is equal to the corresponding term in Schrödinger equation. Consider condition (2)

$$\nabla \cdot \mathbf{k} = 0, \tag{10}$$

which, by our definition (5) is

$$\nabla \cdot (\frac{\nabla \psi}{\psi}) = \frac{\psi \nabla^2 \psi - \|\nabla \psi\|^2}{\psi^2} = 0,$$

therefore for a quantum fluid *with* sources or sinks one has

$$\frac{\|\nabla \psi\|^2}{\psi} = \nabla^2 \psi. \tag{11}$$

In other words, Schrödinger equation is a special case of the nonlinear equation derived in this paper. Condition (11) reveals an important but *hidden assumption* of the current theory of quantum mechanics about wavefunctions; yet it is not unexpected at all, for it is equivalent –if we use the right definition of momentum– to the condition (2) which was already a well-known fact. What obscured this equation so far to be explicitly stated is the incomplete definition of momentum in orthodox quantum mechanics, which, as we remarked in the introduction, cannot handle condition satisfactorily.

We can now explicitly see how linearity of Schrödinger equation arises from nonlinearity of (8), and how an eigenvalue problem which is a marker of quantum *discreteness* and 'quantum jumps' is only a special case to a nonlinear *but continuous* reality. In this light the superposition 'principle' is but a special-case feature of nature and has a limited domain of applicability.

The observation in which *an eigenvalue problem arises from a more general* ***nonlinear*** *equation* is quite a generic one and worthy of emphasis. As a simple example consider how *Helmholtz equation*

$$\nabla^2 \phi = -k^2 \phi$$

can be an approximation to the following nonlinear equation

$$\nabla \cdot (\frac{\nabla \phi}{\phi}) = -k^2,$$

for $\phi \neq 0$, since

$$\nabla \cdot (\frac{\nabla \phi}{\phi}) = \frac{1}{\phi} \nabla^2 \phi - \frac{|\nabla \phi|^2}{\phi^2} = -k^2$$

Multiplying both sides by ϕ yields

$$\nabla^2 \phi - \frac{|\nabla \phi|^2}{\phi} = -k^2 \phi$$

If we apply the *approximation*

$$\frac{|\nabla \phi|^2}{\phi} \approx 0$$

we are led to the original Helmholtz equation.

3.1.1 Galilean invariance

Equation (8) is Galilean-invariant in the same sense that Schrödinger equation itself is[20]: Assuming the gauge dependence of the wavefunction

$$\psi'(\mathbf{x}',t') = e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\psi(\mathbf{x},t),$$

and the Galilean transformation of gradient and time derivatives

$$\nabla = \nabla'$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla'$$

As

$$\frac{\partial\psi'}{\partial t'} = \frac{\partial\psi'}{\partial t} + \mathbf{v}\cdot\nabla'\psi' = \frac{i}{2\hbar}mv^2\psi' + e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla\psi'$$

$$= \frac{i}{2\hbar}mv^2\psi' + e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\left(-\frac{im}{\hbar}\mathbf{v}\psi' + e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\cdot\nabla\psi\right)$$

$$= e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\left(-\frac{i}{2\hbar}mv^2\psi + \frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla\psi\right),$$

under this transformation the left-hand-side of our equation is

$$i\hbar\frac{\partial\psi'}{\partial t'} = e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\left(i\hbar\frac{\partial\psi}{\partial t} + \frac{1}{2}mv^2\psi + i\hbar\mathbf{v}\cdot\nabla\psi\right).$$

Since

$$\nabla'\psi' = \nabla\psi' = e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\left(-\frac{im}{\hbar}\mathbf{v}\psi + \nabla\psi\right)$$

$$\|\nabla'\psi'\|^2 = e^{-2i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\left(-\frac{m^2}{\hbar^2}v^2\psi^2 - 2\frac{im}{\hbar}\psi\mathbf{v}\cdot\nabla\psi + \|\nabla\psi\|^2\right),$$

the right-hand-side is

$$-\frac{\hbar^2}{2m}\frac{\|\nabla'\psi'\|^2}{\psi'} + V\psi' = e^{-i(m\mathbf{v}\cdot\mathbf{x}-mv^2t/2)/\hbar}\left(-\frac{\hbar^2}{2m}\frac{\|\nabla\psi\|^2}{\psi} + V\psi + \frac{1}{2}mv^2\psi + i\hbar\mathbf{v}\cdot\nabla\psi\right)$$

therefore

$$i\hbar\frac{\partial\psi'}{\partial t'} = -\frac{\hbar^2}{2m}\frac{\|\nabla'\psi'\|^2}{\psi'} + V\psi'.$$

3.2 Generalization of Klein-Gordon equation

Our definition of momentum (5) can be readily substituted in $E^2 = p^2c^2 + m^2c^4$ to yield

$$\boxed{\frac{1}{c^2}(\frac{\partial\psi}{\partial t})^2 - \|\nabla\psi\|^2 + (\frac{mc}{\hbar})^2\psi^2 = 0} \tag{12}$$

which can also be written as

$$-\frac{\langle\partial_\mu\psi,\partial^\mu\psi\rangle}{\psi^2}=(\frac{mc}{\hbar})^2 \tag{13}$$

Similar to the case for the non-relativistic equation (3), this equation is simplified to the Klein-Gordon equation as well via the special-relativistic generalization of condition (2)

$$\partial_\mu p^\mu=\partial_\mu k^\mu=0 \tag{14}$$

which is

$$\frac{\psi\Box\psi-\langle\partial_\mu\psi,\partial^\mu\psi\rangle}{\psi^2}=0$$

$$\Rightarrow\frac{\langle\partial_\mu\psi,\partial^\mu\psi\rangle}{\psi^2}=\frac{\Box\psi}{\psi}$$

Substitution in the alternative form (13), we have

$$\frac{\Box\psi}{\psi}=-(\frac{mc}{\hbar})^2$$

multiplication of both sides by ψ yields

$$\left(\Box+(\frac{mc}{\hbar})^2\right)\psi=0$$

i.e. the Klein-Gordon equation.

3.2.1 Lorentz invariance

Equation (12) is Lorentz-invariant for the same reason that Klein-Gordon equation itself is. As our generalization reads

$$\langle\partial_\mu\psi,\partial^\mu\psi\rangle=-\psi^2(\frac{mc}{\hbar})^2,$$

the right-hand-side being a scalar remains invariant under a Lorentz transformation; the left-hand-side is the inner product of Minkowski space which is also invariant under Lorentz transformations.

3.3 Comparison with other relevant nonlinear theories

There are two existing main branches of research that involve nonlinearity in relation to quantum mechanics:

1. *Particular* Nonlinear Schrödinger equations; which are neither considered nor pursued as serious proposals for fundamental modifications of quantum mechanics.

2. *General frameworks*, like de Broglie's[7], GRW theory[12] and Weinberg's[23, 14]; which pursue nonlinearity on a fundamental level, but involve arbitrariness.

Our proposal not only incorporates all the strengths of these two branches but also addresses their weaknesses: it is a *general framework* enjoying an *explicit* generalization of Schrödinger and Klein-Gordon equations without any arbitrariness whatsoever.
The first category (particulars) includes the Nonlinear Schrödinger equation[11]

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + k|\psi|^2\psi,$$

Logarithmic Schrödinger equation[24]

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + b\psi\log|\psi|^2,$$

and other similar equations like Gross-Pitaevskii[25][18]. On the mathematical side, our proposed equation is totally different from such equations in that it is a *first-order* PDE. On the physical side, all such equations are based on various approximations[26, 27] and with no simple derivation. Being approximations and *not generalizations* they all owe their physical role to a more *fundamental* (as opposed to *emergent*) equation. In case of the Gross-Pitaevskii equation for example, the nonlinearity is an emergent one[19] due to interactions between particles.
Our approach on the contrary is based on no approximation nor additional assumption or new parameters: it is based on the most general form of the de Broglie hypothesis that is possible to express in terms of the wavefunction itself using differential calculus. What totally distinguishes our proposed equation is that it is based on a firm physical foundation and simplifies to Schrödinger and Klein-Gordons equations without letting equal to zero any *arbitrary* parameter, but by imposition of a well-defined *dynamical* condition that we *already knew* is assumed in quantum mechanics. For the reasons stated, the particular equations have not the potential of being extended to relativistic quantum mechanics while Special Relativity is easily applied to our proposed neoclassical definition of momentum (5) to yield the elegant generalization of Klein-Gordon equation (12).
Along the lines of the second category that similar to our approach follow nonlinearity as having a fundamental role in quantum physics, the term $\Box\psi/\psi$ in (13) resembles $\Box f/f$ in de Broglie's theory of double solutions[7]. Basically de Broglie considers two waves; usual ψ and the u-wave. de Broglie thought of this new u-wave as representing a 'mobile singularity' intended to represent the particle aspect. The f function is the amplitude of the u-wave. Although de Broglie himself –like the theory that bears his name– also missed the point by neglecting the new (second) term in (6), he did correctly realise the significance of $\Box f/f$ by stating that '*The departure from the older mechanics is always bound up with the presence of the term $\Box f/f$.*'[7]. Apart from this remark, our theory is completely different from de Broglie's theory and does not appeal to the redundant notion of u-wave. More recent approaches include GRW theory[12], Weinberg's attempts[23, 14] and Doebner-Goldin's[28]. As we mentioned earlier, GRW theory introduces two new parameters (collapse rate and

[18]Weinberg[23] has a thorough review of such attempts and their status from the perspective of nonlinearity as a fundamental guiding principle.
[19]Involving scattering length a_s which is not a fundamental constant of physics, the nonlinear term is due to interaction of particles and not a fundamental term present for single particles.

localization distance) which are considered to be new fundamental constants of nature[29]. This is nothing but arbitrariness: new fundamental constants of nature can only be introduced out of absolute necessity; on the verge of desperation (Planck), otherwise we would have as many new 'fundamental' constants as there are trends of theoretical physics. By *Occam's razor* there is no need for new constants when we already have at hand a nonlinear fundamental theory without any arbitrariness. Same can be said about Doebner-Goldin equation[28]

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi+iD\hbar\left(\frac{|\nabla\psi|^2}{|\psi|^2}\psi+\nabla^2\psi\right)$$

due to the presence of arbitrary parameter D. Regarding Weinberg's theory, we maintain that its most important physical aspect is the *homogeneity property* according to which if ψ is a solution then so is $Z\psi$ for arbitrary complex constant Z. This condition is so crucial for Weinberg that his proposed equation for evolution of wavefunction is motivated so as to satisfy it. Indeed our equation does satisfy this condition: as our equation for $Z\psi$ reads

$$\frac{i\hbar Z^2}{2}\frac{\partial\psi^2}{\partial t}=-\frac{\hbar^2}{2m}\nabla(Z\psi)\cdot\nabla(Z\psi)+VZ^2\psi^2=-\frac{\hbar^2}{2m}Z^2\|\nabla\psi\|^2+VZ^2\psi^2,$$

it satisfies the condition for arbitrary[20] Z.

4 Implications and Conclusion

It must have become clear by now that our proposal is neither an alternative nor an interpretation of any existing quantum theory. It is a generalization and as such it has novel physical consequences.

4.1 Generation of Probabilities

Although our physical motivation was based on probabilities and measurements, after the development of our proposal we now face difficulties in maintaining such concepts. Our proposal cannot say anything about the Born Principle as its very statement that '$|\psi|^2$ is the probability distribution of a *superposed* quantum state switching to a single definite *eigenvalue* by performance of a measurement' is blurred by our proposal: according to our view there are neither superpositions nor eigenvalues in general; both are too special cases to deserve reference of a fundamental *principle* of nature. Possible redundancy of concepts like superposition and eigenvalues in turn makes the meaning of 'measurement' and 'probability of outcome of a measurement' unclear. In this light therefore, it is expected that we must look for a new meaning for ψ. In this paper however we follow Newton's maxim of *hypotheses non fingo* and leave the question open to further meticulous investigations. Accordingly we temporarily assume that $|\psi|^2$ *is some sort of probability* in order to demonstrate the following consequence. If we leave (almost) intact the assumption of $\rho=|\psi|^2$ representing

[20]Note that in the system that we are confined to (neoclassical approach), zero complex numbers are already excluded right from the beginning, so 'arbitrary' implicitly means *non-zero* arbitrary.

probabilities, comparison of (1) and (6) implies that the rate of generation of probabilities denoted by π, is given by

$$\pi = \left| \frac{i\hbar}{m} \nabla \cdot \left(|\psi|^2 \frac{\nabla R}{R} \right) \right| \tag{15}$$

Naturally $\pi = 0$ is the situation in which probability is not generated i.e. conserved.

4.2 Nonlinearity and Possibility of Solitons

Notice that equation (8) is dispersive: Consider for example the case for a free particle

$$i \frac{1}{\psi} \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\|\nabla \psi\|^2}{\psi^2}$$

in which we have set $\hbar = m = 1$ for simplicity. By our definition (5) the above equation yields the dispersion relation

$$\omega = -\frac{1}{2} |\mathbf{k}|^2,$$

which is in fact identical with the dispersion relation that Schrödinger equation yields for a free particle. *Unlike* Schrödinger equation however, our equation is nonlinear. As we mentioned in the introduction it is known that dispersion and nonlinearity *together* allow for the possibility of existence of solitons[13]. It is therefore possible to revive the old notion of *wavepackets as particles* should such solutions actually occur.

Now that it is assured that our proposal cannot have any disagreement with the orthodox theory, it is a legitimate question that *what new information this theory is giving us for the important experiments of quantum mechanics* like the double-slit experiment? A proper answer of this question depends on solving equation (8) which the author has not yet been able to solve analytically. In absence of an analytical solution the next best action is to analyse the equation numerically which will be done in a consequent paper.

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