The Holographic Complexity on Extremal Branes with Exceptional Higher Derivative Interactions

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Abstract

The philosophy of presented multiversum doctrina dominum article is related to the coloring of the theoretical framework with respect to holographic complexity on extremal branes in exclusive higher-dimensional representations. We examine holographic complexity in the doubly holographic model introduced in the current literature to study quantum extremal islands. We focus on the holographic complexity volume proposal for boundary subregions in the island phase. Exploiting the Fefferman-Graham expansion of the metric and other geometric quantities near the extremal brane, we derive the leading contributions to the complexity and interpret these in terms of the generalized volume of the island derived from the induced higher-curvature gravity action on the extremal brane. We discuss the interpretation of path integral optimization as a uniformization problem in even dimensions. This perspective allows for a systematical construction of the higher-dimensional path integral complexity in holographic conformal field theories in terms of Q-curvature actions. Motivated by the exceptional results, we propose a generalization of the higher-dimensional derivative actions of exotic extremal branes.

Contents

1 Introduction

In the last few years, an influx of concepts from quantum information theory have led to exciting new insights about quantum gravity, especially within the framework of gauge/gravity duality [3]. One of these concepts that has been a topic of much research is the quantum circuit complexity [4], which quantifies how difficult it is to prepare a target state from a simple reference state, given a particular set of elementary gates. Among the various conjectured holographic duals to circuit complexity, the two most extensively studied are the complexity=volume (CV) [5, 6] and the complexity=action (CA) [7, 8] proposals. The CV conjecture states that the complexity of the state in the boundary theory defined on a time slice $\mathbf{S}$ is dual to the volume of the maximal codimension-one bulk surface anchored to $\mathbf{S}$ on the asymptotic boundary,

$$\mathcal{C}_{\rm V}(\mathbf{S}) = \max_{\partial\mathcal{B}=\mathbf{S}} \left[\frac{V(\mathcal{B})}{G_{\rm N}\,\ell}\right] , \tag{1}$$

where $G_{\rm N}$ is the Newton's constant of bulk gravity theory and ℓ is some undetermined length scale. Various aspects of the CV proposal have been studied on the gravitational side of the duality, *e.g.,* see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The above conjecture assumes that the state in question is a pure state defined on a global time slice, *i.e.,* the time slice $\mathbf{S}$ spans the entire asymptotic boundary.

Motivated by entanglement wedge reconstruction [24, 25, 26, 27, 28, 29], the CV proposal was extended to mixed states produced by reducing a global pure state down to a subregion of the boundary [9, 30]. The subregion-CV conjecture proposes that the complexity of the quantum state defined on a boundary subregion $\mathbf{R}$ is given by the volume of a maximal codimension-one bulk surface extending from $\mathbf{R}$ on the asymptotic boundary to the corresponding Ryu-Takayanagi (RT) surface $\Sigma_{\mathbf{R}}$ in the bulk,

$$\mathcal{C}^{\rm sub}_{\rm V}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}} \left[\frac{V(\mathcal{B})}{G_{\rm N}\,\ell}\right] . \tag{2}$$

Recently, information theoretic ideas have also produced exciting new insights for the resolution of the black hole information paradox [53, 54, 55]. The latter can be quantified by examining the von Neumann entropy of the Hawking radiation [55, 56, 57]. Hawking's original analysis indicated that this entropy increases throughout the evaporation of a black hole since one is simply accumulating more and more thermal radiation. However, Page argued that the entropy of the radiation must be bounded by the black hole entropy for a unitary evolution, so the entropy must in fact decrease over the second half of the evaporation process and reach zero in the final state where the black hole has disappeared. The *Page curve* is then a plot of the entropy of the Hawking radiation as a function of time which exhibits this qualitative behaviour [55, 56].

For simplicity, one assumes that the Hawking radiation is absorbed by a non-gravitational reservoir (the *bath*), which is coupled to the asymptotic boundary of the gravitational region containing the black hole. One finds the entropy of the Hawking radiation in a bath subregion $\mathbf{R}$ is given by the *island rule* [58, 61]

$$S_{\rm EE}(\mathbf{R}) = \min\left\{ \mathop{\rm ext}_{\rm islands} \left(S_{\rm QFT}(\mathbf{R}\cup{\rm islands}) + \frac{A(\partial({\rm islands}))}{4G_{\rm N}} \right) \right\} . \tag{3}$$

That is, $S_{\rm EE}(\mathbf{R})$ is not just given by the entropy of the quantum fields in the bath region, but rather one also considers $\mathbf{R}$ together with subregions (*i.e., islands*) in the gravitating region to minimize the entanglement entropy of the combined subregion. Further, the Bekenstein-Hawking entropy appears as an additional gravitational contribution at the boundary of the islands.

Initially, for an evaporating black hole is minimized without any islands and the calculation matches Hawking's evaluation of the entropy. However, at late times, a new saddle point involving a nontrivial island dominates because the Hawking radiation shares a large amount of entanglement with the quantum fields behind the horizon. In this Page phase of the time evolution, the entropy is controlled by the black hole entropy, which appears in the second term and in this way, the island rule yields the expected unitary Page curve.

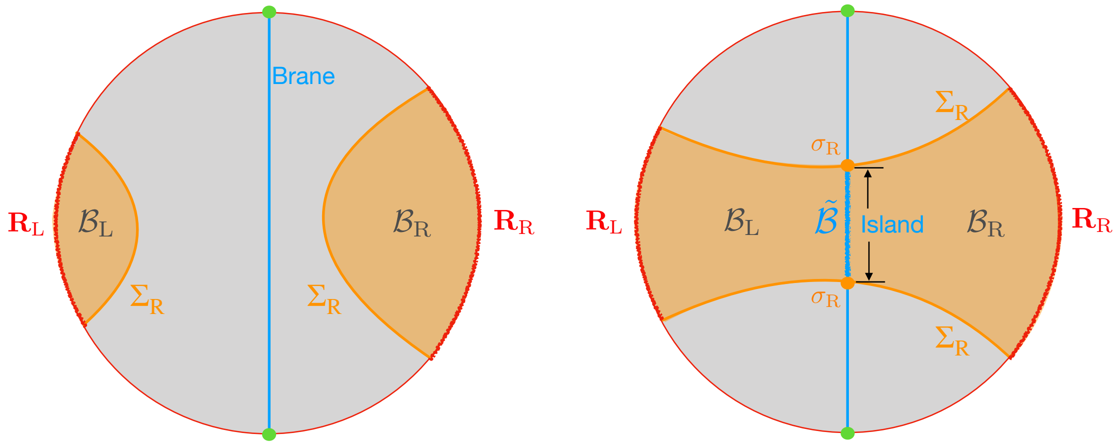

Figure 1: The choice of RT surfaces for the boundary subregion $\mathbf{R} = \mathbf{R}_{\mathrm{L}} \cup \mathbf{R}_{\mathrm{R}}$ on a constant time slice in the presence of the brane (coloured green), showing the island and no-island phases in the right and left panels, respectively. The complexity $\mathcal{C}_{\mathrm{V}}^{\mathrm{sub}}(\mathbf{R})$ in equations is determined by the extremal surface $\mathcal{B} = \mathcal{B}_L \cup \mathcal{B}_R$. In the island phase, the intersection of this surface with the brane defines the 'island' $\widetilde{\mathcal{B}} = \mathcal{B} \cap \text{brane}$.

The island rule has a simple interpretation within certain "doubly-holographic" models in [1, 2, 61, 63]. Of course, the physics can be described with the usual bulk and boundary perspectives of a holographic system. In this case, the *boundary perspective* consists of a d-dimensional CFT coupled to a codimension-one conformal defect, and the *bulk perspective* then becomes $(d+1)$-dimensional gravity on an asymptotically AdS spacetime containing a codimension-one brane, which is anchored at the conformal defect on the asymptotic boundary. This brane back reacts on the bulk spacetime and in an appropriate parameter regime, a third perspective emerges through the Randall-Sundrum mechanism [108, 109, 110]. In this *brane perspective*, the brane supports a theory of d-dimensional gravity coupled to (two copies of) the holographic CFT, and is connected to the CFT on the asymptotic boundary (which becomes the bath) at the position of the defect. We refer the interested reader to [1, 2] for further details on these three perspectives.

A key advantage of this framework is that entanglement entropies in eq. (3) are calculated purely geometrically from the bulk perspective, using the usual rules of holographic entanglement entropy [100, 101, 102, 103, 104, 105]. In particular, the entanglement entropy for a bath or boundary region $\mathbf{R}$ becomes

$$S_{\mathrm{EE}}(\mathbf{R}) = \min \left\{ \underset{\Sigma_{\mathbf{R}}}{\mathrm{ext}} \left(\frac{A(\Sigma_{\mathbf{R}})}{4G_{\mathrm{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\mathrm{brane}}} \right) \right\} , \tag{4}$$

where $\Sigma_{\mathbf{R}}$ is the usual bulk RT surface, while $\sigma_{\mathbf{R}} = \Sigma_{\mathbf{R}} \cap \text{brane}$ is the intersection of the RT surface with the brane. The second term is the Bekenstein-Hawking area contribution that is included when an intrinsic gravitational action (*i.e.,* a DGP term [111]) is included in the brane action [1, 2]. From the brane perspective then, islands simply arise when the minimal RT surfaces in the bulk extend across the brane, as illustrated in the right panel of figure. Further, the transition between the island and no-island phases (*e.g.,* between the Page and Hawking phases of an evaporating black hole) corresponds to a conventional transition found in holographic entanglement entropy between different classes of RT surfaces, *e.g.,* [112, 113, 114, 115]. Let us add that carefully examining near the extremal brane shows that the gravitational contribution in the island rule expands to the Wald-Dong entropy [116, 117, 118, 119] for the higher-curvature gravitational action induced on the brane [1].

In this paper, we extend the examination of the model constructed in [1, 2] to consider holographic complexity, and in particular, the subregion-CV proposal (2). In particular, we focus on the island phase in which case the extremal bulk surface $\mathcal{B}$ also crosses the brane. Following an analysis similar to that of [1] for the holographic entanglement entropy, we employ the FG expansion of the subregion-CV in the vicinity of the brane to recast it as an integral of geometric quantities over the island, *i.e.,* $\widetilde{\mathcal{B}} = \mathcal{B} \cap \text{brane}$.

Then to leading order, eq. (2) yields

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) \simeq \max\left[\frac{d-2}{d-1}\,\frac{V(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell}+\cdots\right], \tag{5}$$

where $G_{\rm eff}$ is the induced Newton's constant for the gravitational theory on the brane. Setting aside the dimension-dependent prefactor, the geometric integral over $\widetilde{\mathcal{B}}$ is naturally interpreted as the holographic complexity of the island region.

However, beyond the volume term, the ellipsis also includes higher curvature corrections. By examining these contributions, we are lead to a generalized CV formula derived from the induced higher-curvature gravity action on the brane. That is, we propose to generalize the complexity=volume conjecture for an arbitrary $(d+1)$-dimensional higher-curvature gravity theory in the bulk as

$$\mathcal{C}_{\rm V}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}}\left[\frac{W_{\rm gen}(\mathcal{B})+W_K(\mathcal{B})}{G_{\rm N}\,\ell}\right]. \qquad (d>2) \tag{6}$$

where $W_{\rm gen}$ is called the generalized volume because this expression reduces to the volume term $V(\mathcal{B})$ for Einstein gravity, and W_K introduces extra corrections involving the extrinsic curvature $\mathcal{K}_{\mu\nu}$ of the hypersurface $\mathcal{B}$. Explicitly, our analysis determines these two contributions as

$$\begin{aligned}
W_{\rm gen}(\mathcal{B}) =& \frac{2}{(d-1)(d-2)}\int_{\mathcal{B}} d^d\sigma\,\sqrt{\det h}\left(1+(d-3)\,\frac{\partial \mathbf{L}_{\rm bulk}}{\partial \mathcal{R}_{\mu\nu\rho\sigma}}n_\mu h_{\nu\rho} n_\sigma\right),\\
W_K(\mathcal{B}) =& \frac{4(d-3)}{(d-1)^2(d-2)}\int_{\mathcal{B}} d^d\sigma\,\sqrt{\det h}\,\frac{\partial^2 \mathbf{L}_{\rm bulk}}{\partial \mathcal{R}_{\mu_1\nu_1\rho_1\sigma_1}\partial \mathcal{R}_{\mu_2\nu_2\rho_2\sigma_2}}\\
&\times\ \left[\mathcal{K}_{\nu_1\sigma_1}\left(h_{\mu_1\rho_1}+(d-2)n_{\mu_1}n_{\rho_1}\right)\mathcal{K}_{\nu_2\sigma_2}\left(h_{\mu_2\rho_2}+(d-2)n_{\mu_2}n_{\rho_2}\right)\right]+\cdots.
\end{aligned} \tag{7}$$

For these expressions, we have rescaled the gravitational Lagrangian so that the gravitational action carries an overall factor: $I_{\rm grav} = \frac{1}{16\pi G_{\rm N}}\int d^{d+1}x\sqrt{-g}\,\mathbf{L}_{\rm bulk}$. Further, $\mathcal{B}$ denotes a spacelike codimension-one bulk hypersurface with unit normal n^μ, induced metric $h_{\mu\rho}$, and extrinsic curvature $\mathcal{K}_{\mu\nu}$. The generalized subregion-CV functional is maximized subject to the constraint that the codimension-one hypersurface $\mathcal{B}$ is anchored at the boundary subregion $\mathbf{R}$ and the corresponding RT surface $\Sigma_{\mathbf{R}}$, *i.e.,* $\partial\mathcal{B} = \mathbf{R}\cup\Sigma_{\mathbf{R}}$.

Our proposal to the generalized CV contains two contributions, in a similar spirit to the Wald-Dong entropy [116, 117, 118, 119]. The generalized volume $W_{\rm gen}$ was first conjectured in [120], which left the precise coefficients of various contributions undetermined. This expression is analogous to the original Wald entropy, which is derived for stationary event horizons on which the extrinsic curvature terms vanish. We fix the coefficients, as shown in the article by carefully examining the higher-curvature corrections. We have introduced a convenient factor of $(d–1)$ in eq. (6), which allows us to combine naturally combine contributions for gravitational terms of differing dimensions as in doubly-holographic models studied here. The term W_K in generalizes the results to surfaces where the extrinsic curvature is non-vanishing, in analogy to Dong's extrinsic curvature corrections to the Wald entropy [119]. These corrections naturally arise here in matching the subleading terms in the FG expansion of the volume of $\mathcal{B}$ in the bulk Einstein gravity case. However, as indicated in eq. (7), we have only matched the corrections which are quadratic in $\mathcal{K}_{\mu\nu}$ and as indicated by the ellipsis, this is only the first term in a longer expansion just as is found in the Wald-Dong entropy [119]. We must also admit that even for the quadratic corrections, there is a high degree of ambiguity and the expression in eq. (7) is only the simplest ansatz consistent with our analysis.

The full analysis leading to these results is presented as follows: In section 2, we exploit the Fefferman-Graham expansion near the brane to show that the leading-order contribution to holographic complexity coming from the island is given by the expression in eq. (5). In the process, we derive the generalized complexity (6) for the effective higher-curvature theory of gravity on the brane. We also argue that the surface $\widetilde{\mathcal{B}}$ on which the complexity is evaluated corresponds to the maximal complexity island. In section

3, we test our conclusions by beginning with a higher-curvature gravity theory in the $(d+1)$-dimensional bulk, *i.e.,* Gauss-Bonnet gravity and $f(\mathcal{R})$ gravity, and explicitly show our proposal (6) consistently yields the same holographic complexity of islands as that derived from the effective gravitational theories on the brane. We present with a discussion of our results and future directions in section 14. In particular, we consider the quantum field theory corrections that implicitly appear when eq. (5) is interpreted from the brane perspective.

2 Holographic Complexity on the Island

In this section, we examine the subregion-CV conjecture in the context of the holographic model constructed in [1, 2]. So we begin by reviewing some of the salient points of the model: As usual, the bulk gravity theory is described by

$$I_{\text{bulk}} = \frac{1}{16\pi G_{\text{bulk}}} \int_{\text{bulk}} d^{d+1}y\sqrt{-g}\left(\frac{d(d-1)}{L^2} + \mathcal{R}[g_{\mu\nu}]\right), \tag{8}$$

where L becomes the radius of curvature for the vacuum AdS_{d+1} spacetime. Here, the bulk theory also includes a codimension-one brane with the action

$$I_{\text{brane}} = -T_o \int d^dx\sqrt{-\tilde{g}}\,, \tag{9}$$

where T_o is the tension and $\tilde{g}_{ij}$ is the induced metric on the brane.

Following [1, 2], we foliate of the bulk geometry with AdS_d slices as in

$$ds^2_{\text{AdS}_{d+1}} = \frac{L^2}{\sin^2\theta}\left(d\theta^2 + ds^2_{\text{AdS}_d}\right). \tag{10}$$

where the AdS_d metric has unit curvature. The solution with the brane is constructed by cutting the above geometry along an AdS_d slice at some $\theta = \theta_{\text{B}}$ near the asymptotic boundary. Joining together two copies of this geometry, as in figure 2, the brane is then represented as the interface between the two. That is, the brane is considered a shell of zero thickness and it's position the spacetime is determined using the Israel junction conditions [121]

$$\Delta(\mathcal{K}_{\text{B}})_{ij} - \tilde{g}_{ij}\Delta\mathcal{K}_{\text{B}} = 8\pi G_{\text{bulk}}\, S_{ij} = -8\pi G_{\text{bulk}}\, T_o\, \tilde{g}_{ij}\,, \tag{11}$$

where S_{ij} is the boundary stress tensor introduced by the brane and $\Delta(\mathcal{K}_{\text{B}})_{ij} \equiv \mathcal{K}^{\text{L}}_{ij} - \mathcal{K}^{\text{R}}_{ij}$. The brane position can be written as

$$\sin^2\theta_{\text{B}} = \frac{L^2}{\ell^2_{\text{B}}} = 2\,\varepsilon\,(1-\varepsilon/2) \qquad \text{where} \;\; \varepsilon \equiv \left(1 - \frac{4\pi G_{\text{bulk}} L T_o}{d-1}\right), \tag{12}$$

and ℓ_{B} is the curvature scale on the brane.

Now by construction, the bulk geometry locally takes the form of AdS_{d+1} spacetime away from the brane. However, the brane's backreaction expands the bulk and with $\theta_{\text{B}} \ll 1$, the brane is pushed towards the asymptotic boundary of eq. (10). Of course, this boundary (at $\theta = 0$) is cut out of the construction, but we may still use the usual Fefferman-Graham (FG) expansion [122, 123] to examine the geometry in the vicinity of the brane. While the explicit construction described above is for the maximally symmetric ground state configuration, in the following, we consider more general configurations where the brane geometry may deviate slightly from the AdS_d geometry.

We begin by writing the metric on an asymptotically AdS_{d+1} spacetime as

$$ds^2 = g_{\mu\nu}\, dy^\mu\, dy^\nu = \frac{L^2}{z^2}\left(dz^2 + g_{ij}(z, x^i)dx^i dx^j\right). \tag{13}$$

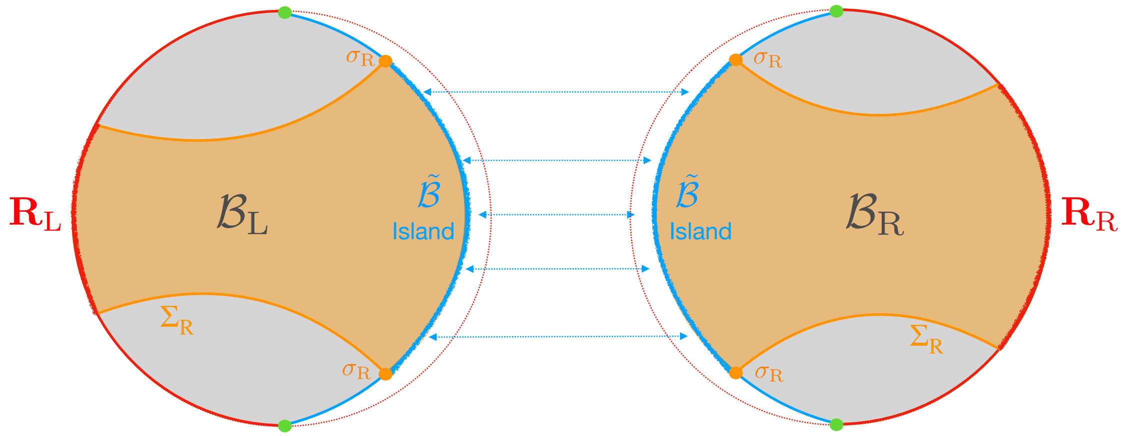

Figure 2: The holographic setup with islands in AdS_{d+1}. The two AdS_{d+1} geometries are cut off at $\theta = \theta_{\mathrm{B}}$ (or $z = z_{\mathrm{B}}$) and glued together with the brane at the junction between the two. The island region emerges on the brane when the RT surfaces $\Sigma_{\mathbf{R}}$ of the boundary subregion $\mathbf{R} = \mathbf{R}_{\mathrm{L}} \cup \mathbf{R}_{\mathrm{R}}$ cross the brane. The maximal volume bulk slice $\mathcal{B} = \mathcal{B}_{\mathrm{L}} \cup \mathcal{B}_{\mathrm{R}}$ crosses the brane, and the intersection of these two surfaces determines the island $\tilde{\mathcal{B}} = \mathcal{B} \cap \text{brane} = \mathcal{B}_{\mathrm{L}} \cap \mathcal{B}_{\mathrm{R}}$.

In these coordinates, we approach the asymptotic boundary for $z \to 0$, and the brane is located at $z = z_{\mathrm{B}} \ll L$. Around the asymptotic boundary, the Fefferman-Graham expansion provides the a series expansion of the metric $g_{ij}(z, x^i)$ in terms of the boundary metric $\overset{(0)}{g}_{ij}$ and the boundary stress tensor $\overset{(d/2)}{g}_{ij} \propto \langle T_{ij} \rangle$ [124, 125], *i.e.*,

$$g_{ij}(z, x^i) = \overset{(0)}{g}_{ij}\left(x^i\right) + \frac{z^2}{L^2}\overset{(1)}{g}_{ij}\left(x^i\right) + \cdots \frac{z^d}{L^d}\left(\overset{(d/2)}{g}_{ij}(x^i) + f_{ij}(x^i)\log\left(\frac{z}{L}\right)\right) + \cdots, \tag{14}$$

where the logarithmic term is present only when d is even. Now $z_{\mathrm{B}}/L \ll 1$ emerges as a natural expansion parameter, which we can apply in the FG expansion to study the geometry near the brane.

Applying the bulk Einstein equations in the FG expansion (14) fixes the expansion coefficients $\overset{(n)}{g}_{ij}$ (with $0 < n < \frac{d}{2}$) in terms of the boundary metric $\overset{(0)}{g}_{ij}$ [124, 125]. For example, the first term in the expansion is given by the Schoutten tensor P_{ij} (for $d > 2$),

$$\overset{(1)}{g}_{ij}\left(x^i\right) = -L^2 P_{ij}[\overset{(0)}{g}] = -\frac{L^2}{d-2}\left(R_{ij}[\overset{(0)}{g}] - \frac{\overset{(0)}{g}_{ij}}{2(d-1)}R[\overset{(0)}{g}]\right), \tag{15}$$

where R_{ij} and R denote the Ricci tensor and Ricci scalar calculated with $\overset{(0)}{g}_{ij}$, respectively. We further note the above expression can also be derived by examining the effect of Penrose-Brown-Henneaux transformations [126], which implies that $\overset{(1)}{g}_{ij}\left(x^i\right)$ is completely determined by the conformal symmetries on the boundary and therefore it is independent of the bulk gravity theory. In contrast, the next term $\overset{(2)}{g}_{ij}$ in the expansion depends on the details of the bulk gravity theory, *e.g.*, see [127, 128]. More precisely, it depends on whether the gravitational action contains interaction with the Riemann tensor squared, as we will see in section 3.

With the assumption that $\theta_{\mathrm{B}} \ll 1$, one application of the FG expansion [122, 123] is to derive the effective action for the gravity theory on the brane [1]

$$\begin{aligned} I_{\text{eff}} \;=\; & \frac{1}{16\pi G_{\text{eff}}}\int d^dx\sqrt{-\tilde{g}}\left[\frac{(d-1)(d-2)}{\ell_{\text{eff}}^2} + \tilde{R}(\tilde{g})\right] \\ & + \frac{1}{16\pi G_{\text{eff}}}\int d^dx\sqrt{-\tilde{g}}\left[\frac{L^2}{(d-4)(d-2)}\left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{d}{4(d-1)}\tilde{R}^2\right) + \cdots\right], \end{aligned} \tag{16}$$

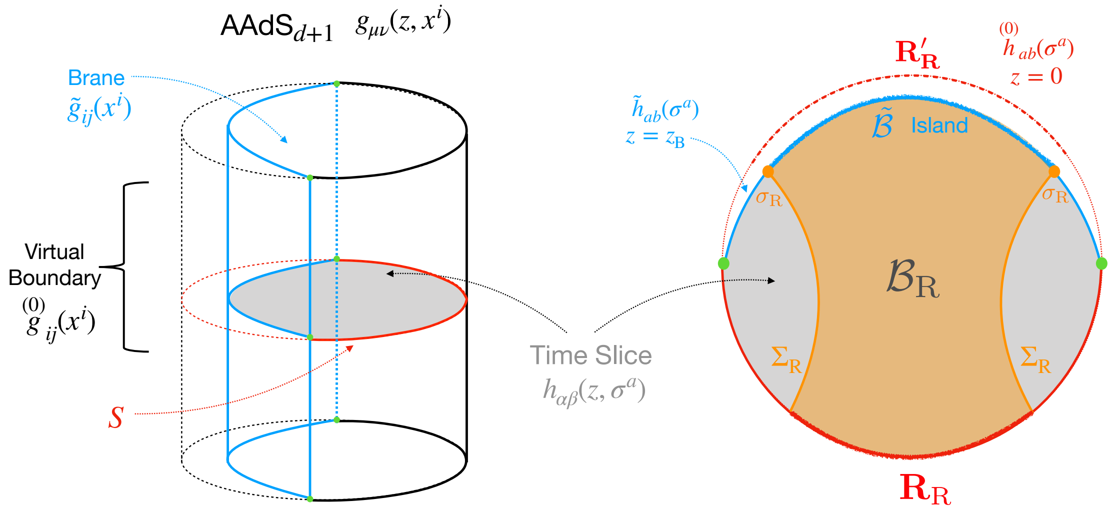

Figure 3: The full asymptotically AdS$_{d+1}$ geometry from the right side of the construction in figure 2. The time slice $\mathbf{S}$ is introduced in the left panel and detailed in the right panel. We explicitly show various metrics for the different regions.

where

$$\frac{1}{G_{\text{eff}}} = \frac{2L}{(d-2)G_{\text{bulk}}}\,, \qquad \frac{1}{\ell_{\text{eff}}^2} = \frac{2}{L^2}\,\varepsilon\,, \tag{17}$$

and $\tilde{g}_{ij}$ is the induced metric on the brane. The UV cutoff in this effective theory is given by $\tilde{\delta} = L$, and this controls the contributions of the higher curvature terms appearing in the second line of eq. (16). Hence we are naturally lead to consider $\theta_{\mathrm{B}} \ll 1$ (or equivalently, $L^2/\ell_{\text{eff}}^2 \ll 1$ or $\varepsilon \ll 1$) as this corresponds to the regime in which the induced brane theory is well approximated by Einstein gravity with a negative cosmological constant.

Similarly, the FG expansion can be applied to understand the contributions of the holographic entanglement entropy (4) in terms of the brane theory, *e.g.,* one finds that the gravitational contribution in the island rule (3) corresponds to the Wald-Dong entropy for the induced action (16) evaluated on the boundaries of the island [1]. In the following, we follow a similar strategy applying the FG expansion to examine the bulk holographic complexity (2) evaluated in the vicinity of the brane and reinterpret the result in terms of the brane theory. In particular, we will find the geometric contributions in the 'island' complexity, and provide a prescription to derive these from the effective action (16).

2.1 Extremal Surfaces Near the Brane

Eq. (2) gives the complexity=volume proposal for a boundary subregion $\mathbf{R}$ as,

$$\mathcal{C}_{\mathrm{V}}^{\text{sub}}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}}\left[\frac{V(\mathcal{B})}{G_{\text{bulk}}\,\ell}\right]. \tag{18}$$

In particular, one extremizes the volume of codimension-one hypersurface $\mathcal{B}$ anchored on the subregion $\mathbf{R}$ on the asymptotic boundary and on the RT surface $\Sigma_{\mathbf{R}}$ in the bulk. Since we are interested in reinterpreting the bulk results in terms of the brane theory, we will assume that we are in the island phase, *i.e.,* the RT surface $\Sigma_{\mathbf{R}}$ crosses the brane, as shown in figure 2. Then, as shown, our boundary subregion $\mathbf{R}$ will generally have components $\mathbf{R}_{\mathrm{L}}$ and $\mathbf{R}_{\mathrm{R}}$ on either side of the conformal defect in the boundary theory. Similarly, we decompose the bulk surface in terms of components on either side of the brane, *i.e.,* $\mathcal{B} = \mathcal{B}_{\mathrm{L}} \cup \mathcal{B}_{\mathrm{R}}$. We also remark that in applying the FG expansion, we extend the left or right geometry to a 'virtual' asymptotic boundary at $z=0$, so that the 'boundary metric' $\overset{(0)}{h}_{ab}$ and other boundary quantities are evaluated at the region $\mathbf{R}'_{\mathrm{R}}$ (and similarly a region $\mathbf{R}'_{\mathrm{L}}$ for the left AdS region) at this virtual boundary, as shown in the right panel of figure 3.

To facilitate our analysis, we introduce d-dimensional coordinates σ^α in $\mathcal{B}$ with letters from the beginning of the Greek alphabet, *i.e.,* α, β, γ which run from 1 to d. Further, we use Gaussian normal coordinates with respect to the intersection $\widetilde{\mathcal{B}} = \mathcal{B} \cap \text{brane}$, with $\zeta = \sigma^d$ being the coordinate normal to the brane. Latin indices a, b, c from the beginning of the alphabet denote the other directions running from 1 to $d-1$, *i.e.,* $\sigma^\alpha = (\zeta, \sigma^a)$.s Taking the parametrization of the bulk hypersurface $\mathcal{B}$ as $y^\mu(\zeta, \sigma^a)$, we can define the induced metric on this surface by

$$h_{\alpha\beta} = \frac{\partial y^\mu}{\partial \sigma^\alpha} \frac{\partial y^\nu}{\partial \sigma^\beta} g_{\mu\nu}[y]\,. \tag{19}$$

As a bulk tensor, we may also write the induced metric as

$$h_{\mu\nu} = [g_{\mu\nu}]_{\mathcal{B}} + n_\mu n_\nu\,, \tag{20}$$

where n^μ is the unit vector normal to $\mathcal{B}$, *i.e.,* $n^\nu n^\mu g_{\mu\nu} = -1$ and $h_{\mu\nu} n^\nu = 0$. Further, it will be convenient to make the following gauge choices:

$$\zeta = \sigma^d = z \qquad \text{and} \qquad h_{za} = 0\,. \tag{21}$$

In order to consider holographic complexity for $(d+1)$-dimensional bulk theory, we are interested in the codimension-1 bulk surface $\mathcal{B}$ with extremal volume in the bulk. Extremizing the volume of hypersurface $\mathcal{B}$ leads to a local equation

$$\text{EOM}^\mu = \frac{1}{\sqrt{h}} \partial_\alpha \left(\sqrt{h} h^{\alpha\beta} \partial_\beta y^\mu \right) + h^{\alpha\beta} \partial_\alpha y^\nu \partial_\beta y^\sigma\, \Gamma^\mu_{\nu\sigma} = 0\,, \tag{22}$$

where $h = \det h_{\alpha\beta}$ and $\Gamma^\mu_{\nu\sigma}$ is the Christoffel symbol associated with the bulk metric $g_{\mu\nu}$. As a vector, the above expression is orthogonal to $\mathcal{B}$ and taking the inner product with n^μ leaves a simple expression in terms of the extrinsic curvature $\mathcal{K}_{\alpha\beta}$ of the submanifold (see eq. (27)),

$$\mathcal{K} = h^{\alpha\beta}\, \mathcal{K}_{\alpha\beta} = 0\,. \tag{23}$$

Since we are interested in the geometry near the asymptotic boundary, above equation can be solved order by order in a Fefferman-Graham expansion for $x^i(z, \sigma^a)$

$$x^i(z, \sigma^a) = \overset{(0)}{x^i}(\sigma^a) + \frac{z^2}{L^2} \overset{(1)}{x^i}(\sigma^a) + \mathcal{O}\left(\frac{z^4}{L^4} \right)\,. \tag{24}$$

Noting that the leading contribution in eq. (22) involves the terms with two z derivatives, we see that the extremization condition does not fix the leading coefficients $\overset{(0)}{x}{}^i$, *i.e.,* the profile of the surface at $z=0$. Alternatively, we can think of this indeterminacy as the profile of the intersection of the extremal surface $\mathcal{B}$ and the brane, which we will refer to as the island $\widetilde{\mathcal{B}} = \mathcal{B} \cap \text{brane}$. As we will emphasize in section 2.3, solving eq. (22) or (23) ensures that the volume of $\mathcal{B}$ is extremized in the bulk, *i.e.,* away from the brane. Producing the correct maximal volume surface in eq. (18) requires a second step where we vary the island profile $\widetilde{\mathcal{B}}$ which maximizes complexity functional on the brane – see eqs. (41) and (64).

Following the analysis in, *e.g.,* [1, 9, 126], the leading order terms in eq. (22) are

$$\frac{2z^2}{L^2}(1-d)\overset{(1)}{x^i} + \frac{1}{\sqrt{\overset{(0)}{h}}} \partial_a \left(\sqrt{\overset{(0)}{h}}\, \overset{(0)}{h}{}^{ab} \partial_b x^i \right) + \overset{(0)}{h}{}^{ab} \Gamma^i_{jk} \partial_a \overset{(0)}{x^j} \partial_b \overset{(0)}{x^k} + \mathcal{O}(z^4) = 0\,. \tag{25}$$

Thus the the first order term in the FG expansion for x^i is given by

$$\overset{(1)}{x^i}(\sigma^a) = \frac{L^2}{2(d-1)} \left(\overset{(0)}{D}{}^a (\partial_a \overset{(0)}{x^i}) + \overset{(0)}{h}{}^{ab} \partial_a \overset{(0)}{x^j} \partial_b \overset{(0)}{x^j}\, \overset{(0)}{\Gamma}{}^i_{jk} \right) = \frac{L^2}{2(d-1)} K \overset{(0)}{n^i}\,, \tag{26}$$

where $\overset{(0)}{D}_a$ denotes the covariant derivative associated with induced metric $\overset{(0)}{h}_{ab}$ on the (implicit) boundary time slice at $z = 0$, K is the trace of extrinsic curvature for this time slice (*i.e.*, $K = \overset{(0)}{g}{}^{ij} K_{ij}$), and $\overset{(0)}{n}{}^i$ denotes the timelike unit normal to the same time slice (*i.e.*, $\overset{(0)}{n}{}^i \overset{(0)}{n}{}^i \overset{(0)}{g}_{ij} = -1$). In order to get the second equality in eq. (26), we have used the trace of Gauss-Weingarten equation, which reads

$$e^j_b \nabla_j (e^i_a) = \Gamma^c_{ab} e^i_c + K_{ab} n^i \,, \tag{27}$$

after taking $e^i_a \equiv \partial_a \overset{(0)}{x}{}^i$. The above result is very similar to the solutions for the extremal RT surface in a $(d+2)$-dimensional bulk model, although in this case, we are working with a codimension-one hypersurface. With the asymptotic solutions, we find the induced metric components on the extremal surface $\mathcal{B}$ read

$$\begin{aligned} h_{zz} &= \frac{L^2}{z^2} \left(1 + \frac{z^2}{L^2} \frac{4 \overset{(1)}{x}{}^i \overset{(1)}{x}{}^j}{L^2} \overset{(0)}{g}_{ij} + \cdots \right) = \frac{L^2}{z^2} \left(1 - \frac{z^2}{(d-1)^2} K^2 + \cdots \right) \,, \\ h_{ab} &= \frac{L^2}{z^2} \left(\overset{(0)}{h}_{ab} + \frac{z^2}{L^2} \overset{(1)}{h}_{ab} + \cdots \right) \,, \end{aligned} \tag{28}$$

with

$$\overset{(0)}{h}_{ab} = \overset{(0)}{g}_{ij} \partial_a \overset{(0)}{x}{}^i \partial_b \overset{(0)}{x}{}^i \,, \qquad \overset{(1)}{h}_{ab} = \overset{(1)}{g}_{ab} + \frac{L^2}{d-1} K K_{ab} \,, \tag{29}$$

where the tensors with indices a, b are associated with those with i, j by using the projection $\partial_a \overset{(0)}{x}{}^i \equiv e^i_a$.

Following the subregion-CV proposal (18), our goal is to find the maximal volume hypersuface $\mathcal{B}$ anchored on the boundary subregion $\mathbf{R}$ and the bulk RT surface $\Sigma_{\mathbf{R}}$, *i.e.*, $\partial \mathcal{B} = \mathbf{R} \cup \Sigma_{\mathbf{R}}$, and then evaluate

$$\mathcal{C}^{\text{sub}}_{\text{V}}(\mathbf{R}) = \frac{V(\mathcal{B})}{G_{\text{bulk}} \ell} = \frac{1}{G_{\text{bulk}} \ell} \int_{\mathcal{B}} d^{d-1}\sigma dz \sqrt{\det h_{\alpha\beta}} \,. \tag{30}$$

In the present calculation with the brane positioned at $z_{\text{B}} \ll L$, we are particularly interested the contributions to the maximal volume coming from the region in the vicinity of the brane. These are less interesting for our purposes and might be eliminated by considering the mutual complexity [49, 129] – see the discussion section. Approaching $z \to 0$, the volume measure reduces to

$$\sqrt{\det h_{\alpha\beta}} = \sqrt{\det \overset{(0)}{h}_{ab}} \left(\frac{L}{z} \right)^d \left(1 - \frac{z^2}{2(d-1)^2} K^2 + \frac{z^2}{2L^2} \overset{(0)}{h}{}^{ab} \overset{(1)}{h}_{ab} + \cdots \right) \,. \tag{31}$$

where we have ignored the contributions from higher order z_{B}/L terms. Performing the z-integral explicitly and introducing $R^a_a[\overset{(0)}{g}] = \overset{(0)}{h}{}^{ab} R_{ab}[\overset{(0)}{g}]$, we can find the leading contributions of the holographic subregion-complexity near the brane

$$\frac{L^d}{G_{\text{bulk}} \ell} \int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma \sqrt{\det \overset{(0)}{h}_{ab}} \left[\frac{1}{(d-1) z_{\text{B}}^{d-1}} + \frac{1}{(d-3) z_{\text{B}}^{d-3}} \left(\frac{d-2}{2(d-1)^2} K^2 - \frac{R^a_a - \frac{1}{2} R}{2(d-2)} \right) + \cdots \right] \,, \tag{32}$$

where the extrinsic curvature and Ricci tensor are all related to boundary geometry at $z = 0$.

We can also evaluate the volume of the island region

$$\begin{aligned} V(\widetilde{\mathcal{B}}) &= \int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma \sqrt{\det \tilde{h}_{ab}} \,, \\ &= L^{d-1} \int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma \sqrt{\det \overset{(0)}{h}_{ab}} \left[\frac{1}{z_{\text{B}}^{d-1}} + \frac{1}{z_{\text{B}}^{d-3}} \left(\frac{K^2}{2(d-1)} - \frac{R^a_a - \frac{1}{2} R}{2(d-2)} \right) + \cdots \right] \,. \end{aligned} \tag{33}$$

with $\tilde{h}_{ab} \equiv h_{ab}(z = z_{\rm B})$ as the induced metric on the intersection $\widetilde{\mathcal{B}} = \mathcal{B} \cap \text{brane}$. Combing eqs. (32) and (33), it is straightforward to rewrite the holographic subregion-complexity (32) as

$$
\begin{aligned}
\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) &= \frac{V(\mathcal{B})}{G_{\rm bulk}\ell} \\
&\simeq \frac{2LV(\widetilde{\mathcal{B}})}{(d-1)G_{\rm bulk}\ell} + \frac{2}{G_{\rm bulk}\ell}\int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma\sqrt{\det \overset{(0)}{h}_{ab}}\,\frac{L^d}{z_{\rm B}^{d-3}}\left(\frac{K^2}{2(d-1)^2(d-3)} - \frac{R^a_a - \frac{1}{2}R}{(d-1)(d-2)(d-3)}\right) + \cdots \\
&\simeq \frac{2LV(\widetilde{\mathcal{B}})}{(d-1)G_{\rm bulk}\ell} + \frac{2L^3}{G_{\rm bulk}\ell}\int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma\sqrt{\det \tilde{h}_{ab}}\left(\frac{\tilde{K}^2}{2(d-1)^2(d-3)} - \frac{\tilde{R}_{ij}\tilde{n}^i\tilde{n}^j + \frac{1}{2}\tilde{R}}{(d-1)(d-2)(d-3)}\right) + \cdots .
\end{aligned}
\tag{34}
$$

where the factor of 2 above originates from the fact that we are integrating over both sides of the island, *i.e.,* we are including the contributions from both $\mathcal{B}_{\rm L}$ and $\mathcal{B}_{\rm R}$. Furthermore, we note that we *do not* need to require a symmetric setup because the near-brane regions from $\mathcal{B}_{\rm L}, \mathcal{B}_{\rm R}$ have the same leading order contributions, despite the fact that the full volume of the subregions $\mathcal{B}_{\rm L}, \mathcal{B}_{\rm R}$ may be different. Of course, while the surfaces $\mathcal{B}_{\rm L}$ and $\mathcal{B}_{\rm R}$ are independent away from the brane, their profiles on the brane coincide, *i.e.,* $\widetilde{\mathcal{B}} = B_{\rm L} \cap B_{\rm R}$. Let us also note here that $\widetilde{\mathcal{B}}$ is anchored to the intersection of the RT surface $\Sigma_{\mathbf{R}}$ with the brane, *i.e.,* $\partial\widetilde{\mathcal{B}} = \sigma_{\mathbf{R}} = \Sigma_{\mathbf{R}} \cap \text{brane}$, but this is precisely the quantum extremal surface (QES) in the brane theory [1, 2].

To arrive at the last line of eq. (34), we recast the boundary terms into terms related to the brane geometry following [1]. First we note that the induced metric on the brane reads

$$
\begin{aligned}
\tilde{g}_{ij}(x^i) &\equiv g^{\rm bulk}_{ij}(z_{\rm B}, x^i) = \frac{L^2}{z_{\rm B}^2}\, g_{ij}(z_{\rm B}, x^i) \approx \frac{L^2}{z_{\rm B}^2}\, \overset{(0)}{g}_{ij}(x^i) + \mathcal{O}\left(z_{\rm B}^0\right) , \\
\tilde{h}_{ab} &\equiv h_{ab}(z = z_{\rm B}) \approx \frac{L^2}{z_{\rm B}^2}\, \overset{(0)}{h}_{ij}(x^i) + \mathcal{O}\left(z_{\rm B}^0\right) ,
\end{aligned}
\tag{35}
$$

as well as using $\tilde{h}_{ij} = \tilde{g}_{ij} + \tilde{n}_i\tilde{n}_j$, where $\tilde{n}^i$ denotes the unit time-like normal to island in the brane. We therefore find

$$
\begin{aligned}
\frac{z_{\rm B}^2}{L^2}\overset{(0)}{h}{}^{ab} e^i_a e^j_b \left(R_{ij}[\overset{(0)}{g}] - \frac{\overset{(0)}{g}_{ij}}{2(d-1)}\mathcal{R}[\overset{(0)}{g}]\right) &\approx \tilde{h}^{ab}\left(\tilde{R}_{ab}[\tilde{g}] - \frac{\tilde{h}_{ab}}{2(d-1)}\tilde{R}[\tilde{g}]\right) \\
&\approx \frac{1}{2}\tilde{R}[\tilde{g}] + \tilde{R}_{ij}[\tilde{g}]\tilde{n}^i\tilde{n}^j ,
\end{aligned}
\tag{36}
$$

by keeping track of the leading contributions in the $z_{\rm B}/L$ expansion. As expected, the leading term in $\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R})$ is the volume of island region. Interestingly, the result in eq. (34) shows that the subleading terms include intrinsic geometric quantities on the brane but also include the extrinsic curvature of the island region $\widetilde{\mathcal{B}}$, *i.e.,* the term proportional to $\tilde{K}^2$. This feature is also found in a similar analysis for holographic entanglement entropy in section 4.3 of [1].

Now examining eq. (34), we see to leading order that we have

$$
\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) = \frac{V(\mathcal{B})}{G_{\rm bulk}\ell} = \frac{2L\,V(\widetilde{\mathcal{B}})}{(d-1)G_{\rm bulk}\ell} + \cdots = \frac{d-2}{d-1}\frac{V(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell} + \cdots ,
\tag{37}
$$

where $\frac{1}{G_{\rm eff}} = \frac{2L}{(d-2)G_{\rm bulk}}$ is the effective Newton's constant for the brane gravity, as given in eq. (17). That is, the complexity=volume formula in the bulk yields a complexity=volume formula on the brane, up to an inconvenient numerical factor. Now this factor could be easily absorbed if we modify the length scale for the CV proposal on the brane, *i.e.,*

$$
\ell' = \frac{d-1}{d-2}\,\ell .
\tag{38}
$$

However, beyond the volume term, eq. (34) also contains higher-order corrections involving the curvature on the brane and the extrinsic curvature of the surface $\widetilde{\mathcal{B}}$. By examining these contributions more carefully in the next subsection, we will be able to interpret them in terms of a generalized CV formula derived from the induced higher-curvature gravity action (16) on the brane. The emergence of this generalized CV expression in the brane theory is then analogous to the appearance of the Wald-Dong entropy in the island rule (3) on the brane discussed in [1].

2.2 Holographic Complexity on the Brane

In this subsection, we show that the sub-leading contributions in eq. (34) can be consistently derived from the induced gravity action in eq. (16) with a simple generalization of the complexity=volume prescription in eq. (18). The question of extending the CV proposal to higher curvature theories of gravity was first considered in [120]. For a gravitational theory in $d+1$ dimensions, their proposal was that the usual volume functional should be replaced by a generalized volume of the following form

$$W_{\rm gen}(\mathcal{B}) = \int_{\mathcal{B}} d^d\sigma\,\sqrt{h}\left(\frac{\partial \mathbf{L}}{\partial R_{ijkl}} h_{jk}\left(\alpha_{d+1} n_i n_l + \beta_{d+1} h_{il}\right) + \gamma_{d+1}\right) \tag{39}$$

where α_{d+1}, β_{d+1} and γ_{d+1} are numerical constants (depending on the boundary dimension d).

However, this suggestion by itself can not provide the extrinsic curvature terms in eq. (34). A similar issue was encountered in extending holographic entanglement entropy to higher curvature theories. In particular, it was shown that replacing the Bekenstein-Hawking entropy with the Wald entropy [116, 117, 118] in the RT prescription will not produce the expected entanglement entropy for the boundary theory [130]. Instead, the correct extension required the addition of 'corrections' involving the extrinsic curvature of the extremal surface in the bulk [119]. Hence we propose the generalized CV prescription for higher curvature gravity theories must include additional K-terms. Explicitly, we suggest that the leading contributions take the form

$$\begin{aligned} W_K(\mathcal{B}) = \int_{\mathcal{B}} d^d\sigma\,\sqrt{h}\,\Bigg[&\frac{\partial^2 \mathbf{L}}{\partial R_{ijkl}\partial R^{mnop}} K_{jl}\left(A_{d+1} h_{ik} + B_{d+1} n_i n_k\right) \\ &\times\; K^{np}\left(A_{d+1} h^{mo} + B_{d+1} n^m n^o\right)\Bigg], \end{aligned} \tag{40}$$

where again A_{d+1} and B_{d+1} are numerical constants.

Correspondingly, we propose that the holographic complexity for the island region on the brane can be derived from

$$\mathcal{C}_{\rm V}^{\rm Island} = \max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}}\left[\frac{\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell'}\right], \tag{41}$$

where $\sigma_{\mathbf{R}} = \Sigma_{\mathbf{R}} \cap \text{brane}$ is the quantum extremal surface on the brane – see figure 2. We have introduced the notation $\widetilde{W}_{\rm gen}$, $\widetilde{W}_K$ to indicate these are quantities defined for the d-dimensional gravity theory on the brane. In the following subsections, we seek to compare $\mathcal{C}_{\rm V}^{\rm Island}$ with the leading terms in the holographic CV found in eq. (34) to fix the numerical coefficients in eqs. (39) and (40). This proposal also requires that we maximize the new functional over all profiles $\widetilde{\mathcal{B}}$ anchored to the QES $\sigma_{\mathbf{R}}$, but we leave the discussion of this point to section 2.3.

2.2.1 Generalized Volume on the Island

Substituting eq. (17) for effective Newton's constant and eq. (38) for the CV length scale on the brane into the last line of eq. (34), the leading contribution to the holographic complexity becomes

$$\begin{aligned}\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) &= \frac{V(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell'} \\ &\quad + \frac{L^2}{G_{\rm eff}\,\ell'}\int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma\,\sqrt{\tilde{h}}\left(\frac{\tilde{K}^2}{2(d-1)(d-3)} - \frac{\frac{1}{2}\tilde{R}[\tilde{g}] + \tilde{R}_{ij}[\tilde{g}]\tilde{n}^i\tilde{n}^j}{(d-2)(d-3)} + \cdots\right).\end{aligned} \tag{42}$$

Now our aim is to show that these results can be derived from our proposal for the complexity of the island in eq. (41) applied to the effective gravitational action (16). In particular, to make this match, we must choose the appropriate numerical constants α_d, β_d, γ_d, A_d and B_d for the d-dimensional brane theory. Here, we focus on the first three coefficients appearing in the generalized volume $\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}})$.

To begin with, we consider a general quadratic Lagrangian as

$$\mathbf{L}_{\rm eff} \equiv 16\pi G_{\rm eff}\mathcal{L} = \tilde{R} - 2\Lambda + \lambda_1\tilde{R}^2 + \lambda_2\tilde{R}_{ij}\tilde{R}^{ij}\,. \tag{43}$$

We want to evaluate the generalized volume for the complexity $\mathcal{C}_{\rm V}^{\rm Island}$ in eq. (41). Using the Kronecker delta of rank-two (*i.e.*, $\delta^{ij}_{mn} = \delta^i_m\delta^j_n - \delta^i_n\delta^j_m$), the derivative with respect to Riemannian tensor is explicitly written as [131]

$$\frac{\partial\tilde{R}_{mnop}}{\partial\tilde{R}_{ijkl}} \equiv (\partial\tilde{R})^{ijkl}_{mnop} = \frac{1}{12}\left(\delta^{ij}_{mn}\delta^{kl}_{op} - \frac{1}{2}\delta^{ik}_{mn}\delta^{lj}_{op} - \frac{1}{2}\delta^{il}_{mn}\delta^{jk}_{op} + \delta^{ij}_{op}\delta^{kl}_{mn} - \frac{1}{2}\delta^{ik}_{op}\delta^{lj}_{mn} - \frac{1}{2}\delta^{il}_{op}\delta^{jk}_{mn}\right). \tag{44}$$

It is then straightforward to get the tensor

$$\frac{\partial\mathbf{L}_{\rm eff}}{\partial R_{ijkl}} = \left(\frac{1}{2} + \lambda_1\tilde{R}\right)2\tilde{g}^{i[k}\tilde{g}^{l]j} + \lambda_2\left(\tilde{R}^{i[k}\tilde{g}^{l]j} + \tilde{R}^{j[l}\tilde{g}^{k]i}\right), \tag{45}$$

where $Z^{[ij]} = \frac{1}{2}(Z^{ij} - Z^{ji})$. One can explicitly evaluate the needed contractions to find

$$\begin{aligned}\frac{\partial\mathbf{L}_{\rm eff}}{\partial R_{ijkl}}\tilde{h}_{jk}\left(\alpha_d\tilde{n}_i\tilde{n}_l + \beta_d\tilde{h}_{il}\right) + \gamma_d &= \gamma_d + \left(\frac{1}{2} + \lambda_1\tilde{R}\right)(d-1)\left(\alpha_d - (d-2)\beta_d\right) \\ &\quad + \frac{\lambda_2}{2}\left(\tilde{R}(\alpha_d - 2(d-2)\beta_d) - \tilde{R}^{ij}\tilde{n}_i\tilde{n}_j(\alpha_d + 2\beta_d)(d-2)\right).\end{aligned} \tag{46}$$

Comparing the above results with eq. (42), and taking the effective action (16) on the brane, *i.e.*, choosing the two coupling constants as

$$\lambda_1 = -\frac{dL^2}{4(d-1)(d-2)(d-4)}\,, \qquad \lambda_2 = \frac{L^2}{(d-2)(d-4)}\,, \tag{47}$$

one finds that the three coefficients in the generalized volume should be fixed to

$$\alpha_d = \frac{2(d-4)}{(d-2)(d-3)}\,, \quad \beta_d = 0\,, \quad \gamma_d = \frac{2}{(d-2)(d-3)}\,. \tag{48}$$

As a recap, the comparison between the leading contributions to the volume of the extremal surface $\mathcal{B}$ in the vicinity of the brane for $(d+1)$-dimensional bulk gravity theory in eq. (42) and the generalized volume on the brane determines the numerical coefficients in the latter as in eq. (48). Hence, the resulting generalized volume reads

$$\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) = \frac{2}{(d-2)(d-3)}\int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma\,\sqrt{\det\tilde{h}_{ab}}\left(1 + (d-4)\frac{\partial\mathbf{L}_{\rm eff}}{\partial\tilde{R}_{ijkl}}\tilde{n}_i\tilde{h}_{jk}\tilde{n}_l\right). \tag{49}$$

Furthermore, we propose that this result of the generalized volume can be used in extending the holographic complexity=volume conjecture for higher curvature gravity theories in general, as in eqs. (6) and (7). In section 3, we will test this proposal further by considering higher curvature gravity in the bulk of our holographic model.

2.2.2 K-term on the Island

As discussed above, the generalized volume (39) by itself fails to provide the full holographic complexity on the island due to the appearance of terms involving the extrinsic curvature $\tilde{K}$ on the brane. Inspired by the Wald-Dong entropy, we suggested the addition of K-terms to the generalized volume. At the second-order, we can produce a covariant quantity by contracting the tensor $\frac{\partial^2 \mathbf{L}_{\text{eff}}}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}}$, with the tensors built from the three independent symmetric tensors $\tilde{K}_{ij}, \tilde{h}_{ij}, \tilde{n}_i\tilde{n}_j$. The simplest choice is the following

$$\widetilde{W}_K(\widetilde{\mathcal{B}}) = \int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma \sqrt{\tilde{h}} \frac{\partial^2 \mathbf{L}_{\text{eff}}}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}} \tilde{K}_{jl} \left(A_d \tilde{h}_{ik} + B_d \tilde{n}_i \tilde{n}_k \right) \tilde{K}^{np} \left(A_d \tilde{h}^{mo} + B_d \tilde{n}^m \tilde{n}^o \right) , \tag{50}$$

where as before, $\mathbf{L}_{\text{eff}} = 16\pi G_{\text{eff}} \mathcal{L}_{\text{eff}}$. Our goal is then to fix the two numerical coefficients A_d, B_d.

To compute $\frac{\partial^2 \mathbf{L}_{\text{eff}}}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}}$, we need to use the second derivative

$$\begin{aligned}
\frac{\partial^2 (\tilde{R}^2)}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}} &= \frac{1}{2} \left(\tilde{g}^{ik} \tilde{g}^{jl} - \tilde{g}^{il} \tilde{g}^{jk} \right) \left(\tilde{g}_{mo} \tilde{g}_{np} - \tilde{g}_{mp} \tilde{g}_{no} \right) , \\
\frac{\partial^2 (\tilde{R}_{i_1 j_1} \tilde{R}^{i_1 j_1})}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}} &= \frac{1}{2} \tilde{g}_{rs} \left((\partial \tilde{R})^{irks}_{mnop} \, \tilde{g}^{jl} - (\partial \tilde{R})^{irls}_{mnop} \, \tilde{g}^{jk} - (\partial \tilde{R})^{jrks}_{mnop} \, \tilde{g}^{il} + (\partial \tilde{R})^{jrls}_{mnop} \, \tilde{g}^{ik} \right) ,
\end{aligned} \tag{51}$$

where the tensor $(\partial \tilde{R})^{ijkl}_{mnop}$ is the first derivative defined in eq. (44).

Applying the second derivative (51) to the effective action in eq. (16), one finds that the proposed $\widetilde{W}_K$ reduces to

$$\begin{aligned}
\widetilde{W}_K(\widetilde{\mathcal{B}}) = \int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma \sqrt{\tilde{h}} \Bigg[& \frac{\lambda_1 \tilde{K}^2}{2} \left((d-2) A_d - B_d \right)^2 \\
& + \frac{\lambda_2}{8} \left(\tilde{K}^2 \left(B_d^2 - 2 A_d B_d + (3d-7) A_d^2 \right) + \tilde{K}_{ij} \tilde{K}^{ij} ((d-3) A_d - B_d)^2 \right) \Bigg] .
\end{aligned} \tag{52}$$

Noting the absence of $\tilde{K}_{ij}\tilde{K}^{ij}$ term in eq. (42), we can fix

$$B_d = (d-3)\, A_d \, . \tag{53}$$

Further, comparing eqs. (42) and (52), the last parameter is fixed as

$$A_d^2 = \frac{4(d-4)}{(d-2)^2 (d-3)} \, . \tag{54}$$

Finally, we can write the K-term (50) as

$$\begin{aligned}
\widetilde{W}_K(\widetilde{\mathcal{B}}) = & \frac{4(d-4)}{(d-2)^2 (d-3)} \int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma \sqrt{\tilde{h}} \frac{\partial^2 \mathbf{L}_{\text{eff}}}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}} \\
& \times \tilde{K}_{jl} \left(\tilde{h}_{ik} + (d-3) \tilde{n}_i \tilde{n}_k \right) \tilde{K}^{np} \left(\tilde{h}^{mo} + (d-3) \tilde{n}^m \tilde{n}^o \right) .
\end{aligned} \tag{55}$$

Although we have a successful match here, we should point out that the K-term defined in eq. (40) was chosen for its simplicity and in a similar spirit to the analogous term appearing in the Wald-Dong entropy. However, it is easy to find many other ways in contracting all the indexes in $\frac{\partial^2 \mathbf{L}_{\text{eff}}}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}}$ with two extrinsic curvatures and combinations of $\tilde{h}^{ij}$ and $\tilde{n}^i \tilde{n}^j$. Some examples would include

$$\begin{aligned}
& \frac{\partial^2 \mathbf{L}_{\text{eff}}}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}} \tilde{K}_{ik} \tilde{K}_{jl} \left(A_1 \, \tilde{g}^{mo} + B_1 \, \tilde{n}^m \tilde{n}^o \right) \left(A_1 \, \tilde{g}^{np} + B_1 \, \tilde{n}^n \tilde{n}^p \right) , \\
& \frac{\partial^2 \mathbf{L}_{\text{eff}}}{\partial \tilde{R}_{ijkl} \partial \tilde{R}^{mnop}} \tilde{K}_i^m \tilde{K}_j^n \left(A_2 \, \tilde{g}_{kl} + B_2 \, \tilde{n}_k \tilde{n}_l \right) \left(A_2 \, \tilde{g}^{op} + B_2 \, \tilde{n}^o \tilde{n}^p \right) .
\end{aligned} \tag{56}$$

Note that in the first case, both extrinsic curvatures are contracted with the indices of a single variation with respect to the Riemann tensor, while in the second, the two indices of each individual extrinsic curvature are contracted with different variations. Note that no terms with these structures appear in the K corrections of the Wald-Dong entropy [119]. However, at present, we do not have a strong reason to rule out these expressions or their linear combinations. This means that in general, there is much more ambiguity in defining $\widetilde{W}_K(\widetilde{\mathcal{B}})$ than indicated in eq. (50) and the numerical coefficients can not be completely fixed. This stands in contrast with the Wald-Dong entropy, for which a unique extrinsic curvature term is derived from the replica trick [119]. Unfortunately, we do not have a proper derivation of the complexity=volume proposal, which we might extend to probe the complexity of theories dual to higher derivative gravity. However, we will test our simple ansatz in section 3 by continuing to show that our calculations are consistent with higher curvature gravity in the bulk.

We should also add that we expect that eq. (55) is only the first in an infinite series of corrections involving the extrinsic curvatures, as appears in the Wald-Dong entropy. Here, we have limited ourselves to the terms quadratic in $\tilde{K}$ because we only evaluated the effective action (16) to include the terms which are quadratic in the curvatures. It may be interesting to extend our calculations to third order, from which we expect to find $\tilde{K}^3$ contributions to $\widetilde{W}_K$.

In summary, we find that the leading contributions from the geometry in the vicinity of the brane from usual subregion-CV proposal for the bulk Einstein gravity suggests a generalized CV formula for the induced gravity theory on the brane, *i.e.,*

$$\mathrm{ext}\left[\frac{V(\mathcal{B})}{G_{\text{bulk}}\,\ell}\right] \simeq \frac{\widetilde{W}_{\text{gen}}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\text{eff}}\,\ell'}\,, \tag{57}$$

where the generalized volume $\widetilde{W}_{\text{gen}}$ and $\widetilde{W}_K$ term are fixed in eqs. (49) and (55), respectively. Further, the scales, ℓ in the bulk and ℓ' on the brane, are related by eq. (38). We should stress that the above identification relies on the extremality of the bulk surface $\mathcal{B}$, which was required in deriving eq. (34). As commented above, we propose that these results can be used to generalize the holographic complexity=volume conjecture for higher curvature gravity theories in general, as in eqs. (6) and (7). Further, we will test this proposal in section 3, by examining our holographic model with higher curvature gravity in the bulk.

2.2.3 DGP Term on the Brane

In a construction analogous to that of Dvali, Gabadadze and Porrati (DGP) [111], one can also add an intrinsic Einstein term to brane action as follows – for details see [1]

$$I_{\text{brane}} = -(T_o - \Delta T)\int d^dx\sqrt{-\tilde{g}} + \frac{1}{16\pi G_{\text{brane}}}\int d^dx\sqrt{-\tilde{g}}\,\tilde{R}\,, \tag{58}$$

which yields the new effective gravitational action on d-dimensional brane as

$$\begin{aligned} I_{\text{eff}} = \frac{1}{16\pi G_{\text{eff}}}\int d^dx\sqrt{-\tilde{g}}\left[\frac{(d-1)(d-2)}{\ell_{\text{eff}}^2} + \tilde{R}(\tilde{g})\right] \\ + \frac{1}{16\pi G_{\text{RS}}}\int d^dx\sqrt{-\tilde{g}}\left[\frac{L^2}{(d-4)(d-2)}\left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{d}{4(d-1)}\tilde{R}^2\right) + \cdots\right]\,. \end{aligned} \tag{59}$$

In the first line of this action, the new effective Newton constant associated with Einstein term is given by

$$\frac{1}{G_{\text{eff}}} = \frac{2L}{(d-2)G_{\text{bulk}}} + \frac{1}{G_{\text{brane}}}\,, \tag{60}$$

while in the second line, $G_{\text{RS}} = (d-2)G_{\text{bulk}}/(2L)$.

This provides an interesting framework to extend our generalized proposal for complexity=volume. In the case of holographic entanglement entropy, one can clearly argue that the DGP term introduces a brane contribution in eq. (4) by simply following the derivations in [103, 132]. Unfortunately, such a derivation is lacking for the CV formula, and so we will simply say that it is natural to expect that with a DGP term, the CV proposal should have a similar extension to include a contribution proportional to the volume of $\widetilde{\mathcal{B}} = \mathcal{B} \cap \text{brane}$. More precisely, if the extremal surface crosses a DGP brane, then eq. (18) would become

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}} \left[\frac{V(\mathcal{B})}{G_{\rm bulk}\,\ell} + \frac{V(\widetilde{\mathcal{B}})}{G_{\rm brane}\,\ell'} \right], \tag{61}$$

where ℓ and ℓ' are the independent 'unknown' length scales for the bulk and brane, as are expected for the CV ansatz.

Now if we examine the leading contributions from the bulk geometry in the vicinity of the brane, as in eq. (37), the above expression yields

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) = \frac{2L\,V(\widetilde{\mathcal{B}})}{(d-1)G_{\rm bulk}\ell} + \cdots + \frac{V(\widetilde{\mathcal{B}})}{G_{\rm brane}\,\ell'} = \frac{V(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell'} + \cdots \tag{62}$$

where to produce the second equality, we have used eq. (38) to relate the two length scales, ℓ and ℓ', and then eq. (60) applied for effective Newton's on the brane. Hence, we see that combining eqs. (38) and (61) produces a consistent framework with which to understand complexity=volume for the brane theory. While we have ignored the higher curvature terms above, it is clear that including the DGP term on the brane leads to the same results as eqs. (49) and (55) with the same dimensionless coefficients for the new gravity theory (59) on the brane. It would be interesting to examine if this approach continues to succeed if one were to extend the brane action (58) with higher curvature terms.

2.3 Maximal Islands

Up to this point, we have shown that with the usual subregion-CV proposal (18) and applying the FG expansion for extremal surfaces in the bulk, integrating the leading contributions in the vicinity of the brane produces a generalized CV formula for the induced theory on the brane. In particular, the new complexity functional (41) is easily derived from the higher-curvature gravity action on the brane (16) using eqs. (49) and (55). We stress that the above identification relies on the extremality of the surface $\mathcal{B}$ in the bulk, which was required in deriving eq. (34). However, at this point, we want to turn to the appearance of the maximization that appears in eq. (41).

Here it is enlightening to return to the relation between the island rule (3) on the brane and the RT prescription (4) in the bulk – see discussions in [1, 2]. Our first observation is that analogous to our analysis above, carefully examining the extremal RT surfaces near the brane shows that the Bekenstein-Hawking formula in the island rule (3) actually expands to the Wald-Dong entropy for the gravity action induced on the brane [1]. As in the above, this requires that we solve the local equations in the bulk which extremize the RT surfaces away from the brane, but in doing so, one produces a family of solutions that are extremal in the bulk (and have the fixed boundary conditions on the asymptotic AdS boundary) but which have different profiles on the brane. Finding the correct solution amongst this family can be characterized in terms of satisfying a particular boundary condition at the brane – see eq. (4.17) in [1]. However, a more pragmatic approach is to simply find the correct solution by varying over the possible profiles on the brane to see which one actually minimizes the entropy functional in eq. (4). This second stage is then precisely the extremization appearing in the island rule (3).

Of course, the same narrative applies here to the holographic complexity. Recall that our boundary state was defined on a region $\mathbf{R} = \mathbf{R}_{\rm L} \cup \mathbf{R}_{\rm R}$, where the subregions $\mathbf{R}_{\rm L,R}$ sit to either side of the conformal defect in the asymptotic boundary, as shown in figure 2. Similarly, we divide the bulk surface $\mathcal{B} = \mathcal{B}_{\rm L} \cup \mathcal{B}_{\rm R}$ into the two components on either side of the brane. For both of these components, we demand that these surfaces are extremal away from the brane by solving eq. (23), subject to the boundary condition that $\mathcal{B}_{\rm L,R}$ are anchored at the corresponding $\mathbf{R}_{\rm L,\,R}$ on the asymptotic boundary, the RT surface $\Sigma_{\mathbf{R}}$ in the bulk, and the island $\widetilde{\mathcal{B}}$ on the brane, *i.e.*, $\partial\mathcal{B}_{\rm L} = \mathbf{R}_{\rm L} \cup \Sigma_{\mathbf{R}} \cup \widetilde{\mathcal{B}}$ (and similarly for the right side). In particular, both surfaces $\mathcal{B}_{\rm L,R}$ intersect the brane along with the common profile $\widetilde{\mathcal{B}}$, however, this profile is left undetermined at this stage. Hence we find a wide family of codimension-one surfaces which are extremal in the bulk, *i.e.,* away from the brane. Then, to find to correct extremal surface, we must finally maximize that volume by varying over the possible profiles. That is, we have decomposed the extremization of $\mathcal{B}$ into two steps:

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) \equiv \max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}} \left(\underset{\mathcal{B}_{\rm L},\mathcal{B}_{\rm R}}{\rm ext} \left[\frac{V(\mathcal{B}_{\rm L}) + V(\mathcal{B}_{\rm R})}{G_{\rm bulk}\ell} \right] \right) . \tag{63}$$

Combined with the near-brane contributions in eq. (57), this equation then becomes

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) = \max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}} \left[\frac{\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\rm eff}\ell'} + \cdots \right] , \tag{64}$$

using the generalized volume and K-term in eqs. (49) and (55), respectively.

The ellipsis in eq. (64) indicates the contributions coming far from the brane, *i.e.,* from regions with $\theta_{\rm B} \ll \theta\ \pi$ with the coordinates in eq. (10). It is interesting to note that the analogous contributions for the holographic entanglement entropy (4) provide the quantum contributions when interpreted in terms of the effective d-dimensional brane perspective, *i.e.,* $S_{\rm QFT}(\mathbf{R}\cup \text{islands})$ in eq. (3). Hence it is natural to expect that the corresponding contribution in the holographic complexity constitutes a (semiclassical) contribution in the bath region R combined with the island $\widetilde{\mathcal{B}}$ on the brane. We return to discuss this point in section 14.

3 Higher Curvature Gravity in the Bulk

In the previous section, we showed how holographic complexity naturally arises for the induced gravity theory on the brane in the doubly holographic model of [1, 2]. However, beginning with the usual complexity=volume conjecture (2) for ordinary Einstein gravity in the bulk, we were lead to a generalization of the CV proposal suitable for higher curvature gravity, such as the induced theory (16) on the brane. Our proposal is that the new functional appearing for the holographic complexity on the brane should in fact serve to provide a generalized complexity=volume conjecture for any higher curvature theory

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}} \left[\frac{W_{\rm gen}(\mathcal{B}) + W_K(\mathcal{B})}{G_{\rm N}\,\ell} \right] , \tag{65}$$

with the functionals given in eq. (7). As indicated, the maximization is performed over all possible codimension-one surfaces $\mathcal{B}$ anchored at the subregion $\mathbf{R}$ on the asymptotic boundary and the corresponding RT surface $\Sigma_{\mathbf{R}}$ in the bulk. Of course, this proposal reduces to the standard CV conjecture (2) when the bulk theory is Einstein gravity.

In this section, we examine a new consistency check for our new proposal by considering higher curvature gravity in the bulk. That is, we start by considering a theory of higher curvature gravity in the $(d+1)$-dimensional bulk and apply eq. (65) for the holographic complexity. Then following the analogous calculations as in section 2, we show that the holographic complexity for the induced theory on the d-dimensional brane takes the same form, *i.e.,*

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) \simeq \mathcal{C}_{\rm V}^{\rm Island} = \max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}} \left[\frac{\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell'} \right] , \tag{66}$$

where the functionals $\widetilde{W}_{\rm gen}$ and $\widetilde{W}_K$ are adapted to the new spacetime dimension and the induced gravity action on the brane.

Our calculations will refer to several different hypersurfaces and the corresponding extrinsic and intrinsic curvatures associated with these surfaces – see figure 4. In order to clarify the notation, we list the different curvatures here:

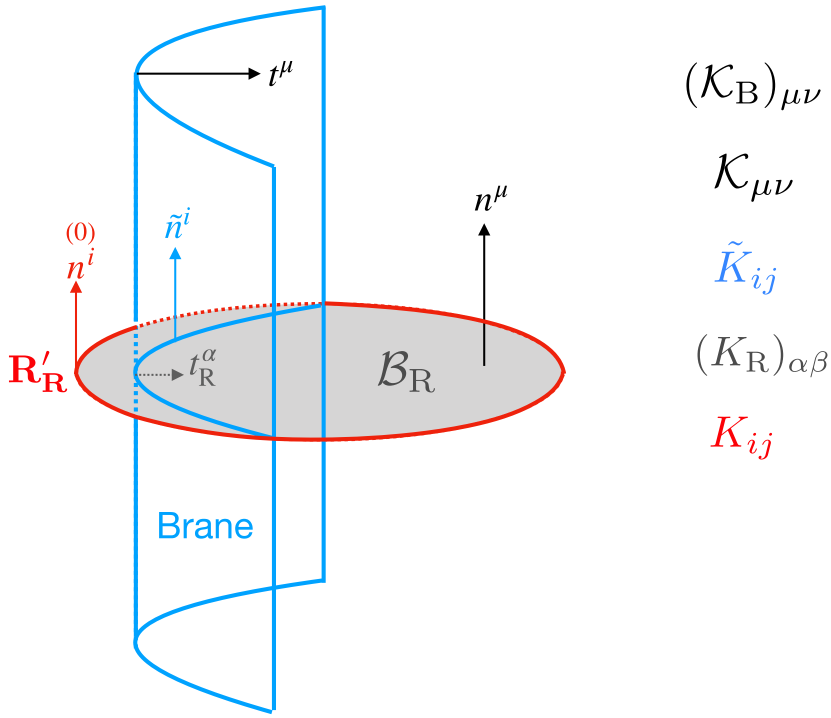

Figure 4: Different hypersurfaces in the doubly holographic system and their corresponding extrinsic curvatures.

- the $(d+1)$-dimensional bulk, with intrinsic curvature $\mathcal{R}[g^{\text{bulk}}_{\mu\nu}]$;
- the spacelike surfaces $\mathcal{B}$ embedded in the $(d+1)$-dimensional bulk, with timelike normal n^μ, extrinsic curvature $\mathcal{K}_{\mu\nu}$ and intrinsic curvature $R_\mathcal{B}[h_{\alpha\beta}]$;
- the brane embedded in the $(d+1)$-dimensional bulk, with spacelike normal t^μ, extrinsic curvature $(\mathcal{K}_\mathrm{B})_{\mu\nu}$ and intrinsic curvature $\tilde{R}[\tilde{g}_{ij}]$;
- the island region $\widetilde{\mathcal{B}} = \mathcal{B} \cap \text{brane}$ (with $\mathcal{B} = \mathcal{B}_\mathrm{L} \cup \mathcal{B}_\mathrm{R}$) thought of as being embedded in the surface $\mathcal{B}_\mathrm{R}$, with spacelike normal t^α_R and extrinsic curvature $(K_\mathrm{R})_{\alpha\beta}$; similarly for $\widetilde{\mathcal{B}}$ embedded in the surface $\mathcal{B}_\mathrm{L}$, we have the spacelike normal t^α_L to the island and extrinsic curvature $(K_\mathrm{L})_{\alpha\beta}$;
- the island region $\widetilde{\mathcal{B}}$ thought of as being embedded in the brane, with timelike normal $\tilde{n}^i$ and extrinsic curvature $\tilde{K}_{ij}$;
- the subregion $\mathbf{R}'$ (where $\mathcal{B}_{\mathrm{L,R}}$ would meet a virtual asymptotic boundary at $z = 0$) embedded in the asymptotic boundary, with timelike normal $\overset{(0)}{n}{}^i$, extrinsic curvature K_{ij} and intrinsic curvature $R_\Sigma[\overset{(0)}{h}_{ab}]$;
- the virtual asymptotic boundary (see above) with intrinsic curvature $R[\overset{(0)}{g}_{ij}]$.

3.1 Holographic Complexity for Gauss-Bonnet Gravity

Our first consistency check with higher curvature gravity consists of having Gauss-Bonnet gravity in the bulk. The bulk gravitation action is therefore given by

$$I^{\text{GB}}_{\text{bulk}} = \frac{1}{16\pi G_{\text{bulk}}} \int d^{d+1}y\,\sqrt{-g}\left[\frac{d(d-1)}{L^2} + \mathcal{R}[g_{\mu\nu}] + \lambda_{\text{GB}}\,\mathcal{L}_{\text{GB}}\right] + I^{\text{GB}}_{\text{surf}}\,, \tag{67}$$

with the Gauss-Bonnet term defined by

$$\lambda_{\rm GB} = \frac{L^2\lambda}{(d-2)(d-3)}\,, \qquad \mathcal{L}_{\rm GB} = \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2\,. \tag{68}$$

Here, we have explicitly included the boundary term $I^{\rm GB}_{\rm surf}$ to emphasize that GB gravity has a well-posed variational principle with Dirichlet boundary conditions $\delta g_{\mu\nu} = 0$ [133]. Similar to the standard Gibbons-Hawking-York term, the extended boundary term is given by [134]

$$I^{\rm GB}_{\rm surf} = \frac{1}{16\pi G_{\rm bulk}}\oint d^dx\sqrt{-\tilde g}\left[2\mathcal{K}_{\rm B} + \frac{4L^2\lambda}{(d-2)(d-3)}\left(\tilde R\mathcal{K}_{\rm B} - 2\tilde R_{ij}\mathcal{K}^{ij}_{\rm B} + J\right)\right]\,, \tag{69}$$

where $\tilde R_{ij}$ and $\mathcal{K}_{ij}$ denote the Ricci tensor and extrinsic curvature associated with the boundary geometry, and J is the trace of

$$J_{ij} \equiv \frac{1}{3}\left(2\mathcal{K}\,\mathcal{K}_{ik}\,\mathcal{K}^k{}_j + \mathcal{K}^{kl}\,\mathcal{K}_{kl}\,\mathcal{K}_{ij} - 2\mathcal{K}_{ik}\,\mathcal{K}^{kl}\,\mathcal{K}_{lj} - \mathcal{K}^2\mathcal{K}_{ij}\right)\,. \tag{70}$$

The presence of the Gauss-Bonnet term modifies the Israel junction conditions (11) determining the position of the brane as [135, 136]

$$\Delta(\mathcal{K}_{\rm B})_{ij} - \tilde g_{ij}\,\Delta\mathcal{K}_{\rm B} + 2\lambda_{\rm GB}\,\Delta\left[\tilde E_{iklj}\mathcal{K}^{kl}_{\rm B} + 3J_{ij}(\mathcal{K}_{\rm B}) - J\,\tilde g_{ij}\right] = 8\pi G_{\rm bulk}\,S_{ij}\,, \tag{71}$$

where the tensor $\tilde E^{ijkl}$ is defined as

$$\tilde E^{ijkl} = 2\tilde R\,\tilde g^{i[k}\tilde g^{l]j} - 4\left(\tilde R^{i[k}\tilde g^{l]j} + \tilde R^{j[l}\tilde g^{k]i}\right) + 2\tilde R^{ijkl}\,. \tag{72}$$

This generalized Israel junction condition can be derived by considering a thin shell and taking the thickness of the shell $\delta z \to 0$ – see [136] for details. Similar to the derivation of the Israel junction condition for Einstein gravity, one can also obtain the generalized Israel junction condition by considering the gravitational action on either side of the brane with the boundary term in eq. (69) at the brane [135]. That is, with these boundary terms, we solve the gravity equations in the bulk away from the brane with some fixed boundary condition for $g_{\mu\nu}$ at the brane (as well as asymptotic infinity, of course). Then we solve the full system by allowing $g_{\mu\nu}$ at the brane surface to vary and gluing the two surfaces together while demanding that the generalized Israel boundary condition in eq. (71) is satisfied. We should note that the latter approach is implicitly adopted in deriving the induced gravity action in eq. (74) – see appendix A of [128]. More specifically, in evaluating the bulk action in the vicinity of the brane, it is essential to include the contribution of the boundary term.

While the length scale L defines the cosmological constant in the action (67), the curvature scale $\tilde L$ of the AdS vacuum solution in the Gauss-Bonnet gravity is

$$\tilde L^2 = \frac{L^2}{f_\infty}\,, \qquad \text{with} \qquad f_\infty = \frac{1-\sqrt{1-4\lambda}}{2\lambda}\,. \tag{73}$$

The induced gravitational action on the brane is given by [128]

$$\begin{aligned} I^{\rm GB}_{\rm eff} \;=\; \frac{1}{16\pi G_{\rm eff}}\int_{\rm brane} d^dx\,\sqrt{-\tilde g}\,\bigg[&\frac{(d-1)(d-2)}{\ell^2_{\rm eff}} + \tilde R[\tilde g] \\ &+\kappa_1\left(\tilde R_{ij}\tilde R^{ij} - \frac{d}{4(d-1)}\tilde R^2\right) + \kappa_2\tilde C_{ijkl}\tilde C^{ijkl} + \cdots\bigg]\,, \end{aligned} \tag{74}$$

where the effective Newton constant and coupling constants are

$$\frac{1}{G_{\rm eff}} = \frac{2\tilde L}{d-2}\frac{1+2\lambda f_\infty}{G_{\rm bulk}}\,, \tag{75}$$

$$\kappa_1 = \frac{\tilde L^2}{(d-2)(d-4)}\,\frac{1-6\lambda f_\infty}{1+2\lambda f_\infty}\,, \qquad \kappa_2 = \frac{\tilde L^2}{(d-3)(d-4)}\,\frac{\lambda f_\infty}{1+2\lambda f_\infty}\,,$$

and $\tilde{C}_{ijkl}$ denotes the Weyl tensor on the brane. We also note that the expression for the scale ℓ_{eff} in eq. (17) is replaced by

$$\frac{1}{\ell_{\text{eff}}^2} = \frac{2}{\tilde{L}^2(1+2\lambda f_\infty)}\left(1-\frac{2}{3}\lambda f_\infty - \frac{4\pi\tilde{L}G_{\text{bulk}}T_o}{d-1}\right). \tag{76}$$

In the following, we adopt our proposal (65) to evaluate the holographic complexity for $(d+1)$-dimensional GB gravity in the bulk and compare the leading terms in the FG expansion near the brane to the complexity of the island in the d-dimensional effective higher-curvature gravity on the brane. As we will see below, the leading terms in the generalized holographic CV for the boundary subregion agree with the proposed complexity (66) of the island. We see this consistency as extra support for our proposal for holographic complexity for higher-curvature gravity theory.

In evaluating the holographic complexity for $(d+1)$-dimensional bulk theory, we consider a codimension-one slice $\mathcal{B}$, with time-like normal n^μ and induced metric $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Following the analysis in the previous section, our first step is to extremize the complexity functional on $\mathcal{B}$ away from the brane, while leaving the profile $\widetilde{\mathcal{B}}$ on the brane undetermined. In order to ensure that this involves a well-defined variational principle, we actually extend eq. (65) to include a surface term

$$\mathcal{C}_{\text{V}}^{\text{sub}}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}}\left[\frac{W_{\text{gen}}(\mathcal{B})+W_K(\mathcal{B})+W_{\text{bdy}}(\partial\mathcal{B}_{\text{L}}\cup\partial\mathcal{B}_{\text{R}})}{G_{\text{bulk}}\ell}\right]. \tag{77}$$

Of course, the generalized volume W_{gen} and the K-term are defined in eq. (7). We do not specify the details of W_{bdy} but its form will become evident in the following. Further, we note that we are evaluating this expression on $\partial\mathcal{B}_{\text{L}}\cup\partial\mathcal{B}_{\text{R}}$. In particular, this contribution appears on (either side of) $\widetilde{\mathcal{B}}$, which is not really a boundary of the full surface $\mathcal{B}=\mathcal{B}_{\text{L}}\cup\mathcal{B}_{\text{R}}$. Hence we are treating W_{bdy} in a manner to the gravitational surface (69), which appears on either side of the surface defined by the brane – see discussion below eq. (72).

Let us begin by evaluating W_{gen} for the GB theory. It is straightforward to obtain

$$\begin{aligned}
&\alpha_{d+1}\frac{\partial(\mathcal{R}+\lambda_{\text{GB}}\mathcal{L}_{\text{GB}})}{\partial\mathcal{R}_{\mu\nu\rho\sigma}}n_\mu h_{\nu\rho}n_\sigma+\gamma_{d+1}\\
&=\frac{2(d-3)}{(d-1)(d-2)}\left(\frac{d}{2}+\lambda_{\text{GB}}(d-2)(\mathcal{R}+2\mathcal{R}^{\mu\nu}n_\mu n_\nu)\right)+\frac{2}{(d-1)(d-2)}\\
&=1+\lambda_{\text{GB}}\frac{2(d-3)}{(d-1)}\left(\mathcal{R}+2\mathcal{R}^{\mu\nu}n_\mu n_\nu\right),
\end{aligned} \tag{78}$$

where the values of $\alpha_{d+1},\gamma_{d+1}$ are given using eq. (48). Now using eq. (51), the W_K term yields

$$\begin{aligned}
&A_{d+1}^2\left(\frac{\partial^2(\mathcal{R}+\lambda_{\text{GB}}\mathcal{L}_{\text{GB}})}{\partial\mathcal{R}_{\mu_1\nu_1\rho_1\sigma_1}\partial\mathcal{R}_{\mu_2\nu_2\rho_2\sigma_2}}\mathcal{K}_{\nu_1\sigma_1}\left(h_{\mu_1\rho_1}+(d-2)n_{\mu_1}n_{\rho_1}\right)\mathcal{K}_{\mu_2\sigma_2}\left(h_{\mu_2\sigma_2}+(d-2)n_{\mu_2}n_{\sigma_2}\right)\right),\\
&=\frac{4\lambda_{\text{GB}}(d-3)}{(d-1)^2(d-2)}\left(\frac{\mathcal{K}^2}{2}-\frac{\mathcal{K}^2}{2}\left((d+1)(d-4)+8\right)+\frac{1}{2}\left(\mathcal{K}^2+(d-1)(d-2)\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\right)\right),\\
&=\frac{2\lambda_{\text{GB}}(d-3)}{(d-1)}\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu}-\mathcal{K}^2\right),
\end{aligned} \tag{79}$$

where A_{d+1}^2 was replaced using eq. (54). Noting Gauss's "Theorema Egregium" for the hypersurface $\mathcal{B}$ with the induced metric $h_{\alpha\beta}$ and intrinsic curvature $R_{\mathcal{B}}$, *i.e.,*

$$R_{\mathcal{B}}[h_{\alpha\beta}] = \mathcal{R}[g_{\mu\nu}] + \left(2\mathcal{R}^{\mu\nu}n_\mu n_\nu - \mathcal{K}^2+\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\right), \tag{80}$$

we can recast the λ_{GB}-terms into the intrinsic geometric quantities of hypersurface $\mathcal{B}$, *i.e.,*

$$W_{\text{gen}}(\mathcal{B})+W_K(\mathcal{B}) = \int_{\mathcal{B}} d^{d-1}\sigma dz\sqrt{\det h_{\alpha\beta}}\left(1+\frac{2L^2\lambda}{(d-1)(d-2)}R_{\mathcal{B}}\right). \tag{81}$$

Given the above result, it is straightforward to derive the desired $W_{\rm bdy}$. Namely, extremizing this generalized volume functional will have a good variational principle if we add the usual 'Gibbons-Hawking' term on the boundary. The island contribution on the brane is then given by

$$W_{\rm bdy}(\widetilde{\mathcal{B}}) = \frac{4L^2\lambda}{(d-1)(d-2)} \int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma \sqrt{\det \tilde{h}}\, (K_{\rm L} + K_{\rm R})\,, \tag{82}$$

where $K_{\rm L}, K_{\rm R}$ denote the trace of the extrinsic curvature of $\widetilde{\mathcal{B}}$ embedded in $\mathcal{B}_{\rm L}, \mathcal{B}_{\rm R}$, respectively. We may note the importance of this term by observing that the GB contribution in eq. (81) is negative for $\lambda > 0$ because $R_{\mathcal{B}}$ is negative. On the other hand, in eq. (75), we see the effective Newton constant on the brane has a positive contribution from the Gauss-Bonnet term. Hence for the corrections of the GB term to the coefficient of the volume term in the holographic complexity on the brane to match, there must be an additional contribution beyond eq. (81). Indeed, we will find the extra contribution from eq. (82) yields the desired match. Similar to the extremizing condition for entanglement entropy in GB gravity (*e.g.,* see [137, 138]), it is straightforward to find that the generalized CV functional (77) for GB gravity is extremized by the following local equation

$$\mathcal{K} + \frac{2L^2\lambda}{(d-2)(d-3)} \left(R_{\mathcal{B}}\, \mathcal{K} - 2R_{\mathcal{B}}^{\alpha\beta}\mathcal{K}_{\alpha\beta} \right) = 0\,. \tag{83}$$

This extremizing condition generalizes eq. (81) for Einstein gravity to GB gravity in the bulk and ensures that $\mathcal{B}$ is extremal away from the brane.

In summary, applying our proposal (77) to GB gravity in the bulk, the generalized holographic complexity becomes

$$\begin{aligned} \mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) = \max_{\partial\mathcal{B} = \mathbf{R}\cup\Sigma_{\mathbf{R}}} \frac{1}{G_{\rm bulk}\ell} \Bigg[& V(\mathcal{B}) \\ & + \frac{2L^2\lambda}{(d-1)(d-2)} \left(\int_{\mathcal{B}} d^{d-1}\sigma dz \sqrt{\det h}\, R_{\mathcal{B}} + 2\int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma\sqrt{\det \tilde{h}}\, (K_{\rm L}+K_{\rm R}) \right) \Bigg]\,, \end{aligned} \tag{84}$$

and the resulting condition for extremality of $\mathcal{B}$ in the bulk is given by eq. (83).

3.1.1 Holographic Complexity from Induced Gravity

Our goal is to compare the near-brane contributions of eq. (84) to the proposed holographic complexity (41) on the brane. Hence taking the effective action on the brane in eq. (74), we must evaluate

$$\mathcal{C}_{\rm V}^{\rm Island} \equiv \max_{\partial\widetilde{\mathcal{B}} = \sigma_{\mathbf{R}}} \left[\frac{\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell'} \right]\,, \tag{85}$$

where the generalized volume and K-term are defined in eqs. (49) and (55). The boundary term $\widetilde{W}_{\rm bdy}$ does not affect the calculation of the complexity of the island. In fact, most of $\mathcal{C}_{\rm V}^{\rm Island}$ is the same as that found in section 2 (see eq. (42)) except for the contributions from $\tilde{C}_{ijkl}\tilde{C}^{ijkl}$ term in (74). Noting the square of Weyl tensor reads

$$\tilde{C}_{ijkl}\tilde{C}^{ijkl} = \tilde{R}_{ijkl}\tilde{R}^{ijkl} - \frac{4}{d-2}\tilde{R}_{ij}\tilde{R}^{ij} + \frac{2}{(d-1)(d-2)}\tilde{R}^2\,, \tag{86}$$

and using eq. (44) again, the following tensor contraction gives

$$\begin{aligned} & \frac{\partial(\tilde{C}_{ijkl}\tilde{C}^{ijkl})}{\partial\tilde{R}_{ijkl}}\tilde{n}_i\tilde{h}_{jk}\tilde{n}_l \\ &= \left(-2\tilde{R}^{ab}\tilde{n}_a\tilde{n}_b - \frac{4}{(d-2)}\frac{1}{2}\left(\tilde{R} - \tilde{R}^{ab}\tilde{n}_a\tilde{n}_b(d-2)\right) + \frac{2}{(d-1)(d-2)}\tilde{R} \right) \\ &= 0\,, \end{aligned} \tag{87}$$

where we show the individual contributions from $\tilde{R}_{ijkl}\tilde{R}^{ijkl}$, $\tilde{R}_{ij}\tilde{R}^{ij}$ and $\tilde{R}^2$, respectively, on the second line. Although the Weyl tensor term does not contribute to the generalized volume, it still plays a role in the K-term. Using eq. (51) and also

$$\frac{\partial^2(\tilde{R}_{i_1j_1k_1l_1}\tilde{R}^{i_1j_1k_1l_1})}{\partial\tilde{R}_{ijkl}\partial\tilde{R}^{mnop}} = 2\frac{\partial\tilde{R}_{mnop}}{\partial\tilde{R}_{ijkl}} \equiv 2(\partial\tilde{R})^{ijkl}_{mnop}\,, \tag{88}$$

we find the new contribution to $\tilde{W}_{\tilde{K}}$ from the Weyl-tensor-squared term in the induced action is given by

$$\begin{aligned}
&\kappa_2 A_d^2\frac{\partial^2(\tilde{C}_{ijkl}\tilde{C}^{ijkl})}{\partial\tilde{R}_{i_1j_1k_1l_1}\partial\tilde{R}_{i_2j_2k_2l_2}}\tilde{K}_{j_1l_1}\left(\tilde{h}_{i_1k_1}+(d-3)\tilde{n}_{i_1}\tilde{n}_{k_1}\right)\tilde{K}_{j_2l_2}\left(\tilde{h}_{i_2k_2}+(d-3)\tilde{n}_{i_2}\tilde{n}_{k_2}\right)\\
&=\kappa_2 A_d^2\left[\frac{1}{2}\left(\tilde{K}^2+(d-2)(d-3)\tilde{K}_{ij}\tilde{K}^{ij}\right)-\frac{4}{(d-2)}\frac{\tilde{K}^2}{8}(d(d-5)+8)+\frac{\tilde{K}^2}{(d-1)(d-2)}\right]\\
&=\frac{2\lambda f_\infty \tilde{L}^2}{(d-2)(d-3)(1+2\lambda f_\infty)}\left(\tilde{K}^{ij}\tilde{K}_{ij}-\frac{\tilde{K}^2}{d-1}\right).
\end{aligned}\tag{89}$$

Collecting these results, we finally find that the holographic complexity on the island takes the following form

$$\begin{aligned}
\mathcal{C}_{\rm V}^{\rm Island} = \max_{\partial\tilde{\mathcal{B}}=\sigma_{\mathbf{R}}}\ &\frac{1}{G_{\rm eff}\ell'}\Big[V(\tilde{\mathcal{B}})\\
&+\frac{\tilde{L}^2}{2}\frac{1-6\lambda f_\infty}{1+2\lambda f_\infty}\int_{\tilde{\mathcal{B}}}d^{d-1}\sigma\sqrt{\tilde{h}}\left(\frac{\tilde{K}^2}{(d-1)(d-3)}-\frac{\tilde{R}[\tilde{g}]+2\tilde{R}_{ij}[\tilde{g}]\tilde{n}^i\tilde{n}^j}{(d-2)(d-3)}\right)\\
&+\frac{2\tilde{L}^2}{(d-2)(d-3)}\frac{\lambda f_\infty}{1+2\lambda f_\infty}\int_{\tilde{\mathcal{B}}}d^{d-1}\sigma\sqrt{\tilde{h}}\left(\tilde{K}^{ij}\tilde{K}_{ij}-\frac{\tilde{K}^2}{d-1}+\mathcal{O}(z_{\rm B}^4)\right)\Big].
\end{aligned}\tag{90}$$

Of course, the above result reproduces the holographic complexity derived from Einstein gravity in the bulk, *i.e.,* eq. (42), after setting $\lambda=0$.

3.1.2 Holographic Complexity from Near-Brane Region

To compare eq. (90) with eq. (84), we must integrate the latter over the bulk region near the brane. Hence as in the previous section, we turn to the FG expansion and evaluate quantities for $z=z_{\rm B}\ll L$.

From the FG expansion of the induced metric $h_{\alpha\beta}$ on the time slice $\mathcal{B}$, *i.e.,*

$$h_{zz}=\frac{\tilde{L}^2}{z^2}+\delta h_{zz}\,,\qquad h_{ab}=\frac{\tilde{L}^2}{z^2}\overset{(0)}{h}_{ab}+\delta h_{ab}\,, \tag{91}$$

one can derive the FG expansion of the Ricci tensor on $\mathcal{B}$ as

$$(R_{\mathcal{B}})^{zz}=-\frac{(d-1)z^2}{\tilde{L}^4}+\cdots\,,\quad (R_{\mathcal{B}})^{ab}=-\frac{(d-1)z^2}{\tilde{L}^4}\overset{(0)}{h}{}^{ab}+\frac{z^4}{\tilde{L}^4}(R_\Sigma)^{ab}\big[\overset{(0)}{h}\big]+\cdots\,. \tag{92}$$

We can see that these curvatures correspond very nearly to those of AdS$_d$ with a curvature scale $\tilde{L}$. Hence, the extremality condition (83) for GB gravity simply reduces to

$$\left(1-2\lambda_{\rm GB}\frac{(d-1)(d-2)}{\tilde{L}^2}+\mathcal{O}(z^2)\right)\mathcal{K}=0\,. \tag{93}$$

Using the expansion of $\mathcal{K}$ in eq. (25), we find the leading order terms in the FG expansion of the embedding of $\mathcal{B}$

$$\overset{(1)}{x^i}(\sigma^a) = \frac{\tilde{L}^2}{2(d-1)} K n^i \,, \tag{94}$$

which is essentially the same as for Einstein gravity. Similarly, we can find the expansion of the induced metric on $\mathcal{B}$

$$\begin{aligned} \delta h_{zz} &= \overset{(1)}{h}_{zz} + \mathcal{O}(z^2) = -\frac{\tilde{L}^2}{(d-1)^2} K^2 + \mathcal{O}(z^2) \,, \\ \delta h_{ab} &= \overset{(1)}{h}_{ab} + \mathcal{O}(z^2) = \overset{(1)}{g}_{ab} + \frac{\tilde{L}^2}{d-1} K K_{ab} + \mathcal{O}(z^2) \,. \end{aligned} \tag{95}$$

To find the subleading contributions in $R_{\mathcal{B}}$, we consider the FG-expansion as a perturbation on the metric $\overset{(0)}{h}_{\alpha\beta}$ and calculate the perturbation of the Ricci scalar by

$$\delta R_{\mathcal{B}} = -(R_{\mathcal{B}})^{\alpha\beta}\delta h_{\alpha\beta} + \nabla^\alpha\nabla^\beta \delta h_{\alpha\beta} - \nabla^\alpha\nabla_\alpha \delta h^\beta_\beta \,. \tag{96}$$

Keeping in mind that the terms with more z-derivatives dominate in the small z expansion, one can get the expansions of the Christoffel symbols as

$$\Gamma^z_{zz} \approx -\frac{1}{z} \quad \Gamma^z_{ab} \approx \frac{1}{z}\overset{(0)}{h}_{ab} \quad \Gamma^a_{bz} \approx -\frac{1}{z}\delta^a_b \,. \tag{97}$$

The Ricci scalar near the asymptotic boundary is given by

$$\begin{aligned} R_{\mathcal{B}}[h_{\alpha\beta}] &= -\frac{d(d-1)}{\tilde{L}^2} + \frac{z^2}{\tilde{L}^2} R_\Sigma[\overset{(0)}{h}_{ab}] + \delta R_{\mathcal{B}} \\ &= -\frac{d(d-1)}{\tilde{L}^2} + \frac{z^2}{\tilde{L}^2}\left(R_\Sigma + \frac{(d-1)(d-2)}{\tilde{L}^2}\overset{(1)}{h}_{zz} + \frac{2(d-2)}{\tilde{L}^2}\overset{(0)}{h}{}^{ab}\overset{(1)}{h}_{ab} \right) + \mathcal{O}(z^4) \\ &\approx -\frac{d(d-1)}{\tilde{L}^2} + \frac{z^2}{\tilde{L}^2}\left[R_\Sigma[\overset{(0)}{h}_{ab}] - 2\overset{(0)}{h}{}^{ab} R_{ab} + R + \frac{(d-2)}{(d-1)} K^2 \right] \,, \end{aligned} \tag{98}$$

with Ricci tensor R_{ab} associated with boundary metric $\overset{(0)}{g}_{ij}$. We can use the Gauss-Codazzi equation

$$(R_\Sigma)_{abcd} = R_{abcd} - K_{ac}K_{bd} + K_{ad}K_{bc} \,, \tag{99}$$

to rewrite the expansion of $R_{\mathcal{B}}[h_{\alpha\beta}]$ as

$$R_{\mathcal{B}}[h_{\alpha\beta}] \approx -\frac{d(d-1)}{\tilde{L}^2} + \frac{z^2}{\tilde{L}^2}\left(K_{ab}K^{ab} - \frac{1}{(d-1)}K^2 \right) + \mathcal{O}(z^4) \,. \tag{100}$$

Note that the Ricci tensor terms in $R_{\mathcal{B}}[h_{\alpha\beta}]$ at order $\mathcal{O}(z^2)$ are absent, which is similar to the contributions of the Weyl tensor term on the brane as shown in eq. (87).

Lastly, we deal with the extrinsic curvature term associated with $K_{\rm R}$ in eq. (84). The unit normal $(t_{\rm R})_\alpha$ to the island $\widetilde{\mathcal{B}}$ embedded on the hypersurface $\mathcal{B}_{\rm R}$ is

$$(t_{\rm R})_\alpha = -\sqrt{h_{zz}(z_{\rm B})}\delta^z_\alpha \,. \tag{101}$$

From the definition of the extrinsic curvature, *i.e.,* $(K_{\rm R})_{ab} = D_a(t_{\rm R})_b$, its trace (in Gaussian normal coordinate) is given by

$$
\begin{aligned}
K_{\rm R} &= -\frac{h^{ab}}{2\sqrt{h_{zz}}}\frac{\partial h_{ab}}{\partial z}\bigg|_{z=z_{\rm B}} \\
&\approx h^{ab}(z_{\rm B})\frac{\tilde{L}}{z_{\rm B}^2}\left(1-\frac{z_{\rm B}^2}{2\tilde{L}^2}\overset{(1)}{h}_{zz}\right)\overset{(0)}{h}_{ab}+\mathcal{O}(z_{\rm B}^4) \\
&\approx \frac{(d-1)}{\tilde{L}}\left(1-\frac{z_{\rm B}^2}{2\tilde{L}^2}\overset{(1)}{h}_{zz}\right)-\frac{z_{\rm B}^2}{\tilde{L}^3}\overset{(0)}{h}{}^{ab}\overset{(1)}{h}_{ab} \\
&\approx \frac{(d-1)}{\tilde{L}}-\frac{\tilde{L}}{2}\left(\frac{\tilde{K}^2}{(d-1)}-\frac{\tilde{R}+2\tilde{R}_{ij}\tilde{n}^i\tilde{n}^j}{(d-2)}\right)+\mathcal{O}(z_{\rm B}^4)\,,
\end{aligned}
\tag{102}
$$

where we have recast all geometric quantities as the ones living on the brane in the last line by using eq. (35) again. Of course, we also find a similar result for $K_{\rm L}$.

Finally, substituting eqs. (102), and (98) into the proposed generalized CV for GB gravity, *i.e.,* eq. (84), we can explicitly perform the z-integral with lower bound $z_{\rm B}$ and obtain the leading contributions as

$$
\begin{aligned}
\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}_{\rm L}\cup\mathbf{R}_{\rm R}) \approx \max_{\partial\tilde{\mathcal{B}}=\sigma_{\mathbf{R}}}\Bigg[&\frac{2\tilde{L}V(\tilde{\mathcal{B}})}{G_{\rm bulk}\ell(d-1)}(1+2\lambda f_\infty) \\
&+\frac{\tilde{L}^3(1-6\lambda f_\infty)}{(d-1)(d-3)G_{\rm bulk}\ell}\int_{\tilde{\mathcal{B}}}d^{d-1}\sigma\sqrt{\tilde{h}}\left(\frac{\tilde{K}^2}{(d-1)}-\frac{\tilde{R}[\tilde{g}]+2\tilde{R}_{ij}[\tilde{g}]\tilde{n}^i\tilde{n}^j}{(d-2)}\right) \\
&+\frac{4\lambda f_\infty\tilde{L}^3}{(d-1)(d-2)(d-3)G_{\rm bulk}\ell}\int_{\tilde{\mathcal{B}}}d^{d-1}\sigma\sqrt{\tilde{h}}\left(\tilde{K}^{ij}\tilde{K}_{ij}-\frac{\tilde{K}^2}{d-1}+\mathcal{O}(z_{\rm B}^4)\right)\Bigg]\,.
\end{aligned}
\tag{103}
$$

However, we see this is exactly the expression in eq. (90) derived for the induced action on the brane, by noting the relation $\ell'=\frac{d-1}{d-2}\ell$ and $\frac{1}{G_{\rm eff}}=\frac{2\tilde{L}}{d-2}\frac{1+2\lambda f_\infty}{G_{\rm bulk}}$. Note that we have counted the double contributions from both sides of the bulk surface $\mathcal{B}=\mathcal{B}_{\rm L}\cup\mathcal{B}_{\rm R}$ which give rise to the same contributions around the island region. Therefore, our generalized CV proposal for higher-curvature gravity theory produces consistent results between the bulk gravity theory and brane gravity theory, *i.e.,*

$$
\mathcal{C}_{\rm V}^{\rm Island}\simeq\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R})\,,
\tag{104}
$$

where the maximization of the same functionals over the island region $\tilde{\mathcal{B}}$ is considered on both sides as discussed in section 2.3.

3.2 Holographic Complexity for $f(\mathcal{R})$ Gravity

In this subsection, we apply the same consistency test with $f(\mathcal{R})$ gravity in the bulk to check our proposal. In contrast to the GB theory in the previous subsection, there are extra propagating degrees of freedom in this higher curvature theory [139, 140, 141, 142], *i.e.,* $f(\mathcal{R})$ gravity is properly referred to as a higher derivative theory. We must emphasize the importance of this feature since we saw in the previous section that to properly treat our brane in the limit of zero thickness, the bulk gravity theory should have a good boundary value problem. However, this issue is easily resolved for $f(\mathcal{R})$ gravity by recasting it as a scalar-tensor theory – see below.

We consider the $(d+1)$-dimensional bulk theory with the action

$$
I_{\rm bulk}^{f}=\frac{1}{16\pi G_{\rm bulk}}\int d^{d+1}y\sqrt{-g}\left(\frac{d(d-1)}{L^2}+f(\mathcal{R})\right)\,.
\tag{105}
$$

In principle, one should consider adding a surface term to this action (*e.g.,* see [143]) but we will not need to consider the details of this contribution here. Given the above action, it is straightforward to find the equation of motion:

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu} + \left(g_{\mu\nu}\nabla^\sigma\nabla_\sigma - \nabla_\mu\nabla_\nu\right) f'(\mathcal{R}) - \frac{g_{\mu\nu}}{2}\left(f(\mathcal{R}) + \frac{d(d-1)}{L^2}\right) = 8\pi G_{\text{bulk}} T_{\mu\nu}\,. \tag{106}$$

In the absence of matter (*i.e.,* with $T_{\mu\nu} = 0$), we will assume that this equation is solved by an AdS$_{d+1}$ spacetime whose curvature scale $\tilde{L}$ is related to L by

$$-\frac{d(d-1)}{L^2} = f(\mathcal{R}_0) + \frac{2d}{\tilde{L}^2} f'(\mathcal{R}_0) \qquad \text{where} \;\; \mathcal{R}_0 = -\frac{d(d-1)}{\tilde{L}^2}\,. \tag{107}$$

As emphasized above, $f(\mathcal{R})$ gravity is a fourth-derivative theory but is classically equivalent to a second-derivative scalar-tensor theory *e.g.,* [140, 141]. To be precise, by introducing a scalar field Φ, we can define the classically equivalent scalar-tensor theory with action

$$\begin{aligned} I^{st}_{\text{bulk}} =& \frac{1}{16\pi G_{\text{bulk}}}\int d^{d+1}y\sqrt{-g}\left(\frac{d(d-1)}{L^2} + f(\Phi) + f'(\Phi)\left(\mathcal{R} - \Phi\right)\right) \\ &+ \frac{1}{8\pi G_{\text{bulk}}}\oint d^dx\sqrt{-\tilde{g}}\,\mathcal{K}_{\text{B}} f'(\Phi)\,. \end{aligned} \tag{108}$$

Here, we have explicitly introduced the surface term here which produces a well-posed variational principle with Dirichlet boundary conditions *i.e.,* $\delta\Phi = 0 = \delta g_{\mu\nu}$. The equation of motion for the scalar field reads

$$f''(\Phi)\left(\mathcal{R} - \Phi\right) = 0\,. \tag{109}$$

Imposing the on-shell condition $\Phi = \mathcal{R}$ (assuming $f''(\mathcal{R}) \neq 0$), the action in eq. (108) obviously reduces to eq. (105) for $f(\mathcal{R})$ gravity. On the other hand, varying the metric yields the field equations

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} \;\;=&\;\; \frac{1}{f'(\Phi)}\left[\nabla_\mu\nabla_\nu f'(\Phi) - g_{\mu\nu}\Box f'(\Phi) - \frac{1}{2}g_{\mu\nu}\left(\Phi f'(\Phi) - f(\Phi) - \frac{d(d-1)}{L^2}\right)\right] \\ &+ \frac{8\pi G_{\text{bulk}}}{f'(\Phi)}\,T_{\mu\nu}\,. \end{aligned} \tag{110}$$

Upon substituting the on-shell condition $\Phi = \mathcal{R}$, these equations of motion reduce to the fourth-order equations (106) derived by varying the original $f(\mathcal{R})$ action. Noting the coefficient associated with the matter stress tensor $T_{\mu\nu}$, we introduce the "effective Newton constant" for the $(d+1)$-dimensional scalar-tensor theory as

$$\frac{1}{\widehat{G}_{\text{eff}}} = \frac{f'(\Phi)}{G_{\text{bulk}}}\,, \tag{111}$$

due to the coupling between gravity and the scalar field Φ. When the matter terms are absent, the bulk spacetime remains the same AdS$_{d+1}$ as above with $\Phi_0 = \mathcal{R}_0$, and in the case, the "effective Newton constant" is actually a constant

$$\frac{1}{\widehat{G}_{\text{eff}}} = \frac{f'(\Phi_0)}{G_{\text{bulk}}} = \frac{f'(\mathcal{R}_0)}{G_{\text{bulk}}}\,. \tag{112}$$

More generally, we can considering an asymptotically AdS$_{d+1}$ spacetime and one finds that the FG expansion for the Ricci scalar $\mathcal{R}$ up to the fourth order takes the form

$$\mathcal{R}\left[g_{\mu\nu}\right] = \mathcal{R}_0 + \mathcal{O}(z_{\text{B}}^6)\,. \tag{113}$$

by doing a similar calculation to those in the previous subsection. Hence with the on-shell condition, we have $\Phi = mR = \mathcal{R}_0 + \mathcal{O}(z_{\text{B}}^6)$. Further the trace of the extrinsic curvature $\mathcal{K}_{\text{B}}$ at the brane is given by

$$\mathcal{K}_{\text{B}} = \frac{1}{\tilde{L}}\left[d + \frac{\tilde{L}^2}{2(d-1)}\tilde{R} + \frac{\tilde{L}^4}{2(d-1)(d-2)^2}\left(\tilde{R}_{ij}\tilde{R}^{ij} - \frac{d}{4(d-1)}\tilde{R}^2\right)\right] + \mathcal{O}\left(z_{\text{B}}^6\right)\,. \tag{114}$$

Now integrating out the radial direction in the bulk action in the vicinity of the brane, we obtain the induced gravitational action on the brane as [144]

$$I_{\rm ind} = \frac{1}{16\pi G_{\rm eff}} \int_{\rm brane} d^dx\sqrt{-\tilde{g}} \left[\frac{(d-1)(d-2)}{\ell_{\rm eff}^2} + \tilde{R} + \kappa_1 \left(\tilde{R}_{ij}\tilde{R}^{ij} - \frac{d}{4(d-1)}\tilde{R}^2\right) + \mathcal{O}\left(z_{\rm B}^6\right)\right] , \tag{115}$$

where the various coupling constants are given by

$$\begin{aligned} \frac{1}{\ell_{\rm eff}^2} &= \frac{2}{\tilde{L}^2}\left(1 - \frac{4\pi\tilde{L}\widehat{G}_{\rm eff}T_o}{(d-1)}\right) = \frac{2}{\tilde{L}^2}\left(1-\frac{4\pi\tilde{L}G_{\rm bulk}T_o}{(d-1)f'(\mathcal{R}_0)}\right) , \\ \frac{1}{G_{\rm eff}} &= \frac{2\tilde{L}}{(d-2)\widehat{G}_{\rm eff}} = \frac{2\tilde{L}}{d-2}\frac{f'(\mathcal{R}_0)}{G_{\rm bulk}} , \qquad \kappa_1 = \frac{\tilde{L}^2}{(d-2)(d-4)} . \end{aligned} \tag{116}$$

In producing this result, we have introduced the surface term on either side of the brane, as discussed below eq. (72). The induced action on the brane from $f(\mathcal{R})$ gravity in the bulk is similar to that from Einstein gravity on the bulk in eq. (16) except for the corrections on the coupling constants from $f'(\mathcal{R}_0)$.

Our goal is to show that the relation

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) \simeq \mathcal{C}_{\rm V}^{\rm Island} \equiv \max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}} \left[\frac{\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell'}\right] , \tag{117}$$

also holds for $f(\mathcal{R})$ gravity in the bulk and its induced gravity on the brane. Thanks to the similarity between the induced action in eq. (115) and that for Einstein gravity in the bulk, *i.e.*, eq. (16), it is easy to find that the generalized CV on the brane with this induced gravity theory is given by

$$\begin{aligned} \mathcal{C}_{\rm V}^{\rm Island} = \max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}} &\left[\frac{V(\widetilde{\mathcal{B}})}{G_{\rm eff}\ell'}\right. \\ &\left. + \frac{\tilde{L}^2}{2G_{\rm eff}\ell'}\int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma\,\sqrt{\tilde{h}}\left(\frac{\tilde{K}^2}{(d-1)(d-3)} - \frac{\tilde{R}[\tilde{g}] + 2\tilde{R}_{ij}[\tilde{g}]\tilde{n}^i\tilde{n}^j}{(d-2)(d-3)} + \cdots\right)\right] . \end{aligned} \tag{118}$$

We expect that our proposal can provide the same result as eq. (118) by considering the generalized CV in $(d+1)$-dimensional bulk with $f(\mathcal{R})$ gravity. However, due to the higher-derivative terms, it is much easier to consider the holographic complexity directly in the equivalent scalar-tensor theory (108) because the gravitational part is only described by the Einstein gravity. Correspondingly, the generalized volume term reduces to a volume term and the K-term simply vanishes. The one subtlety is that we apply our proposal (65) to the scalar-tensor theory with the "effective Newton constant" $\widehat{G}_{\rm eff} = G_{\rm bulk}/f'(\Phi)$. However, noticing that $\widehat{G}_{\rm eff}$ may be a locally varying quantity on the asymptotically AdS spacetime, we should put the factor $\frac{1}{\widehat{G}_{\rm eff}}$ inside the integrals for $W_{\rm gen}, W_K$. Then the generalized CV complexity reads

$$\begin{aligned} \mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) &= \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}} \int_{\mathcal{B}} d^d\sigma\,\sqrt{\det h_{\alpha\beta}}\frac{1}{\widehat{G}_{\rm eff}\ell}\left(\alpha_{d+1}\frac{\partial\mathbf{L}_{\rm bulk}}{\partial\mathcal{R}_{\mu\nu\rho\sigma}}n_\mu h_{\nu\rho}n_\sigma + \gamma_{d+1}\right) , \\ &= \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}} \left[\int_{\mathcal{B}} d^{d-1}\sigma dz\sqrt{\det h_{\alpha\beta}}\frac{f'(\Phi)}{G_{\rm bulk}\ell}\left(\frac{d}{2}\alpha_{d+1} + \gamma_{d+1}\right)\right] , \end{aligned} \tag{119}$$

where $\mathbf{L}_{\rm bulk} \equiv 16\pi\widehat{G}_{\rm eff}\mathcal{L}_{\rm bulk}$, both W_K and $W_{\rm bdy}$ vanish due to the absence of higher curvature terms in eq. (108). Substituting the values of α_{d+1} and γ_{d+1} derived from eq. (48), one can find that the expression in round parentheses reduces to one. Then extremizing the holographic complexity in the scalar-tensor theory results in

$$\begin{aligned} &\mathop{\rm ext}_{\mathcal{B}_{\rm L},\mathcal{B}_{\rm R}} \left[\frac{1}{G_{\rm bulk}\ell}\int_{\mathcal{B}} d^{d-1}\sigma dz\sqrt{\det h_{\alpha\beta}}\,f'(\Phi)\right] \\ &\simeq \frac{2\tilde{L}^d f'(\mathcal{R}_0)}{G_{\rm bulk}\ell}\int_{\widetilde{\mathcal{B}}} d^{d-1}\sigma\sqrt{\det \overset{(0)}{h}_{ab}}\left[\frac{1}{(d-1)z_{\rm B}^{d-1}} + \frac{1}{(d-3)z_{\rm B}^{d-3}}\left(\frac{d-2}{2(d-1)^2}K^2 - \frac{R_a^a - \frac{1}{2}R}{2(d-2)}\right)\right] , \end{aligned} \tag{120}$$

where, once again $\mathcal{B}=\mathcal{B}_{\rm L}\cup\mathcal{B}_{\rm R}$ and $\widetilde{\mathcal{B}}=\mathcal{B}_{\rm L}\cap\mathcal{B}_{\rm R}$ and we also used the on-shell condition in eq. (109) and the series expansion $f'(\Phi)=f'(\mathcal{R})\approx f'(\mathcal{R}_0)+\mathcal{O}(z_{\rm B}^6)$. Using the geometric quantities of the brane and noting the maximization over $\widetilde{\mathcal{B}}$, we can finally obtain the generalized CV for the $f(\mathcal{R})$ gravity in the bulk as

$$\begin{aligned}\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R})=\max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}}&\left[\frac{2\tilde{L}f'(\mathcal{R}_0)}{G_{\rm bulk}\ell(d-1)}\left(V(\widetilde{\mathcal{B}})\right.\right.\\&\left.\left.+\frac{\tilde{L}^2}{2(d-3)}\int_{\widetilde{\mathcal{B}}}d^{d-1}\sigma\sqrt{\tilde{h}}\left(\frac{\tilde{K}^2}{d-1}-\frac{\tilde{R}[\tilde{g}]+2\,\tilde{R}_{ij}[\tilde{g}]\tilde{n}^i\tilde{n}^j}{d-2}+\mathcal{O}(z_{\rm B}^4)\right)\right)\right],\end{aligned}\tag{121}$$

Comparing eqs. (118) and (121), we also find the equivalence between the holographic complexity derived from $f(\mathcal{R})$ gravity and its induced gravity theory on the brane, *i.e.*,

$$\mathcal{C}_{\rm V}^{\rm Island}\simeq\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R})\,,\tag{122}$$

where we have used the relations $\ell'=\frac{d-1}{d-2}\ell$ and $\frac{1}{G_{\rm eff}}=\frac{2\tilde{L}}{d-2}\frac{f'(\mathcal{R}_0)}{G_{\rm bulk}}$. Once again, this equivalence supports that our proposed holographic complexity for higher-derivative gravity theory produces consistent results.

4 Complexity=Action

We start our analysis studying in more details the holographic complexity=action proposal of [7, 8], which entails evaluating the action

$$\begin{aligned}I_{\rm WDW}&=I_{\rm bulk}+I_{\rm GHY}+I_{\rm joints}+I_\kappa+I_{\rm ct}\\&=\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{|g|}\left(R+\frac{d(d-1)}{\ell^2}\right)+\frac{1}{8\pi G_N}\int_{\rm regulator}d^dy\sqrt{|h|}\,K\\&\quad+\frac{1}{8\pi G_N}\int_{\rm joints}d^{d-1}y\sqrt{\sigma}\,a_{\rm joint}+\frac{1}{8\pi G_N}\int_{\partial\rm WDW}d\lambda\,d^{d-1}y\sqrt{\gamma}\,\kappa\\&\quad+\frac{1}{8\pi G_N}\int_{\partial\rm WDW}d\lambda\,d^{d-1}y\sqrt{\gamma}\,\Theta\,\log\left(L_{\rm ct}\Theta\right)\,.\end{aligned}\tag{123}$$

This includes: $I_{\rm bulk}$, the Einstein-Hilbert action with negative cosmological constant and $I_{\rm GHY}$, the Gibbons-Hawking-York term defined on the AdS boundary regulator surface. In the second line: $I_{\rm joints}$, the contribution of the intersection of the null boundaries of the WDW patch with other hypersurfaces (which we specify better below), and I_κ, which has support on the null boundaries of the WDW patch and vanishes when these are affinely parameterized, as in our case. The term in the last line $I_{\rm ct}$ is known as the *counterterm* [164]. It is also localized on the boundary of the WDW patch and is expressed in terms of Θ, its expansion. This was first proposed in [164] and removes the ambiguity intrinsic to the parametrization of the WDW null boundaries, but it introduces an arbitrary length scale $L_{\rm ct}$. In static background geometries, the role of this counterterm does not influence significantly the holographic CA, see [11]. Nevertheless, for dynamical spacetimes as the ones analyzed in [13, 14], the situation is different: there the inclusion of the counterterm in the total gravitational action is a key ingredient in order to obtain results consistent with general properties of circuit complexity. For example, in the one-sided geometry of [13], the counterterm is essential to obtain the expected late time growth rate in $d>3$ and a positive rate in $d=3$. In the two-sided case, the counterterm is needed to replicate the switchback effect [14]. The inclusion of the counterterm also modifies the structure of divergences of holographic complexity, as first pointed out in the current literature and was observed to play a crucial role in the cancellations occurring for CA in the study of the first law of complexity.

Bulk term. We first write explicitly

$$I_{\text{bulk}} = \frac{1}{16\pi\, G_N} \int_{\text{WDW}} d^3x\, \sqrt{-g} \left(R + \frac{2}{\ell^2} \right) = -\frac{1}{2\, G_N \ell^2} \int dt\, dr\, r\,. \tag{124}$$

where we used the on-shell relations $R = -\frac{6}{\ell^2}$ and $R = 6\Lambda$ and performed the angular integration. Exploiting the left-right symmetry of the WDW patch, we divide its right half in three zones I − III, as labeled in fig. **??**, each with its own integration extrema. For instance in region I, for fixed $r_{m1} \leq r \leq r_+$, we have $t_{\text{min}} \leq t \leq t_{\text{max}}$. By symmetry $t_{\text{min}} = 0$, while t_{max} can be determined observing that the locations (t_{max}, r) and $(t_b/2, r = \infty)$ share the same v coordinate. This fixes $t_{\text{max}} = t_b/2 - r^*(r)$ in region I. All together, we obtain

$$I_{\text{bulk}} = 2 \left(I^{\text{I}}_{\text{bulk}} + I^{\text{II}}_{\text{bulk}} + I^{\text{III}}_{\text{bulk}} \right) \tag{125}$$

with

$$I^{\text{I}}_{\text{bulk}} = -\frac{1}{2\, G_N \ell^2} \int_{r_{m1}}^{r_+} dr\, r \left(\frac{t_b}{2} - r^*(r) \right) \tag{126}$$

$$I^{\text{II}}_{\text{bulk}} = \frac{1}{G_N \ell^2} \int_{r_+}^{r_{\text{max}}} dr\, r\, r^*(r) \tag{127}$$

$$I^{\text{III}}_{\text{bulk}} = \frac{1}{2\, G_N \ell^2} \int_{r_{m2}}^{r_+} dr\, r \left(\frac{t_b}{2} + r^*(r) \right)\,, \tag{128}$$

where r_{m1}, r_{m2} are given implicitly by eq. (**??**) and r_{max} denotes a radial cutoff introduced to regularize these expressions. Thus

$$I_{\text{bulk}} = \frac{1}{G_N \ell^2} \left\{ \frac{t_b}{4} \left(r_{m1}^2 - r_{m2}^2 \right) + \int_{r_{m1}}^{r_{\text{max}}} dr\, r\, r^*(r) + \int_{r_{m2}}^{r_{\text{max}}} dr\, r\, r^*(r) \right\} \tag{129}$$

The UV divergent terms of the bulk action do not contribute to the complexity growth rate. In fact r_{m1} and r_{m2} evolve according to equation, but r_{max} is constant in time. As we shall see the same remains true also for the other contributions to the WDW action (123).

Performing explicitly the integrals in (129) we obtain the expression

$$I_{\text{bulk}} = \frac{1}{4\, G_N} \left\{ 2 \left(r_{m1} + r_{m2} - 4 r_{\text{max}} \right) - r_+ \log \frac{(r_+ + r_{m1})(r_+ + r_{m2})}{(r_+ - r_{m1})(r_+ - r_{m2})} \right\} + O\left(\frac{1}{r_{\text{max}}} \right)\,, \tag{130}$$

where we used (**??**) and expanded in $r_{\text{max}} \to \infty$. This correctly reduces to the non rotating BTZ result of [11] for $r_-, r_{m1} \to 0$.

GHY terms. Next we evaluate the GHY term in (123) for the timelike cutoff surface at $r = r_{\text{max}}$

$$I_{\text{GHY}} = \frac{1}{8\pi\, G_N} \int_{r = r_{\text{max}}} d^2y\, \sqrt{-h}\, K\,. \tag{131}$$

Here $K = h^{ab} K_{ab}$ is the trace of the extrinsic curvature $K_{ab} = \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} \nabla_\mu n_\nu$, and n_ν the outward directed normal to the cutoff surface. These read

$$n_\mu dx^\mu = \frac{dr}{\sqrt{f(r_{\text{max}})}}\,, \qquad K = \frac{2\, r_{\text{max}}^2 - r_+^2 - r_-^2}{\ell^2\, r_{\text{max}} \sqrt{f(r_{\text{max}})}}\,. \tag{132}$$

Taking into account the right-left symmetry of the problem, and the fact that the time integration along the cutoff surface $r = r_{\text{max}}$ is restricted by the null boundaries of the WDW, we have

$$\begin{aligned} I_{\text{GHY}} &= \frac{\left(2 r_{\text{max}}^2 - r_+^2 - r_-^2 \right)}{2\, G_N \ell^2} \int_{\frac{t_b}{2} + r^*(r_{\text{max}})}^{\frac{t_b}{2} - r^*(r_{\text{max}})} dt = -\frac{r^*(r_{\text{max}}) \left(2 r_{\text{max}}^2 - r_+^2 - r_-^2 \right)}{G_N \ell^2} \\ &= \frac{2\, r_{\text{max}}}{G_N} + O\left(\frac{1}{r_{\text{max}}} \right)\,. \end{aligned} \tag{133}$$

The GHY term only yields a divergent contribution to the total action, and thus does not contribute to the complexity growth rate.

Joints terms. There are different joints with null surfaces contributing to the action (123). Null-null joints at the future and past tip of the WDW patch, and time-null joints formed at the intersection of the WDW patch with the cutoff surface at $r_{\rm max}$. Adopting the conventions of [11, 13], we have the following rules

$$\begin{aligned}
&\text{Time-Null joint:} \quad a_{\rm joint} = \epsilon \log |n_1 \cdot k_2| \quad \text{with } \epsilon = -\text{sign}\,(n_1 \cdot k_2)\, \text{sign}\left(\hat{t}_1 \cdot k_2\right) \\
&\text{Null-Null joint:} \quad a_{\rm joint} = \epsilon \log \left|\tfrac{k_1 \cdot k_2}{2}\right| \quad \text{with } \epsilon = -\text{sign}\,(k_1 \cdot k_2)\, \text{sign}\left(\hat{k}_1 \cdot k_2\right) .
\end{aligned} \tag{134}$$

Here k_i, n_i are respectively null and spacelike normal one-forms outward-directed from the relevant boundary of the WDW patch. The auxiliary null and timelike vectors $\hat{k}_i$, $\hat{t}_i$ are defined in the tangent space of the appropriate boundary region, pointing outward from it and orthogonal to the joint.

Let us start from the future null-null joint at the tip of the WDW patch, where $t = 0$ and $r = r_{m1}$. This contributes to the total gravitational action (123) with

$$\begin{aligned}
I_{\rm joints}^{\rm Null-Null} &= \frac{1}{8\pi G_N} \int_{r=r_{m1}} dy\, \sqrt{\sigma} \log \left| \frac{k_L \cdot k_R}{2} \right| \\
&= \frac{1}{4\, G_N}\, r_{m1} \log \left(-\frac{\alpha^2}{f(r_{m1})} \right) .
\end{aligned} \tag{135}$$

To obtain this result we used $\sigma = r^2$ and the following right and left null normals at the future joint of the WDW patch

$$k_{R\,\mu} = \left(\alpha\,,\, \frac{\alpha}{f}\,,\, 0 \right) , \qquad k_{L\,\mu} = \left(-\alpha\,,\, \frac{\alpha}{f}\,,\, 0 \right) . \tag{136}$$

Adding the analogous contribution coming from the bottom joint, we have for null-null joints

$$I_{\rm joints}^{\rm Null-Null} = \frac{1}{4\, G_N} \left\{ r_{m1} \log \left(-\frac{\alpha^2}{f(r_{m1})} \right) + r_{m2} \log \left(-\frac{\alpha^2}{f(r_{m2})} \right) \right\} . \tag{137}$$

Next, we evaluate the time-null joints term at the cutoff surface. Consider the right cutoff surface $r = r_{\rm max}$ and the joint term in its future at $t = \frac{t_b}{2} - r^*(r_{\rm max})$. Using the normal n_μ from (132) and $k_{\mu R}$ from (136), gives

$$\begin{aligned}
I_{\rm joints}^{\rm Time-Null} &= -\frac{1}{8\pi G_N} \int_{r=r_{\rm max}} dy\, \sqrt{\sigma}\, \log |n \cdot k| \\
&= -\frac{1}{4G_N}\, r_{\rm max} \log \left(\frac{\alpha\, \ell}{r_{\rm max}} \right) + O\left(\frac{1}{r_{\rm max}} \right) .
\end{aligned} \tag{138}$$

This divergent term is independent from the boundary time.

The other three time-null joints at the cutoff surface yield identical contributions. All together, including the null-null terms, we therefore have

$$I_{\rm joints} = -\frac{1}{4\, G_N} \left\{ 4 r_{\rm max} \log \left(\frac{\alpha\, \ell}{r_{\rm max}} \right) - r_{m1} \log \left(-\frac{\alpha^2}{f(r_{m1})} \right) - r_{m2} \log \left(-\frac{\alpha^2}{f(r_{m2})} \right) \right\} + O\left(\frac{1}{r_{\rm max}} \right) . \tag{139}$$

Counterterms. To evaluate the last contribution to the gravitational action (123), let us consider first the right future null boundary of the WDW patch. The counterterm action $I_{\rm ct}$ for this contribution evaluates to

$$\begin{aligned}
I_{\rm ct}^{RF} &= \frac{1}{8\pi\, G_N} \int d\lambda\, dy\, \sqrt{\gamma}\, \Theta\, \log\left(|L_{\rm ct}\, \Theta| \right) \\
&= \frac{1}{4\, G_N} \int_{r_{m1}}^{r_{\rm max}} dr\, \log \left(\frac{L_{\rm ct}\, \alpha}{r} \right) \\
&= \frac{1}{4\, G_N} \left\{ r_{\rm max} \left[1 + \log \left(\frac{L_{\rm ct}\, \alpha}{r_{\rm max}} \right) \right] - r_{m1} \left[1 + \log \left(\frac{L_{\rm ct}\, \alpha}{r_{m1}} \right) \right] \right\} ,
\end{aligned} \tag{140}$$

In deriving this expression we used that the normal vector to the surface implicitly defines a parametrization through $\partial_\lambda = k^\mu \partial_\mu$, together with the explicit form of the one-dimensional induced metric $\gamma = e^\mu e^\nu g_{\mu\nu} = r^2$, which defines $\Theta = \partial_\lambda \log \sqrt{\gamma}$. In particular, this yields $dr = \alpha d\lambda$ and $\Theta = \alpha \partial_r \log \sqrt{\gamma} = \frac{\alpha}{r}$.

Given the left-right symmetry, the left future null boundary gives an identical contribution. It is also straightforward to check that the past boundaries lead to an analogous result with $r_{m1} \to r_{m2}$. Putting everything together:

$$I_{\rm ct} = \frac{1}{2\,G_N}\left\{2\,r_{\rm max}\left[1+\log\left(\frac{L_{\rm ct}\,\alpha}{r_{\rm max}}\right)\right] - r_{m1}\left[1+\log\left(\frac{L_{\rm ct}\,\alpha}{r_{m1}}\right)\right] - r_{m2}\left[1+\log\left(\frac{L_{\rm ct}\,\alpha}{r_{m2}}\right)\right]\right\}. \tag{141}$$

The counterterm will thus give a non-vanishing contribution both to CA itself and to its growth rate. We will analyze in what follows how this counterterm contribution modifies the results of [**?**], obtained without the counterterm action later introduced in [164].

5 Circuit Complexity: Rotating TFD State

After working out different holographic measures of complexity in rotating black hole settings, we would like to study the corresponding complexity in the boundary theory. For concreteness we focus on the holographic dual of rotating BTZ, *i.e.* the rotating TFD state

$$|rTFD\rangle = \frac{1}{\sqrt{Z\left(\beta,\Omega\right)}}\sum_n e^{-\beta(E_n+\Omega J_n)/2} e^{-i(E_n+\Omega J_n)t}\,|E_n,J_n\rangle_L\,|E_n,J_n\rangle_R\,, \tag{142}$$

describing an entangled state of the two identical CFT_2 on the right and left asymptotic boundaries of the black hole geometry. Here E_n and J_n label energy and momentum eigenstates, β matches the inverse Hawking temperature of the dual black hole and Ω is the angular velocity. In writing the dynamics in (142), we have taken a symmetric time $t_R = t_L = t/2$, as to match the holographic model, and evolved with the deformed Hamiltonian on both sides. Another possibility would be to evolve with the undeformed Hamiltonian only, that is

$$|rTFD\rangle = \frac{1}{\sqrt{Z\left(\beta,\Omega\right)}}\sum_n e^{-\beta(E_n+\Omega J_n)/2} e^{-iE_n t}\,|E_n,J_n\rangle_L\,|E_n,J_n\rangle_R\,. \tag{143}$$

We will consider the two options in what follows.

In both cases, turning-off the potential Ω, one obtains

$$|TFD\rangle = \frac{1}{\sqrt{Z\left(\beta\right)}}\sum_n e^{-\beta E_n/2} e^{-iE_n t}\,|E_n\rangle_L\,|E_n\rangle_R\,, \tag{144}$$

representing the TFD state dual to the (non-spinning) BTZ black hole.

Ideally, one would like to evaluate complexity for this state in a holographic CFT_2, but a general definition of complexity in QFT (and CFT) is still lacking and the majority of results available so far concerns Gaussian states in free theories.

In order to make a qualitative comparison with the holographic results, we will follow the approach of [**?**] and consider as a toy model that of a free scalar field. As we will show explicitly, it is then easy to give an effective description of the rotating TFD state (142) in terms of the non-rotating one (144), and make use of the available Gaussian state results. This is analogous to what happens for the charged TFD studied, which can also be given an effective description in terms of (144).

Rotating TFD. We consider a simple model where right and left degrees of freedom are described by two identical copies of a (1+1)-dimensional free scalar QFT on a circle of length L, each with Hamiltonian

$$H = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\left[\frac{\pi^2}{2}+\frac{m^2}{2}\phi^2+\frac{1}{2}(\partial_x\phi)^2\right] = \sum_k \omega_k\,\left(a_k^\dagger a_k + \frac{1}{2}\right)\,, \tag{145}$$

and angular momentum operator

$$J = -\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, \pi\partial_x\phi = \sum_k p_k \left(a_k^\dagger a_k + \frac{1}{2}\right)\, . \tag{146}$$

In writing the r.h.s. of these expressions we have used the mode decompositions at $t=0$

$$\phi = \sum_k \frac{1}{\sqrt{2L\,\omega_k}}\left(e^{ip_kx}a_k + e^{-ip_kx}a_k^\dagger\right) \qquad \pi = -i\sum_k \sqrt{\frac{\omega_k}{2L}}\left(e^{ip_kx}a_k - e^{-ip_kx}a_k^\dagger\right) \tag{147}$$

with $p_k = \frac{2\pi}{L}k$ and $\omega_k = \sqrt{p_k^2+m^2}$. For each mode, modulo the shift in the zero-point energy, both H and J are proportional to the particle number operator $N_k = a_k^\dagger a_k$,

$$N_k\left|n\right\rangle_k = n_k\left|n\right\rangle_k\,, \qquad \left|n\right\rangle_k = \frac{(a_k^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle\, . \tag{148}$$

Mode-by-mode we can therefore simultaneously label Hamiltonian and momentum eigenstates in terms of the particle number eigenstates $\left|n\right\rangle_k$

$$H\left|n\right\rangle_k = E_{k,n}\left|n\right\rangle_k = \omega_k\left(n+\frac{1}{2}\right)\left|n\right\rangle_k\,, \qquad J\left|n\right\rangle_k = J_{k,n}\left|n\right\rangle_k = p_k\left(n+\frac{1}{2}\right)\left|n\right\rangle_k\,. \tag{149}$$

Given the free QFT structure, which yields modes factorization, the TFD state can be written as the product of TFD states of single right-left couples of harmonic oscillators, each labeled by the mode number k

$$\left|rTFD\right\rangle = \bigotimes_k \left|rTFD\right\rangle_k\, . \tag{150}$$

Making the eigenvalues structure explicit, the single mode states then take the form

$$\begin{aligned}\left|rTFD\right\rangle_k &= \frac{1}{\sqrt{Z_k(\beta,\Omega)}}\sum_n e^{-\left(\frac{\beta}{2}+it\right)(E_n+\Omega J_n)}\left|n\right\rangle_{k,L}\left|n\right\rangle_{k,R} && (151)\\ &= \frac{1}{\sqrt{Z_k(\beta,\Omega)}}\sum_n e^{-\left(\frac{\beta}{2}+it\right)(\omega_k+\Omega\, p_k)\left(n+\frac{1}{2}\right)}\left|n\right\rangle_{k,L}\left|n\right\rangle_{k,R}\,, && (152)\end{aligned}$$

with normalization factor

$$Z_k(\beta,\Omega) = \frac{e^{-\frac{\beta}{2}(\omega_k+\Omega\,p_k)}}{1-e^{-\beta(\omega_k+\Omega\,p_k)}}\, . \tag{153}$$

Defining for every single mode an effective inverse temperature and time as

$$\beta_k = \beta\left(1+\Omega\,\frac{p_k}{\omega_k}\right), \qquad t_k = t\left(1+\Omega\,\frac{p_k}{\omega_k}\right)\,, \tag{154}$$

it is then immediate to see that the rotating TFD state can be effectively written as a TFD state with no rotation

$$\left|rTFD\right\rangle_k = \frac{1}{\sqrt{Z_k(\beta_k,\Omega=0)}}\sum_n e^{-\left(\frac{\beta_k}{2}+it_k\right)\omega_k\left(n+\frac{1}{2}\right)}\left|n\right\rangle_{k,L}\left|n\right\rangle_{k,R}\, . \tag{155}$$

We shall notice that as long as $|\Omega|<1$ the effective inverse temperature (154) is non-negative, and only vanishes in the limiting case where $|\Omega|\to 1$ with $m\to 0$. Also, $t=0$ maps to $t_k=0$, and this will be important when computing the complexity of formation. A completely similar reasoning goes through if we choose to time-evolve with the undeformed Hamiltonian as in (143). The only difference being that the effective representation (155) would only involve an effective inverse temperature, but not an effective time. This simple identification, valid for each mode k, allows to borrow and adapt the results for non-rotating TFD states.

Before reviewing the results, let us mention that a similar identification can be performed in the charged, non-rotating case. There however the absolute value of the chemical potential, through the identification of the effective temperature, sets a lower bound for the mass parameter m. This in particular prevents from taking the $m\to 0$ limit in the charged case.

TFD complexity. We have shown that single mode rotating TFD states admit an effective description in terms of non-rotating TFD states. Here we briefly review the complexity analysis for the TFD state (144).

The analysis of the complexity follows and extends the work of [146], which adapted Nielsen's approach to complexity to free scalar fields. The latter starts with a continuum representation of the unitary transformation

$$U(\sigma) = \overleftarrow{\mathcal{P}} \exp\left[-i\int_0^\sigma ds \sum_I Y^I(s)\, K_I\right] \quad \text{with} \quad U(0) = \mathbb{1}, \quad \text{and} \quad U(1) = U_{\mathrm{T}}\,, \tag{156}$$

acting on states and connecting the reference and target states

$$|\psi_{\mathrm{T}}\rangle = U(1)\,|\psi_{\mathrm{R}}\rangle\;. \tag{157}$$

The unitary is constructed in terms of a basis of Hermitian operators K_I, the gate's generators, applied along the circuit parametrized by s as specified by the control functions Y^I. For practical reasons, the set of generators is normally taken to be finite and to realize a closed algebra. Nielsen's approach then assigns a cost to each circuit through a functional

$$\mathcal{D}[U] = \int_0^1 ds\; F\left(U(s), Y^I(s)\right) \tag{158}$$

specified in terms of a local cost function F, and defines the complexity of going from a reference to a target state as the cost associated to the circuit that minimizes the functional, namely

$$\mathcal{C}(U_{\mathrm{T}}) = \min_U \mathcal{D}[U]. \tag{159}$$

In this approach $U(\sigma)$ defines a trajectory in the space of unitaries, with $Y^I(\sigma)$ the components of its tangent vector. The problem of computing complexity is then analogous to solving for the motion of a particle in the geometry emerging from the group structure provided by the gate set, with Lagrangian specified by F.

In the representation, the target state was the non-rotating TFD state, which is the product of single modes TFD states, each corresponding to a TFD state of a pair of harmonic oscillators at fixed k

$$|TFD\rangle = \bigotimes_k |TFD\rangle_k = \bigotimes_k \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\left(\frac{\beta}{2}+it\right)\omega_k\left(n+\frac{1}{2}\right)}\,|n\rangle_{k,L}\,|n\rangle_{k,R}\;. \tag{160}$$

Following [146], the reference state was chosen to be a completely unentangled state obtained as the ground state of (two copies of) a ultralocal Hamiltonian where the spatial derivative term is absent. That is, the ground state of an Hamiltonian with a fixed frequency μ for all modes

$$H = \sum_k \mu\left(a_k^\dagger a_k + \frac{1}{2}\right)\;. \tag{161}$$

To connect the TFD state to the reference state, [**?**] considered circuits built with gates K_I quadratic in the canonical variables associated to each of the entangled pairs of harmonic oscillators making the TFD state. Introducing a UV regulator in the field theory yields a finite number of such gates. A simple way to regularize the theory in the setup at hand is to consider a finite number of modes $\tilde{N}$. In such a case, in the analysis of the relevant group structure turns out to be $Sp(2\tilde{N},\mathbb{R})$. The construction of the generators also introduces an arbitrary gate scale μ_g, which together with the reference state scale μ and the mode frequency ω_k characterize the complexity model.

The cost function on which focused their analysis is the so called $\kappa = 2$

$$F_{\kappa=2} = \sum_I \left|Y^I\right|^2 \tag{162}$$

which is independent of the specific basis for the gates generators. Importantly, for this cost function, when the reference and gate scales are set equal, $\mu_g = \mu$, the optimal circuit does not mix modes with different k, and the minimal length circuit for each mode is generated by repeatedly applying a single generator. In geometrical terms, in this case the optimal circuit computing complexity for each k corresponds to a straight-line geodesic on $Sp(2,\mathbb{R})$. The resulting complexity evaluated is

$$\mathcal{C}_{\kappa=2} = \frac{1}{4}\sum_k \log^2\left(f_k^{(+)} + \sqrt{\left(f_k^{(+)}\right)^2 - 1}\right) + \log^2\left(f_k^{(-)} + \sqrt{\left(f_k^{(-)}\right)^2 - 1}\right) \tag{163}$$

with

$$f_k^{(\pm)} = \frac{1}{2}\left(\frac{\mu}{\omega_k} + \frac{\omega_k}{\mu}\right)\cosh 2\alpha_k \pm \frac{1}{2}\left(\frac{\mu}{\omega_k} - \frac{\omega_k}{\mu}\right)\sinh 2\alpha_k \cos\omega_k t\,, \tag{164}$$

$$\alpha_k = \frac{1}{2}\log\left(\frac{1+e^{-\beta\omega_k/2}}{1-e^{-\beta\omega_k/2}}\right). \tag{165}$$

Let us reiterate that the mode factorization for the circuit allows to obtain the TFD complexity as the the sum of complexities evaluated for each mode separately. This is crucial in view of using the effective description of the rotating TFD (154)-(155) to evaluate complexity in terms of the non-rotating TFD results. In the rest of our work we will thus only consider the situation where the gate scale is set equal to the reference scale.

The basis-dependent cost function

$$F_1 = \sum_I \left|Y^I\right| \tag{166}$$

was also considered to evaluate the length of the straight-line circuit. That is, did not solve explicitly for the optimal circuit for the F_1 cost function, but simply evaluated the length of the straight-line circuit with this measure. Nonetheless, this still provides an upper bound on computational complexity of the TFD state. Interestingly, [?] found that the straight-line circuit provides a qualitative matching with the holographic complexity results for the TFD state when working in the so called physical basis.

In what follows we will then only explore the corresponding result for the F_1 cost:

$$\begin{aligned}
\mathcal{C}_1 = \frac{1}{2}\sum_k \; & \sqrt{2}\left|\log\left(f_k^{(+)} + \sqrt{\left(f_k^{(+)}\right)^2-1}\right)\cos\theta_k^{(+)} + \log\left(f_k^{(-)} + \sqrt{\left(f_k^{(-)}\right)^2-1}\right)\cos\theta_k^{(-)}\right| \\
& + \left|\log\left(f_k^{(+)} + \sqrt{\left(f_k^{(+)}\right)^2-1}\right)\sin\theta_k^{(+)} + \log\left(f_k^{(-)} + \sqrt{\left(f_k^{(-)}\right)^2-1}\right)\sin\theta_k^{(-)}\right| \\
& + \left|\log\left(f_k^{(+)} + \sqrt{\left(f_k^{(+)}\right)^2-1}\right)\cos\theta_k^{(+)} - \log\left(f_k^{(-)} + \sqrt{\left(f_k^{(-)}\right)^2-1}\right)\cos\theta_k^{(-)}\right| \\
& + \left|\log\left(f_k^{(+)} + \sqrt{\left(f_k^{(+)}\right)^2-1}\right)\sin\theta_k^{(+)} - \log\left(f_k^{(-)} + \sqrt{\left(f_k^{(-)}\right)^2-1}\right)\sin\theta_k^{(-)}\right|,
\end{aligned} \tag{167}$$

with

$$\tan\theta_k^{(\pm)} = \frac{1}{2}\left(\frac{\mu}{\omega_k} + \frac{\omega_k}{\mu}\right)\cot\omega_k t \pm \frac{1}{2}\left(\frac{\mu}{\omega_k} - \frac{\omega_k}{\mu}\right)\frac{1}{\tanh 2\alpha_k \sin\omega_k t}. \tag{168}$$

We will also be interested in the complexity of formation, the difference between the rotating TFD state complexity at $t=0$ and that of two copies of the vacuum state

$$\Delta\mathcal{C} \equiv \mathcal{C}(|rTFD(0)\rangle - \mathcal{C}(|0\rangle_L\,|0\rangle_R)\,, \tag{169}$$

This takes a particular simple form for the two cost functions we are considering and is independent from the reference scale μ, namely

$$\Delta\mathcal{C}_1 = 2\sum_k |\alpha_k| \,, \qquad\qquad \Delta\mathcal{C}_{\kappa=2} = 2\sum_k \alpha_k^2 \,. \tag{170}$$

This concludes our summary of the main results that we will use next to evaluate the complexity of rotating TFD states making use of the effective description (154)-(155) of the single mode rotating TFD in terms of a non rotating TFD state.

6 Holographic Partition Function

In this section, we discuss the gauge fixing issue, adopting the axial gauge. Since we are interested in the holographic study, we perform the gauge fixing in such a way that the holographic nature of the AdS_5 gauge theory becomes manifest: we derive the gauge-fixed holographic partition function $Z^{\mathrm{g.f.}}$. Our discussion in this section is influenced by the work of researchers. We, however, generalize it to arbitrary UV and IR boundary conditions. In addition, when the UV-BC is chosen to be Neumann for some non-trivial subgroup $H_0 \subset G$, we show that there is a residual gauge redundancy, requiring further (brane-localized) gauge fixing.

Let us consider a very general situation. We take the bulk gauge group to be G. We imagine a general BC by which G is broken down to $H_0 \subset G$ on the UV brane and to $H_1 \subset G$ on the IR brane. In this case, the full 5D gauge symmetry is given by

$$G_B = \{g(x,z) \in G \mid \hat{g} \equiv g(x,z_0) \in H_0, \bar{g} \equiv g(x,z_1) \in H_1\}. \tag{171}$$

The dual 4D CFT then has a global symmetry group G, which is spontaneously broken to H_1 by confinement at the scale associated with the IR brane location. $H_0 \subset G$ on the UV brane is dual to the fact that the H_0 part of G is weakly gauged, featuring an explicit breaking of G by gauging. The inclusion of the 5D CS action further incorporates the anomaly structure into the 4D dual gauge theory.

We denote the Lie algebra of G as $\mathbf{g}$. Similarly, $\mathbf{h}_0$ is the Lie algebra of H_0 and we denote the space generated by the coset G/H_0 generators to be $\mathbf{k}_0$. Likewise, $\mathbf{h}_1$ is the Lie algebra of H_1 and $\mathbf{k}_1$ denotes the space spanned by the generators of the coset G/H_1. Generators of $\mathbf{g}$ are written as $T^A \in \mathbf{g}, A = 1, \ldots, \mathrm{Dim}[\mathbf{g}]$. Likewise, $T^i_m \in \mathbf{h}_m$ and $T^a_m \in \mathbf{k}_m, m = 0,1$ are the unbroken and broken generators, respectively.

Before gauge fixing, the holographic partition function of a gauge theory of eq. (171) takes the form

$$Z\left[B^a\right] = \int \mathcal{D}B^i \; \mathcal{Z}\left[B^A\right], \tag{172}$$

$$\mathcal{Z}\left[B^A\right] = \int \mathcal{D}A_\mu(x,z)\Big|^{\hat{A}=B}_{\bar{A}:\left[\substack{(F)^i_1=0 \\ (A)^a_1=0}\right.} \; \mathcal{D}A_z(x,z)\; e^{iS[A_\mu(x,z),A_z(x,z)]}. \tag{173}$$

Let us explain the notation we used in these expressions. First, $\mathcal{Z}\left[B^A\right]$ is the partition function with UV boundary value of the bulk gauge field A taken to be B^A. Here, we suppressed the Lorentz index for the sake of brevity. The superscript A runs over *all* generators. This statement about the UV-BC is also written schematically as the superscript "$\hat{A} = B$" in eq. (173). The subscript in the same equation denotes instead the IR-BC. $(F)^i_1 = 0$ means $F_{\mu z} = 0$ for $T^i_1 \in \mathbf{h}_1$. The expression $(A)^a_1 = 0$ can be understood in the same way. As it is, $\mathcal{Z}\left[B^A\right]$ is the holographic partition function with Dirichlet UV-BC for all generators. In order to obtain the partition function corresponding to eq. (171), we need to promote the background source for the $T^i_0 \in \mathbf{h}_0$ to dynamical fields. This is done by path integrating over the fields B^i. Once this is done, then the partition function depends only on B^a, the background fields associated with $T_0^a \in \mathbf{k}_0$. The final result of this whole procedure is summarized by eq. (172) and (173).

As a next step, we want to incorporate the axial gauge, $A_z = 0$. As usual, this is done by inserting the following gauge-fixing factor into eq. (172):

$$\begin{aligned} 1 &= \int \mathcal{D}A_z\, \delta(A_z) = \int \mathcal{D}g\ \mathrm{Det}\left[\frac{\delta A_z^g}{\delta g}\right]\ \delta\left(A_z^g\right) \\ &= \int \mathcal{D}\Sigma_1(x)\ \mathcal{D}h_1(x)\ \mathcal{D}h_0(x)\ \mathcal{D}g|_{\bar{g}=\Sigma_1\circ h_1}^{\hat{g}=h_0}\ \mathrm{Det}\left[D_z(A_z)\right]\ \delta\left(A_z^g\right), \end{aligned} \tag{174}$$

where $A^g \equiv g\,(d+A)\,g^{-1}$ is the gauge transformation of the gauge connection A by $g \in G$. In here and in the following, we adopt the conventions of appendix **??**. To obtain the last line, we used the fact that for a compact group G and a closed subgroup H the integration over the group manifold can be split into H and G/H parts with proper left invariant Haar measures [**? ?**]. In particular, for the coset part, the G-invariant measure on G/H can be expressed as [**?**]

$$\mathcal{D}\Sigma = \prod_a \left(\Sigma^{-1}d\Sigma\right)^a, \tag{175}$$

where $\left(\Sigma^{-1}d\Sigma\right)^a$ is defined via $\left(\Sigma^{-1}d\Sigma\right)_k = \left(\Sigma^{-1}d\Sigma\right)^a T^a$, $T^a \in \mathbf{k}$. The G-invariance is seen by noting that under an arbitrary $g \in G$, $\left(\Sigma^{-1}d\Sigma\right)_k$ transforms as In the above, $h(g,\Sigma) \in H$ is defined by $g\Sigma = \Sigma^g(g,\Sigma)h(g,\Sigma)$. If we had used a naive (non-invariant) measure $\prod_a d\xi_a$ (recall $\Sigma = e^{-\xi_a T^a}$), then G-invariance of the quantum theory could be restored by adding a proper term $\int \mathcal{L}(\xi)$ to the action. The role of this term is to precisely cancel the non-invariance of the naive measure.

The gauge redundancy by G_B can be singled out by the following change of variable: $g \to g = \tilde{\Lambda} \circ g'$, where $\hat{\tilde{\Lambda}} = \tilde{\Lambda}(x, z_0) = 1$ and $\bar{\tilde{\Lambda}} = \tilde{\Lambda}(x, z_1) = \Sigma_1(x)$. After inserting the transformed version of eq. (174) into eq. (172) we get

$$Z[B^a] = \int \mathcal{D}B^i\ \mathcal{Z}\left[B^A\right], \tag{176}$$

$$\begin{aligned} \mathcal{Z}\left[B^A\right] = \int &\left[\mathcal{D}h_0\ \mathcal{D}h_1\ \mathcal{D}g'|_{\bar{g}'=h_1}^{\hat{g}'=h_0}\right]\ \mathcal{D}\Sigma_1(x)\mathcal{D}A_\mu(x,z)|^{\hat{A}=B}_{\bar{A}:\left[\begin{smallmatrix}(F)_1^i=0\\ (A)_1^a=0\end{smallmatrix}\right.}\ \mathcal{D}A_z(x,z) \\ &\times\ \mathrm{Det}\left[D_z(A_z)\right]\ \delta\left(A_z^{\tilde{\Lambda}\circ g'}\right)\ e^{iS[A_\mu(x,z),A_z(x,z)]}. \end{aligned} \tag{177}$$

Notice that $\left[\mathcal{D}h_0\ \mathcal{D}h_1\ \mathcal{D}g'|_{\bar{g}'=h_1}^{\hat{g}'=h_0}\right]$ is nothing but the integration over G_B, the gauge redundancy we hope to remove from the path integral.

We proceed further with a change of variable: $A' = A^{\tilde{\Lambda}\circ g'}$. This simplifies the argument of the delta function, making the evaluation of the A_z-integral trivial. Taking into account the changes of UV-BC, IR-BC, and performing the A_z-integral using the delta function, we arrive at

$$\begin{aligned} Z[B^a] = \int_{G_B} &\left[\mathcal{D}h_0\ \mathcal{D}h_1\ \mathcal{D}g'|_{\bar{g}'=h_1}^{\hat{g}'=h_0}\right]\ \int \mathcal{D}B^i\ \mathcal{D}\Sigma_1(x)\mathcal{D}A'_\mu(x,z)|^{\hat{A}'=B^{h_0}}_{\bar{A}':\left[\begin{smallmatrix}\left((F'_1)^{\Sigma_1^{-1}}\right)^i=0\\ \left((A'_1)^{\Sigma_1^{-1}}\right)^a=0\end{smallmatrix}\right.} \\ &\times\ e^{iS\left[(A'_\mu)^{(\tilde{\Lambda}\circ g')^{-1}}(x,z),0^{(\tilde{\Lambda}\circ g')^{-1}}\right]}. \end{aligned} \tag{178}$$

To get eq. (178) we used the fact that, upon A_z-integration, $\mathrm{Det}\left[D_z(A_z)\right] \to \mathrm{Det}[\partial_z]$ and dropped this irrelevant constant. We also used the H_1-invariance of the IR-BC to simplify its form. Notice that now the second argument (fifth component) in the action is a pure gauge contribution, which in general does not vanish.

Let us now analyze how the latest form of the partition function depends on h_0 and g'. First of all, when H_0 is not trivial, there is a residual gauge freedom that needs to be fixed. The existence of this residual gauge redundancy is seen by the appearance of the extra integration over h_0 compared to the case with a trivial H_0. A more careful statement can be made as follows. The bulk axial gauge fixing coincides with the condition $A'_z = g(\partial_z + A_z)g^{-1} = 0$. This is a condition on g, selecting a specific gauge orbit. The solution to this equation is the Wilson line

$$g = P \exp\left(\int_{z_0}^{z} dz' \, A_z\left(x, z'\right)\right), \tag{179}$$

stretched from the UV brane to a point in the bulk at z. An equivalent solution is the Wilson line from the IR brane to a bulk point at z. Our argument can be applied to both cases. This g successfully removes any A_z component everywhere in the bulk except at $z = z_0$ where g becomes 1. For the Dirichlet UV-BC, this is not an issue since the UV brane preserves no gauge symmetry. If, however, the UV-BC involves a non-trivial H_0, this indicates that the bulk axial gauge fixing is not complete, and we need to add a brane-localized gauge fixing term to eliminate the residual gauge freedom. Since the details of this extra gauge fixing do not affect our discussion below in any crucial way, we simply set $h_0 = 1$ and drop the integration over h_0. This is also equivalent to properly reinterpreting the gauge fixing constraint in eq. (174), so that instead of integrating over the entire gauge manifold, we pick a specific gauge orbit.

The advantage of this approach is that it also allows us to more easily study the g' transformation appearing in the action. We know that there are different contributions to $S[A_\mu, A_5]$, namely the gauge and CS action. The former is by assumption gauge invariant. The latter in general is not, but nevertheless we can use the fact that its variation only contributes as a boundary term to the variation of the full action, and therefore only depends on the boundary values of g'. Thus, by setting $h_0 = 1$, and by assuming from now on that the CS action satisfies $\omega_5^{(0)}(A_h) = 0$, i.e. the CS action (hence the associated anomaly $\omega_4^{(1)}$) vanishes when the gauge field A is restricted to its H_1 part, we can conclude that the full action $S[A_\mu, A_5]$ is invariant under any $g' \in G_B$ transformation. The property $\omega_5^{(0)}(A_h) = 0$ is referred to as an anomaly-free embedding (AFE) of $H_1 \subset G$.

After dropping the g' transformation from eq. (178), we make one last change of variable, which moves the Σ_1 dependence from the IR-BC to the UV-BC. We set $A_\mu = (A'_\mu)^{\Sigma_1^{-1}}$ to obtain

$$Z[B^a] = \int_{G_B} \left[\mathcal{D}h_0 \, \mathcal{D}h_1 \, \mathcal{D}g'\Big|^{\hat{g}'=h_0}_{\bar{g}'=h_1}\right] \int \mathcal{D}B^i \, \mathcal{D}\Sigma_1(x)\mathcal{D}A_\mu(x,z)\Big|^{\hat{A}=B^{\Sigma_1^{-1}}}_{\bar{A}:\left[\substack{F_1^i=0 \\ A_1^a=0}\right.} \quad e^{iS\left[A^\Lambda(x,z)\right]}, \tag{180}$$

where now $A_M = \{A_\mu, 0\}$ indicates a 5D vector with vanishing fifth component, and $\Lambda = \tilde{\Lambda}^{-1} \circ \Sigma_1$, with $\hat{\Lambda} = \Sigma_1$ and $\bar{\Lambda} = 1$. At this point, we want to make a couple of comments. First, we note that the integrand becomes completely independent of any G_B element and the integral over G_B (the infinite gauge redundancy) will be cancelled between the numerator and the denominator in any observable computation. Therefore, as usual, we can simply drop that factor. Second, since we have chosen the axial gauge, we do not need ghost fields to exponentiate the determinant factor. Third, the transformation parameter $\Lambda(x, z)$ can be thought of as an interpolating function between a coset element $\hat{\Lambda}$ and a trivial element $\bar{\Lambda}$. As we mentioned before, such an element exists whenever $\pi_4(G/H) = 0$.

We finally arrive at the gauge-fixed holographic partition function for arbitrary choice of UV and IR BC:

$$Z^{\text{g.f.}}\left[B^A\right] = \int \mathcal{D}B^i \, \mathcal{D}\Sigma_1(x)\mathcal{D}A_\mu(x,z)\Big|^{\hat{A}=B^{\Sigma_1^{-1}}}_{\bar{A}:\left[\substack{F_1^i=0 \\ A_1^a=0}\right.} \quad e^{iS\left[A^\Lambda(x,z)\right]}. \quad \text{(general UV-BC, IR-BC)} \tag{181}$$

Using this general formula, we can obtain results for special cases. Two particularly relevant ones are (i) pure Dirichlet UV-BC and pure Neumann IR-BC and (ii) pure Dirichlet UV-BC and mixed IR-BC with a G/H_1 symmetry breaking pattern. In the first case, we simply remove the integration over B^i and Σ_1 and set $\Sigma_1 = 1$ (hence $\Lambda = 1$ as well). In the second case, while we remove the B^i-integral, we keep Σ_1 and Λ as they are. We obtain

$$Z^{\text{g.f.}}\left[B^A\right] = \int \mathcal{D}A_\mu(x,z)|^{\hat{A}=B}_{\bar{A}:F^A=0}\; e^{iS[A(x,z)]}, \quad \text{(D-UV-BC, N-IR-BC)} \tag{182}$$

$$Z^{\text{g.f.}}\left[B^A\right] = \int \mathcal{D}\Sigma_1(x)\mathcal{D}A_\mu(x,z)|^{\hat{A}=B^{\Sigma_1^{-1}}}_{\bar{A}:\left[\begin{smallmatrix} F_1^i=0 \\ A_1^a=0\end{smallmatrix}\right.}\; e^{iS\left[A^\Lambda(x,z)\right]}. \quad \text{(D-UV-BC, } G/H_1\text{-IR-BC)} \tag{183}$$

7 The Holographic Formalism of Unbroken Symmetry

Having developed the necessary formalism to study gauge theories in a slice of AdS$_5$ holographically, we now turn to the holography of anomaly inflow. In this section, we focus on the case where the IR-BC is purely Neumann. In its 4D dual CFT, this corresponds to the global symmetry (either weakly gauged or not) unbroken by the vacuum condensate. The case of mixed IR-BC with the breaking pattern G/H_1 will be the subject of the next section.

7.1 Purely Global Symmetry

We first study the case with pure Dirichlet UV-BC. The dual CFT then has a purely global symmetry G, without any gauging. In particular, the relevant partition function is eq. (182) and the B^A associated with all $T^A \in \mathbf{g}$ are non-dynamical background fields.

In order to study the (in)variance of the theory, we check how the partition function transforms as we vary the source fields (see appendix **??**). Considering an infinitesimal transformation $g \approx 1 - \alpha$, the partition function transforms as

$$\begin{aligned} Z^{\text{g.f.}}\left[\left(B^A\right)^\alpha\right] &= \int \mathcal{D}A_\mu(x,z)|^{\hat{A}=B^\alpha}_{\bar{A}:F^A=0}\; e^{iS_0[A]+iS_{\text{CS}}[A]} \\ &= \int \mathcal{D}A_\mu(x,z)|^{\hat{A}=B}_{\bar{A}:F^A=0}\; e^{iS_0[A]+iS_{\text{CS}}[A^\alpha]}, \end{aligned} \tag{184}$$

where we made a change of variable $A \to A^\alpha$ and used the G-invariance of the gauge action S_0 and of the IR-BC. In addition, given a UV-localized group element $g(x) \approx 1 - \alpha(x)$ acting on the source B, we extended it to a 5D one as $\alpha(x,z) = \alpha(x)$. In the case of G/H_1 discussed in section 8, an extra subtlety appears regarding the extension of a 4D local gauge group element to a 5D one. From the second line of this equation, it is clear that the invariance of the partition function when the UV-BC is purely Dirichlet is fully determined by the transformation of the CS action

$$\int_{5\text{D}} \delta_\alpha \omega_5^{(0)}(A) = \int_{5\text{D}} d\omega_4^{(1)}(\alpha, A) \tag{185}$$

we obtain

$$Z^{\text{g.f.}}\left[\left(B^A\right)^\alpha\right] = e^{-ic\int_{\text{UV}}\omega_4^{(1)}(\alpha,B)} \int \mathcal{D}A_\mu(x,z)|^{\hat{A}=B}_{\bar{A}:F^A=0}\; e^{iS_0[A]}\; e^{ic\int_{\text{IR}}\omega_4^{(1)}(\alpha,\bar{A})}\; e^{iS_{\text{CS}}[A]}. \tag{186}$$

Here, since the UV brane-localized variance term is independent of A_μ, we factor it out of the path integral. We also observe that the partition function is not invariant under the transformation. Before we proceed any further, however, we first need to discuss one problem. That is, under the transformation, the integrand picks up an IR brane-localized variance term. Recalling that the bulk gauge symmetry G is unbroken at the IR brane, such an IR brane-localized variance is not acceptable. In fact, there is an alternative to this view point. Since the 5D gauge theory is intrinsically non-renormalizable, it is at best an effective field theory. An effective gauge theory with non-vanishing gauge anomaly can still be consistently quantized below a cut-off scale and the associated cut-off scale can be estimated by the knowledge of the anomaly. A study of a 5D $U(1)$ gauge theory along this line was presented. In order to remedy this issue, we introduce an IR-localized effective action, $\Gamma_{\mathrm{IR}}\left[\bar{A}\right]$, which under an infinitesimal transformation A^{α} shifts by an opposite anomaly factor to cancel the bulk-generated anomaly factor:

$$e^{i\Gamma_{\mathrm{IR}}\left[\bar{A^{\alpha}}\right]} = e^{-ic\int_{\mathrm{IR}}\omega_4^{(1)}(\alpha,\bar{A})}e^{i\Gamma_{\mathrm{IR}}\left[\bar{A}\right]}. \tag{187}$$

Such an effective action may be obtained by first introducing an anomalous set of 4D Weyl fermions charged under G and localized on the IR brane. Integrating out the fermions then generates $\Gamma_{\mathrm{IR}}\left[\bar{A}\right]$:

$$e^{i\Gamma_{\mathrm{IR}}\left[\bar{A}\right]} \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{iS_{\mathrm{IR}}\left[\bar{A},\psi,\bar{\psi}\right]}. \tag{188}$$

In this case, however, the coefficient c cannot be arbitrary, and in fact, must be an integral multiple of the coefficient of the chiral anomaly due to a single Weyl fermion. This leads to the *quantization* condition for the CS level c. This issue will be discussed in section **??**.

The UV-localized variance term, on the other hand, is perfectly fine: G is completely broken on the UV brane. In terms of the modified partition function $\tilde{Z}^{\text{g.f.}}$ with Γ_{IR} inserted, the transformation rule for the holographic partition function is therefore given by

$$\begin{aligned} \tilde{Z}^{\text{g.f.}}\left[B^A\right] &= \int \mathcal{D}A_\mu(x,z)|^{\hat{A}=B}_{\bar{A}:F^A=0}\ e^{iS_0[A]+iS_{\mathrm{CS}}[A]}\ e^{i\Gamma_{\mathrm{IR}}\left[\bar{A}\right]}, \\ \tilde{Z}^{\text{g.f.}}\left[\left(B^A\right)^{\alpha}\right] &= e^{-ic\int_{\mathrm{UV}}\omega_4^{(1)}(\alpha,B)}\ \tilde{Z}^{\text{g.f.}}\left[B^A\right]. \end{aligned} \tag{189}$$

In order to obtain the dual 4D CFT interpretation, we now view $\tilde{Z}^{\text{g.f.}}\left[B^A\right]$ as the partition function of a 4D CFT with classical source $\left(B^A\right)_\mu$ coupled to the CFT current operators J^μ. The phase factor in eq. (189) shows that the global symmetry G of the CFT is anomalous. This anomalous global symmetry G is unbroken by confinement, a fact dual to the pure Neumann IR-BC. Furthermore, the appearance of both the UV- and IR-localized anomaly factors in eq. (186), with same magnitude but opposite signs, encodes what we might call a *'t Hooft anomaly matching*. In order to see this explicitly, as is usually done, let us first weakly gauge G. This is done by switching the UV-BC to be purely Neumann. Under this change, what we called the source fields B before now turn into dynamical fields, and the UV-localized variance needs to be cancelled just like we did for the IR-localized anomaly factor. We proceed as we did for the IR-localized anomaly term, by introducing a UV-localized effective action $\Gamma_{\mathrm{UV}}[B]$, which again may be obtained by integrating out UV-localized 4D Weyl fermions. In order to achieve gauge anomaly cancellation, we require

$$e^{i\Gamma_{\mathrm{UV}}[B^{\alpha}]} = e^{+ic\int_{\mathrm{UV}}\omega_4^{(1)}(\alpha,B)}\ e^{i\Gamma_{\mathrm{UV}}[B]}. \tag{190}$$

The full partition function then becomes

$$\tilde{Z}^{\text{g.f.}} = \int \mathcal{D}B^A\ e^{i\Gamma_{\mathrm{UV}}[B]}\ \mathcal{D}A_\mu(x,z)|^{\hat{A}=B}_{\bar{A}:F^A=0}\ e^{iS_0[A]+iS_{\mathrm{CS}}[A]}\ e^{i\Gamma_{\mathrm{IR}}\left[\bar{A}\right]}. \tag{191}$$

As expected, the partition function is independent of any B fields, and the symmetry property of the theory is tested by making a change of variable (or field redefinition) in the form of a G-transformation and check whether it results in an anomalous change to the original theory or not.

The 4D interpretation goes as follows. The 4D confining gauge theory (in fact deformed CFT) has a weakly gauged symmetry G. In addition to the CFT preons, there is an anomalous set of external fermions (UV-localized fermions that induce Γ_{UV}) charged under G. While CFT and external sector are not individually G-anomaly free, these two contributions, nonetheless, cancel, making the gauging of G legal. Importantly, these external fermions need not couple to the confining CFT gauge force, and they are spectator fermions. Along the renormalization group (RG) flow, while the chiral anomaly does not get renormalized, the CFT sector undergoes confinement. By assumption, G is not broken, and the original anomaly of the CFT preons should be reproduced by a spectrum of massless composite fermions. These composite fermions are the IR brane-localized fermions we introduced in 5D to cancel the IR-localized variance. For these reasons, the choice of Neumann IR-BC and the resulting requirement of cancelling the IR-localized variance term by 4D fermions on the IR brane is the holographic realization of 't Hooft anomaly matching.

One perhaps interesting feature deduced from the above discussion is that the anomaly inflowed from the bulk CS theory must be the one that has vanishing mixed anomaly between global G and confining gauge group G_s of the CFT. In order to make this point clear, we may consider a $U(1)$ CS theory in the bulk. The variances induced on the boundaries are dual to a $U(1)^3$ anomaly of the global symmetry. If this global $U(1)$ current had a Adler-Bell-Jackiw (ABJ) type anomaly with the confining gauge force of the CFT, then the spectators would be necessarily coupled to the strong interaction as well, and as a result, 't Hooft argument for the anomaly matching would not hold. However, we have seen that, with Neumann IR-BC, anomaly inflow by the bulk CS action always comes with an IR-localized variance term in addition to the UV variance term: 't Hooft anomaly matching is automatically at play. As we discuss in section 7.2, this fact holds even if G is weakly gauged.

7.2 Partially Gauged Symmetry

In this section, we consider the anomaly inflow with a mixed UV-BC in which a subgroup $H_0 \subset G$ takes a Neumann BC, while the coset G/H_0 has a Dirichlet BC. In order to make the discussion as concrete as possible, and also in part to demonstrate the usage of the formalism, we consider a product group $G = G_1 \times G_2$, where $G_1 = U(1)$ and G_2 is any compact simple Lie group. We choose UV-BC such that one factor group takes Neumann while the other takes Dirichlet BC. This choice is interesting because it admits a non-trivial mixed CS action, hence a mixed anomaly interpretation in the dual 4D picture. A lot of qualitative features we describe below apply to more general cases and an explicit analysis with arbitrary choice of $G \to H_0$ can be achieved straightforwardly.

In the 4D dual description, the CFT has a global $G = G_1 \times G_2$ symmetry and one factor group (either G_1 or G_2) is weakly gauged. We present both cases, one with gauged G_1 and the other with gauged G_2. With pure Neumann IR-BC, none of these symmetries are broken at the confinement scale. The anomaly inflow from the mixed CS action will get a 4D interpretation in terms of G_1-G_2 mixed anomaly, while inflow by the pure CS actions corresponds to pure G_1 and/or G_2 anomalies.

Let us choose $G_2 = SU(2)$. Any other choice of G_2 will require a very similar discussion. One exceptional property of the group $SU(2)$ is that it has vanishing $d^{abc} \propto \mathrm{Tr}[T^a\{T^b, T^c\}]$ and the pure $SU(2)$ CS action vanishes identically. This feature is dual to the fact that there is no perturbative $SU(2)$ anomaly in 4D. The bulk CS action consists of two contributions: pure $U(1)$ and mixed $U(1)$-$SU(2)$. The *form* of the mixed CS action can be obtained by first embedding $U(1)$ and $SU(2)$ into a simple compact Lie group G_{GUT}. Once this is done, then the GUT gauge field A can be written as a sum $A = V + W$ in terms of the $U(1)$ gauge field V and the $SU(2)$ gauge field W. The mixed CS action can be read off from the CS action of A [**?**]. In terms of the *canonical* $\omega_5^{(0)}(A)$, after a couple of integrations by parts, we get (including the pure $U(1)$ CS action)

$$S_{\mathrm{CS}}[V, W] = c_1 \int_{5\mathrm{D}} V dV dV + c_{12} \int_{5\mathrm{D}} 3\mathrm{Tr}\left[V F_W^2\right] + d\mathrm{Tr}\left[\left(2VWdW + \frac{3}{2}VW^3\right)\right], \tag{192}$$

where $F_W = dW + W^2$ is the $SU(2)$ field strength 2-form. Notice that the last term, obtained as a result of integration by parts, is a brane-localized term. The virtue of this form for the CS action is that the bulk CS actions are manifestly $SU(2)$-invariant and any non-trivial $SU(2)$ transformations are from the boundary terms. It may be worth mentioning that a priori the two CS levels c_1 and c_{12} are independent and are subject to separate quantization conditions.

Under $U(1)$ and $SU(2)$ transformations, the CS action changes as

$$\begin{aligned}\delta_{\alpha_1} S_{\rm CS} = c_1 \int_{\rm IR} \alpha_1 d\bar{V} d\bar{V} + c_{12} \int_{\rm IR} \left(3{\rm Tr}\left[\alpha_1 F_{\bar{W}}^2\right] - {\rm Tr}\left[\alpha_1 \left(2d\bar{W}d\bar{W} + \frac{9}{2} d\bar{W}\bar{W}^2\right)\right]\right) \\ - c_1 \int_{\rm UV} \alpha_1 d\hat{V} d\hat{V} - c_{12} \int_{\rm UV} \left(3{\rm Tr}\left[\alpha_1 F_{\hat{W}}^2\right] - {\rm Tr}\left[\alpha_1 \left(2d\hat{W}d\hat{W} + \frac{9}{2} d\hat{W}\hat{W}^2\right)\right]\right),\end{aligned} \tag{193}$$

$$\begin{aligned}\delta_{\alpha_2} S_{\rm CS} = c_{12} \int_{\rm IR} {\rm Tr}\left[2\bar{V} d\alpha_2 \left(d\bar{W} - 2\bar{W}^2\right) + \frac{9}{2}\bar{V}\left(d\alpha_2 \bar{W}^2\right)\right] \\ - c_{12} \int_{\rm UV} {\rm Tr}\left[2\hat{V} d\alpha_2 \left(d\hat{W} - 2\hat{W}^2\right) + \frac{9}{2}\hat{V}\left(d\alpha_2 \hat{W}^2\right)\right].\end{aligned} \tag{194}$$

Next we discuss two cases, one with gauged G_1 and the other with gauged G_2 in turn. We first consider the case with Neumann UV-BC for $U(1)$ and Dirichlet UV-BC for $SU(2)$. The relevant partition function is

$$Z^{\rm g.f.}\left[B^A\right] = \int \mathcal{D}B^i \, \mathcal{D}A_\mu|_{\bar{A}:F^A=0}^{\hat{A}=B} \, e^{iS_0 + iS_{\rm CS}}, \tag{195}$$

with $S_{\rm CS}$ given in eq. (192). In this case, $B^i = \hat{V}$ and $B^a = \hat{W}$. Since the full $G = G_1 \times G_2$ is unbroken on the IR brane, we need to cancel the anomaly factors there. In the 4D dual description, the c_1-term in eq. (193) corresponds to the $U(1)^3$ anomaly, while the c_{12}-terms in eq. (193) and (194) represent the mixed anomaly. One way to eliminate the IR-localized variance is to introduce IR-localized Weyl fermions charged under both $U(1)$ and $SU(2)$. The requirement is that this set is anomalous in such a way that their $U(1)^3$ and mixed anomalies cancel the CS-generated variance terms. In order to present another possibility, however, we take a slightly different path.

We can add a local counter terms $dB_4(A_0, A_1)$ to the bulk CS term in such a way that the CS action becomes invariant under an $H_0 \subset G$ transformation. In particular, this works well for the product group. Applying this to the mixed CS action, in the current example, we take $A_0 = V$ and $A_1 = A = V + W$. The shifted mixed CS action $\omega_5^{(0)} \to \tilde{\omega}_5^{(0)}(V, A) = \omega_5^{(0)}(A) - \omega_5^{(0)}(V) + dB_4(V, A)$ is then invariant under a $U(1)$ transformations. To be more precise, we first note that $\omega_5^{(0)}(A)$ contains the pure $U(1)$ CS action as well as the mixed CS action. Using a short notation for the mixed CS action as $\omega_5^{(0)}(\text{mixed}) = \omega_5^{(0)}(A) - \omega_5^{(0)}(V)$, what we really do is

$$\begin{aligned} S_{\rm CS} &= c_1 \int \omega_5^{(0)}(V) + c_{12} \int \omega_5^{(0)}(\text{mixed}) \\ &\to c_1 \int \omega_5^{(0)}(V) + c_{12} \int \tilde{\omega}_5^{(0)}(V, A) \\ &= c_1 \int \omega_5^{(0)}(V) + c_{12} \int \left[\omega_5^{(0)}(\text{mixed}) + dB_4(V, A)\right]. \end{aligned} \tag{196}$$

Notice that in $\tilde{\omega}_5^{(0)}(V, A)$ there is a cancellation between two $U(1)$ CS actions, and effectively the procedure is equivalent to adding a local counter terms dB_4 to the original mixed CS action. An important property we recover is that this shifted CS action is invariant under a $U(1)$ transformation. We will write the shifted CS action as $\tilde{S}_{\rm CS} = S_{\rm CS} + S_{\rm CT}$, where $S_{\rm CT}$ is the action for the counter terms. In order to show the $U(1)$-invariance more explicitly, we first note that from the explicit form of the counter term B_4 in this case is given by

$$\begin{aligned} S_{\rm CT} &= c_{\rm CT} \int_{\rm 5D} dB_4(V, A), \\ B_4(V, A) &= {\rm Tr}\left[VWdW + \frac{1}{2}VW^3\right]. \end{aligned} \tag{197}$$

One may notice that these two terms in B_4 are exactly the same as the boundary terms in eq. (192), only the relative coefficients differ. Also, eventually, $c_{\rm CT} = c_{12}$ as is evident from eq. (196). Here, we use a separate notation temporarily to make one important point below. Explicit computations show that with counter term added, the shifted CS action transforms according to

$$\begin{aligned}\delta_{\alpha_1}\tilde{S}_{\rm CS} &= c_1\int_{\rm IR}\alpha_1 d\bar{V}d\bar{V} + c_{12}\int_{\rm IR}\left(1-\frac{c_{\rm CT}}{c_{12}}\right){\rm Tr}\left[\alpha_1 F_{\bar{W}}^2\right] \\ &\quad - c_1\int_{\rm UV}\alpha_1 d\hat{V}d\hat{V} - c_{12}\int_{\rm UV}\left(1-\frac{c_{\rm CT}}{c_{12}}\right){\rm Tr}\left[\alpha_1 F_{\hat{W}}^2\right], \end{aligned}\tag{198}$$

$$\delta_{\alpha_2}\tilde{S}_{\rm CS} = c_{12}\int_{\rm IR}\left(2+\frac{c_{\rm CT}}{c_{12}}\right){\rm Tr}\left[\bar{V}d\alpha_2 d\bar{W}\right] - c_{12}\int_{\rm UV}\left(2+\frac{c_{\rm CT}}{c_{12}}\right){\rm Tr}\left[\hat{V}d\alpha_2 d\hat{W}\right].\tag{199}$$

It is observed that the mixed anomaly terms are such that in units of c_{12} the "sum" of $U(1)$ and $SU(2)$ variations is fixed, $(1-c_{\rm CT}/c_{12})+(2+c_{\rm CT}/c_{12})=3$, regardless of the size of the counter term. In contrast, the "difference" is not fixed, and in fact is proportional to the size of the counter term [**?**]. We also confirm that with $c_{\rm CT}=c_{12}$, the shifted *mixed* CS action is indeed invariant under $U(1)$ transformations: all the mixed anomaly is attributed to the $SU(2)$ currents.

We now introduce a set of IR-localized Weyl fermions charged under both $U(1)$ and $SU(2)$ such that their chiral anomaly cancels the CS-induced variance terms. Equivalently, we add $\Gamma_{\rm IR}\left[\bar{V},\bar{W}\right]$ which transforms as

$$\begin{aligned} e^{i\Gamma_{\rm IR}\left[\bar{V}^{\alpha_1},\bar{W}\right]} &= e^{-ic_1\int_{\rm IR}\alpha_1 F_{\bar{V}}^2}\, e^{i\Gamma_{\rm IR}\left[\bar{V},\bar{W}\right]}, \\ e^{i\Gamma_{\rm IR}\left[\bar{V},\bar{W}^{\alpha_2}\right]} &= e^{-ic_{12}\int_{\rm IR} 3{\rm Tr}\left[\bar{V}d\alpha_2 d\bar{W}\right]}\, e^{i\Gamma_{\rm IR}\left[\bar{V},\bar{W}\right]}. \end{aligned}\tag{200}$$

Considering the UV-localized variance term, since only the $U(1)$ factor takes Neumann BC, we only need to cancel the pure $U(1)^3$ variance term. Hence, on the UV brane, we add a set of Weyl fermions charged under $U(1)$ only. The UV brane-localized effective action, upon integrating out these fermions, is then required to transform as

$$e^{i\Gamma_{\rm UV}\left[\hat{V}^{\alpha_1}\right]} = e^{+ic_1\int_{\rm UV}\alpha_1 F_{\hat{V}}^2}\, e^{i\Gamma_{\rm UV}\left[\hat{V}\right]}.\tag{201}$$

Moving on to the 4D dual interpretation, at the UV scale, the theory consists of a CFT sector and a set of external fermions. The CFT has a global symmetry group $G=SU(2)\times U(1)$, of which the $U(1)$ factor is weakly gauged. The external fermions are charged under the $U(1)$ gauge force. The $U(1)^3$ gauge anomaly is cancelled between the two contributions from the CFT and the external sector. There is a non-vanishing $U(1)$-$SU(2)$ mixed anomaly of ABJ type, which comes only from the CFT sector. Naively, depending on the UV regulator, the mixed anomaly can be shared among gauge and global currents. In particular, if the gauge current is anomalous, we have an issue with gauging the $U(1)$ factor. However, we added local counter terms proportional to $B_4(V,A)$, so that we moved all the mixed anomaly to the global $SU(2)$ currents. In this way, the gauged $U(1)$ symmetry is free of any mixed anomalies. Thanks to this feature, the external fermions need not carry the global $SU(2)$ quantum numbers. This is an analog of what occurs in QCD: there the anomaly computed from the Feynman diagrams (*consistent anomaly*) results in the non-conservation of both vector and axial-vector currents. However, by adding an appropriate counter term (*Bardeen's counter term*), the vector current becomes conserved and all the mixed anomaly is moved to the axial-vector current (*covariant anomaly*).

As the theory RG runs to the IR scale, the CFT sector confines and the anomalies are matched by massless composite fermions. Notice that in the current example, the $U(1)$ factor is physically gauged. Nevertheless, anomaly matching arguments go through since $U(1)$ is not a confining force. As for the $SU(2)$ part, as usual, we formally weakly gauge it, and introduce extra external spectator fermions to cancel the mixed anomaly. This mixed anomaly is also reliably reproduced by the massless composite fermions in the IR. Our holographic study of anomaly inflow, therefore, shows that in a confining gauge theory, the anomaly associated with a weakly gauged symmetry in the UV (i.e. $U(1)$ gauge anomaly carried by the composite sector) as well as the ABJ anomaly (i.e. mixed $U(1)$-$SU(2)$) are matched by the composite spectrum in the IR.

In the case $G = G_1 \times G_2$ with gauged G_2, we take $A_0 = W$ and $A_1 = A$. The counter term $B_4(W, A)$ is similarly given by

$$B_4(W, A) = -\mathrm{Tr}\left[2VWdW + \frac{3}{2}VW^3\right]. \tag{202}$$

This exactly cancels the boundary terms in eq. (192) and the shifted CS action becomes manifestly $SU(2)$ invariant. Under a $U(1)$ transformation, we get

$$\delta_{\alpha_1}\tilde{S}_{\mathrm{CS}} = c_1 \int_{\mathrm{IR}} \alpha_1 F_{\bar{V}}^2 + c_{12}\int_{\mathrm{IR}} 3\mathrm{Tr}\left[\alpha_1 F_{\bar{W}}^2\right] - (\mathrm{UV}), \tag{203}$$

where the UV-localized variance terms are obtained from the IR-localized terms with the replacement $\bar{V}, \bar{W} \to \hat{V}, \hat{W}$. In this case, there is no gauge anomaly. All the anomalies are on the global current, either pure global $U(1)^3$ or ABJ-type mixed anomalies. On the IR, since G is preserved, we again need to include brane-localized fermions. On the UV, on the other hand, thanks to the counter term B_4, no $SU(2)$ variance term shows up and there's no need to add anything.

The 4D interpretation is straightforward. The CFT has a global symmetry group $G = SU(2) \times U(1)$ and $SU(2)$ is weakly gauged. While there are global $U(1)^3$ and mixed anomalies, a proper counter term is added in such a way that the gauged $SU(2)$ is free of any mixed anomaly. We can again formally weakly gauge the $U(1)$ part and add spectator fermions. In the IR, when the CFT confines, composite fermions achieve 't Hooft anomaly matching, the 4D dual of IR-localized fermions. Once again, our holographic study of anomaly inflow indicates that an ABJ anomaly in the UV, when the gauged external legs are weakly interacting, is matched by the composite spectrum in the IR. While in the previous section it was $U(1)$ that was weakly gauged, here it is a non-Abelian group, namely $SU(2)$, and the same analysis can be performed with any non-Abelian group.

8 Spontaneously Broken Symmetry and Wess-Zumino-Witten Action

In this section, we consider the possibility that the IR-BC breaks G down to $H_1 \subset G$. In its holographic 4D CFT dual, this corresponds to the spontaneous breaking of the symmetry group G down to H_1 by confinement. If the UV-BC is Dirichlet for all G, the bulk gauge group is dual to a *global* symmetry of the 4D CFT. On the other hand, choosing Neumann UV-BC for all G corresponds to a *weakly gauged* symmetry. A slightly less trivial case can be analyzed by choosing Dirichlet UV-BC for some of the generators, and Neumann UV-BC for the rest. For instance, if we choose Neumann UV-BC for a subgroup $H_0 \subset H_1$, and Dirichlet for the rest, the dual picture is that of a CFT with global symmetry G spontaneously broken to H_1 by vacuum condensate, and a subgroup $H_0 \subset H_1$ is weakly gauged. This is very much like what happens in QCD. There, G is the chiral symmetry $G = SU(3)_L \times SU(3)_R$, which is broken to $H_1 = SU(3)_V$ by the quark condensate. Moreover, $U(1)_{\mathrm{EM}} \subset H_1$ is weakly gauged.

Using the result of section 6, we start with eq. (181), which we report below again for convenience:

$$Z^{\mathrm{g.f.}}\left[B^A\right] = \int \mathcal{D}B^i\, \mathcal{Z}\left[B^A\right] = \int \mathcal{D}B^i\, \mathcal{D}\Sigma_1(x)\mathcal{D}A_\mu(x,z)\Big|^{\hat{A}=B^{\Sigma_1^{-1}}}_{\bar{A}:\left[\begin{smallmatrix} F_1^i=0 \\ A_1^a=0\end{smallmatrix}\right.} e^{iS\left[A^\Lambda(x,z)\right]}, \tag{204}$$

where $\hat{\Lambda} = \Sigma_1$ and $\bar{\Lambda} = 1$. Before we delve into a detailed discussion of the two separate cases (Dirichlet UV-BC vs Neumann UV-BC) let us study the gauge transformation properties of $\mathcal{Z}\left[B^A\right]$. Once this is understood, it's easier to focus on a specific case.

The action consists of the gauge kinetic term, S_0, and of the CS action, which for now we set to its *canonical* version $S_{\mathrm{CS}} = c\int \omega_5^{(0)}(A)$. Also, for the sake of simplicity, we will just write $\bar{A}$ for the IR-BC. For example, IR-BC after a gauge transformation by $h \in H$ will be denoted as $\bar{A}^h$, and this means

$(F_1^h)^i = 0$ for the unbroken generators and $(A_1^h)^a = 0$ for the broken ones. Under $\hat{g} \in G$ on the UV brane, $\mathcal{Z}\left[B^A\right]$ transforms as

$$
\begin{aligned}
\mathcal{Z}\left[\left(B^A\right)^{\hat{g}}\right] &= \int \mathcal{D}\Sigma_1 \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{\Sigma_1^{-1}\circ\hat{g}}} \; e^{iS_0\left[A^\Lambda\right]+iS_{\rm CS}\left[A^\Lambda\right]} \\
&= \int \mathcal{D}\Sigma_1^{\hat{g}} \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{(\Sigma^{\hat{g}})^{-1}\circ\hat{g}}} \; e^{iS_0\left[A^{\Lambda^g}\right]+iS_{\rm CS}\left[A^{\Lambda^g}\right]} \\
&= \int \mathcal{D}\Sigma_1 \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{h(\hat{g},\Sigma)\circ\Sigma^{-1}}} \; e^{iS_0\left[A^{\Lambda^g}\right]+iS_{\rm CS}\left[A^{\Lambda^g}\right]}.
\end{aligned}
\tag{205}
$$

In the second line, we made a change of integration variable $\Sigma_1 \to \Sigma_1^{\hat{g}}$, and used the fact that the integration measure on G/H_1 is the left invariant Haar measure to get the third line.

We mention that we extend a given $\hat{g}(x) \approx 1-\hat{\alpha}(x)$ on the UV brane to 5D such that $g(x, z_0) = \hat{g}$ and $g(x, z_1) = \bar{g} \in H_1$ [**?**]. The reason for the latter condition is simply that on the IR brane H_1 is the only unbroken gauge group. Such an extension of gauge element exists provided $\pi_4(G/H)$ is trivial, which we assume in the paper. This can be understood by first decomposing $g = \Sigma_1 h,\ \Sigma_1 \in G/H_1,\ h \in H_1$, and realizing that the desired extension is equivalent to the deformation that takes a coset element on the UV brane into a trivial element, i.e. H_1-element, on the IR brane.

In order to make the overall transformation more manifest, next we make a change of variable: $A \to A^{h(g,\Sigma)}$. The partition function becomes

$$
\begin{aligned}
\mathcal{Z}\left[\left(B^A\right)^{\hat{g}}\right] &= \int \mathcal{D}\Sigma_1 \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{\Sigma_1^{-1}}} \; e^{iS_0[A]+iS_{\rm CS}\left[A^{\Lambda^g\circ h}\right]} \\
&= \int \mathcal{D}\Sigma_1 \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{\Sigma_1^{-1}}} \; e^{iS_0[A]+iS_{\rm CS}\left[A^{g\circ\Lambda}\right]},
\end{aligned}
\tag{206}
$$

where we used the H_1-invariance of the IR-BC and the G-invariance of S_0. We also used $g\Lambda = \Lambda^g h(g,\Lambda)$ to rewrite the argument of the CS action. Hence, we see that the theory is not invariant under a given $g \in G$ transformation, and in particular, its non-invariance comes from the bulk CS action. In order to understand the form of the transformation, it is sufficient to study the infinitesimal version. Under $\hat{g} \approx 1 - \hat{\alpha}$, we get

$$
\begin{aligned}
\mathcal{Z}\left[\left(B^A\right)^{\hat{g}}\right] &= \int \mathcal{D}\Sigma_1 \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{\Sigma_1^{-1}}} \; e^{iS_0[A]+iS_{\rm CS}\left[A^\Lambda\right]+i\delta_\alpha S_{\rm CS}\left[A^\Lambda\right]} \\
&= e^{-ic\int_{\rm UV}\omega_4^{(1)}(\hat{\alpha},B)} \int \mathcal{D}\Sigma_1 \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{\Sigma_1^{-1}}} \; e^{iS_0[A]+iS_{\rm CS}\left[A^\Lambda\right]} \; e^{+ic\int_{\rm IR}\omega_4^{(1)}(\bar{\alpha},A_h)}.
\end{aligned}
\tag{207}
$$

We used that $A_\mu(x, z_0) = \hat{A}_\mu = B_\mu^{\Sigma_1^{-1}}$ and $A_\mu(x, z_1) = \bar{A}_\mu = A_h$, where A_h is the restriction of the gauge field to $\mathbf{h}_1$. It may be worth mentioning that with $(A)^{g\circ\Lambda} = (A^\Lambda)^g$ we denote the whole gauge transformation of A^Λ as a single gauge connection, i.e. $\left(A^\Lambda\right)^g = g\left(d + A^\Lambda\right)g^{-1}$.

In order to restore 5D consistency, we need to modify the theory to remove the IR-localized variance term. This may be attained by adding an appropriately anomalous set of localized 4D fermions. Alternatively, the problem is solved if the subgroup H_1 is an anomaly-free embedding (AFE). Anomaly-free embedding means $\omega_5^{(0)}(A_h) = 0$, i.e. the CS action (hence associated anomaly $\omega_4^{(1)}$) vanishes when the gauge field A is restricted to its H_1 part. In what follows, we take the second path. In addition, we also promote $\omega_5^{(0)}(A) \to \tilde{\omega}_5^{(0)}(A_h, A)$ so that the bulk CS action is invariant under H_1 transformations. When it comes to the UV-localized variance term, the required amendment and associated dual interpretation depends on the UV-BC.

8.1 Purely Global Symmetry

When all of G satisfies Dirichlet BC on the UV brane, from a 5D perspective, there is no induced UV surface terms as a result of a gauge transformation. Hence, once we cure the IR brane-localized non-invariance, the 5D theory is consistent. Denoting the shifted CS action as $\tilde{S}_{\rm CS} = c\int \tilde{\omega}_5^{(0)}(A_h, A)$, eq. (207) becomes

$$\begin{aligned} Z^{\rm g.f.}\left[B^A\right] &= \int \mathcal{D}\Sigma_1 \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{\Sigma_1^{-1}}}\ e^{iS_0[A]+i\tilde{S}_{\rm CS}\left[A^\Lambda\right]} \\ \to Z^{\rm g.f.}\left[\left(B^A\right)^{\hat{\alpha}}\right] &= e^{-ic\int_{\rm UV}\tilde{\omega}_4^{(1)}(\hat{\alpha},B)}\ Z^{\rm g.f.}\left[B^A\right]. \end{aligned} \tag{208}$$

We remark that while the theory is not invariant under $\hat{\alpha} \in \mathbf{k}_1$ transformations, the anomalous phase $\tilde{\omega}_4^{(1)}(\hat{\alpha}, B)$ vanishes for $\hat{\alpha} \in \mathbf{h}$ transformations.

In the dual 4D CFT, we interpret this as an anomalous global symmetry G of the CFT. More precisely, the global symmetry G is spontaneously broken at the confinement scale by the vacuum condensate, and while the unbroken group H_1 is free of anomalies (thanks to the counter term we added), the anomaly associated with G/H_1 is captured by the above phase factor. Since $H_1 \subset G$ is already anomaly-free, it can be gauged if wanted. In 5D, this is equivalent to the statement that we can freely switch the UV-BC to Neumann without needing extra modifications.

We have seen that if $H_1 \subset G$ is an AFE, no IR brane-localized state is needed. Given that the UV phase has a non-trivial anomaly, this raises a question about anomaly matching. In QCD, the chiral anomaly in the UV quark phase is matched in the IR hadronic phase by the gauged WZW term of Nambu-Goldstone bosons (NGBs). It is then natural to ask if this feature can be seen in the current framework. After all, since we take Dirichlet UV-BC for all G, all of G/H_1 describes physical NGBs in the dual 4D CFT. In light of the discussion given, it can be shown that the answer to this question is yes, but with some subtleties. To see this, we first rewrite the shifted CS action in the partition function as

$$\tilde{\omega}_5^{(0)}\left(\left(A^\Lambda\right)_h, A^\Lambda\right) \to \tilde{\omega}_5^{(0)}(A_h, A) + \left(\tilde{\omega}_5^{(0)}\left(\left(A^\Lambda\right)_h, A^\Lambda\right) - \tilde{\omega}_5^{(0)}(A_h, A)\right) = \tilde{\omega}_5^{(0)}(A_h, A) + \frac{\mathcal{L}_{\rm WZW}}{c}. \tag{209}$$

First of all, the new version is trivially equal to the original CS action. In order to see how the expression in parenthesis is indeed the wanted WZW action, we note that it vanishes trivially as $\Sigma_1 \to 1$, and it depends only on the (UV) boundary value. Hence, it satisfies two of the three conditions for it to qualify as the WZW action. For the last requirement, i.e. solving the anomalous Ward identity, it is sufficient to show that $\tilde{\omega}_5^{(0)}(A_h, A)$, as part of the partition function, is invariant under G-transformations. This is achieved rather easily. One just recalls that under an arbitrary $g \in G$ transformation, the change of the partition function is captured by $A \to A^{h(g,\Sigma_1)}$ in the CS terms. For $\tilde{\omega}_5^{(0)}(A_h, A)$ this corresponds to just $\tilde{\omega}_5^{(0)}((A_h)^{h(g,\Sigma_1)}, A^{h(g,\Sigma_1)})$, which is invariant according to the representation. Therefore, the expression in parenthesis is indeed the WZW term. For the same reason, in the splitting $\tilde{\omega}_5^{(0)}(A_h, A) + \mathcal{L}_{\rm WZW}/c$, the first term is invariant under any $g \in G$ transformation, and the non-invariance is completely encoded in the WZW term.

Crucially, in the partition function, e.g. eq. (208), the path integral variable $\Sigma_1(x)$ depends only on the 4D spacetime coordinates, and upon integrating over the bulk, we get the holographic effective action which depends on $\Sigma_1(x)$ as well as on the boundary value B. In particular, as we just showed, the WZW action only depends on the boundary value, and it can be taken out of the integral over $A_\mu(x,z)$:

$$\begin{aligned} Z^{\rm g.f.}\left[B^A\right] &= \int \mathcal{D}\Sigma_1(x)\ e^{iS_{\rm WZW}\left[B^A,\Sigma_1\right]} \int \mathcal{D}A_\mu|_{\bar{A}}^{\hat{A}=B^{\Sigma_1^{-1}}}\ e^{iS_0[A]+i\tilde{S}_{\rm CS}[A]} \\ &= \int \mathcal{D}\Sigma_1(x)\ e^{iS_{\rm WZW}\left[B^A,\Sigma_1\right]}\ e^{iS_{\rm h,0}\left[B^A,\Sigma_1\right]}. \end{aligned} \tag{210}$$

In the above, $S_{\rm h,0}$ is the holographic effective action obtained by integrating out the bulk degrees of freedom. With pure Dirichlet UV-BC, this action describes the chiral perturbation theory of massless NGBs. The physical NGB fields $\Sigma_1(x)$ correspond to the Wilson line of the zero mode A_5, and hence constitute the low energy degrees of freedom. From this discussion, therefore, it is clear that, in the deep IR after integrating out all massive hadronic states (i.e. integrating out the bulk in 5D), the anomaly of the global symmetry is maintained by the WZW term of NGBs.

We wish to emphasize the differences of the WZW action we obtained in the holographic context compared to the discussion given in the context of $2n$-dimensional spacetime. There, the WZW term is given as $\tilde{\omega}^{(0)}_{2n+1}(A_h, A) - \tilde{\omega}^{(0)}_{2n+1}\left((A^{\Lambda^{-1}})_h, A^{\Lambda^{-1}}\right)$, and the non-invariance comes from the first term, while the second term is invariant. Here, on the other hand, the WZW term is given by eq. (209), and the non-invariance is from $\tilde{\omega}^{(0)}_5(A^\Lambda_h, A^\Lambda)$, the naive shifted CS term being invariant. Of course, this is in part because while in the $g \in G$ transformation is incorporated directly in terms of g, here it is effectively in terms of $h(g, \Sigma_1)$.

A slightly more explicit expression for the WZW action is obtained as follows. Since we assume that H_1 is an anomaly-free embedding (AFE) in G, the d-symbol, $d^{ijk} \propto \mathrm{Tr}[\{T^i, T^j\}T^k]$, vanishes for H_1. Using the properties listed, we can write

$$\begin{aligned} \omega^{(0)}_5(A^\Lambda) &= \tilde{\omega}^{(0)}_5(0, A^\Lambda) = \tilde{\omega}^{(0)}_5(\Lambda^{-1}d\Lambda, A) \\ &= \omega^{(0)}_5(A) - \omega^{(0)}_5(\Lambda^{-1}d\Lambda) + dB_4(\Lambda^{-1}d\Lambda, A). \end{aligned} \tag{211}$$

To get the second equality, we used $\tilde{\omega}^{(0)}_{2n+1}((A_h)^g, A^g) = \tilde{\omega}^{(0)}_{2n+1}(A_h, A)$ with $g = \Lambda^{-1}$. The WZW action then is obtained to be

$$S_{\mathrm{WZW}}[\Sigma_1, B]/c = -\int_{5D} \omega^{(0)}_5(\Lambda^{-1}d\Lambda) + \int_{\partial(5D)} B_4\left(\Lambda^{-1}d\Lambda, A\right) + B_4\left((A^\Lambda)_h, A^\Lambda\right) - B_4(A_h, A). \tag{212}$$

We can use this to extract the pure NGB WZW terms. For this, we just set $A = 0$. Since B_{2n} vanishes if one or both of the arguments are set to zero, we get

$$S_{\mathrm{WZW}}[\Sigma_1, B = 0]/c = -\int_{5D} \omega^{(0)}_5\left(\Lambda^{-1}d\Lambda\right) + \int_{\partial(5D)} B_4((\Lambda d\Lambda^{-1})_h, \Lambda d\Lambda^{-1}). \tag{213}$$

For small $\xi(x)^a$, we expand $\Sigma_1(x) = e^{-\xi(x)} \approx 1 - \xi^a(x)X^a$, and in particular, $\Sigma_1 d\Sigma_1^{-1} = d\xi + \mathcal{O}(\xi^2)$. The point is that infinitesimally, $\Sigma_1 d\Sigma_1^{-1} = d\xi \in \mathbf{k}$ (i.e. $(\Lambda d\Lambda^{-1})_h = 0$), and the B_4 term does not contribute. Using $U = \Lambda^{-1}d\Lambda \approx -d\xi$ straightforward steps lead to

$$\begin{aligned} S_{\mathrm{WZW}}[\xi, B = 0]/c &= -(-1)^5 \frac{1}{10}\int_{5D} \mathrm{Tr}\left[(d\xi)^5\right] + \mathcal{O}(\xi^6) \\ &= -\frac{1}{10}\int_{\mathrm{UV}} \mathrm{Tr}\left[\xi(x)(d\xi(x))^4\right] + \mathcal{O}(\xi^6). \end{aligned} \tag{214}$$

A direct comparison with the existing literature can be made by noting that for chiral symmetry, the parametrization $\Sigma = e^{-2i\xi}$ is used (note the factor of 2). Also, as we show the overall coefficient consistent with the quantization condition is $c = \frac{\kappa}{24\pi^2}$ with $\kappa \in \mathbb{Z}$. It is in this normalization that the bulk non-invariance can be cancelled by integral multiple of fermion chiral anomaly. (Q) Note that this normalization is *different* from the general expression obtained in [**?**], i.e. K_n. However, in the same paper, they do have $\frac{1}{24\pi^2}$ for non-Abelian chiral anomaly. Taking into account the factor 2^5 from the difference in GB parametrization, we get

$$S_{\mathrm{WZW}}[\xi, B = 0] = -\frac{2}{15\pi^2}\kappa \int_{\mathrm{UV}} \mathrm{Tr}\left[\xi(x)(d\xi(x))^4\right] + \mathcal{O}(\xi^6), \qquad \xi = \frac{\Pi(x)}{F_\pi}. \tag{215}$$

8.2 Gauged Symmetry

In this section, we gauge some part of the group G, and analyze the resulting anomaly inflow. This is done by taking Neumann UV-BC for some of the generators of $\mathbf{g}$. We will first consider the case in which we only gauge a subgroup $H_0 \subset H_1$. This is an analog of QCD, and we will obtain a *gauged* version of the WZW action as the low energy effective action, as we expect. The gauged WZW action includes terms that match ABJ and 't Hooft anomalies. We then move on to analyze the most general possibility.

We consider a situation in which a subgroup $H_0 \subset G$ which is not a proper subset of H_1 is gauged. This is a prototypical situation for dynamical symmetry breaking of the electroweak group of the SM. Here, we focus on the topological terms in such a theory, and show, among other things, that the (would-be) NGBs associated with the gauged generators can be removed by means of field redefinitions.

Let us first consider the case in which the UV-BC for the subgroup $H_0 \subset H_1 \subset G$ is taken to be Neumann, while the rest G/H_0 is set to be Dirichlet. The relevant partition function is eq. (204) with the shifted CS action and $B^i_\mu \in \mathbf{h}_0$. As we discussed in section 8.1, the shifted CS action is invariant under any H_1 transformations. This in turn implies that under any gauged $H_0 \subset H_1$ transformations, the shifted action is automatically invariant as well. Therefore, we do not need to add any UV brane-localized effective action: with $\omega_5^{(0)}(A) \to \tilde{\omega}_5^{(0)}(A_h, A)$ the 5D theory is consistent. Using eq. (207), the change in the partition function under an infinitesimal G/H_1 transformation $\hat{\gamma} \in \mathbf{k}_1$ is found to be

$$\begin{aligned} Z^{\text{g.f.}}\left[(B^a)^{\hat{\gamma}}\right] &= \int \mathcal{D}\left(B^i\right)^{\hat{\gamma}} \int \mathcal{D}\left(\Sigma_1\right)^{\hat{\gamma}} \mathcal{D}A_\mu|_{\hat{A}}^{\hat{A}=B^{\left(\Sigma_1{}^{\hat{\gamma}}\right)^{-1}\circ\hat{\gamma}}} e^{iS_0[A]+\tilde{S}_{\text{CS}}\left[A^\Lambda(x,z)\right]} \\ &= \int \mathcal{D}B^i \int \mathcal{D}\Sigma_1 \mathcal{D}A_\mu|_{\hat{A}}^{\hat{A}=B^{h(\hat{\gamma},\Sigma_1)\circ\Sigma_1^{-1}}} e^{iS_0[A]+\tilde{S}_{\text{CS}}\left[A^\Lambda(x,z)\right]}. \end{aligned} \tag{216}$$

Here, we changed the integration variables $B^i \to (B^i)^{\hat{\gamma}}$ and $\Sigma_1 \to \Sigma_1^{\hat{\gamma}}$ and used the invariance of the left-invariant Haar measure. In the second line, we used $g\Sigma_1 = \Sigma_1^g\, h(g, \Sigma_1)$. To extract the anomaly phase, we further redefine $A_\mu \to A_\mu^{h(\hat{\gamma},\Sigma_1)}$ and get

$$\begin{aligned} Z^{\text{g.f.}}\left[(B^a)^{\hat{\gamma}}\right] &= \int \mathcal{D}B^i\, e^{-ic\int_{\text{UV}} \tilde{\omega}_4^{(1)}\left(\hat{\gamma},B^i,B^a\right)} \int \mathcal{D}\Sigma_1 \mathcal{D}A_\mu|_{\hat{A}}^{\hat{A}=B^{\Sigma_1^{-1}}} e^{iS_0[A]+\tilde{S}_{\text{CS}}\left[A^\Lambda(x,z)\right]} \\ &= \int \mathcal{D}B^i\, e^{-ic\int_{\text{UV}} \tilde{\omega}_4^{(1)}\left(\hat{\gamma},B^i,B^a\right)}\, \mathcal{Z}\left[B^A\right]. \end{aligned} \tag{217}$$

Alternatively, we can write this directly in terms of the WZW action, where the partition function is expressed as

$$Z^{\text{g.f.}}\left[B^A\right] = \int \mathcal{D}B^i \int \mathcal{D}\Sigma_1\, e^{iS_{\text{WZW}}\left[B^A,\Sigma_1\right]}\, e^{iS_{\text{h},0}\left[B^A,\Sigma_1\right]}. \tag{218}$$

Recall that $S_{\text{h},0}$ is the holographic effective action and is invariant under any G transformation (see eq. (210)): any variance comes from the WZW action. Eq. (217) turns into

$$Z^{\text{g.f.}}\left[(B^a)^{\hat{\gamma}}\right] = \int \mathcal{D}B^i\, e^{-ic\int_{\text{UV}} \tilde{\omega}_4^{(1)}\left(\hat{\gamma},B^i,B^a\right)} \int \mathcal{D}\Sigma_1\, e^{iS_{\text{WZW}}\left[B^A,\Sigma_1\right]}\, e^{iS_{\text{h},0}\left[B^A,\Sigma_1\right]}. \tag{219}$$

This form of transformation makes the following 4D dual interpretation very transparent.

In the 4D dual CFT, we have a global symmetry G which is broken down to $H_1 \subset G$ by confinement. In addition, $H_0 \subset H_1$ is weakly gauged. In the strongly interacting CFT, we regulate the UV-divergences such that H_1 is anomaly-free. The advantage of such a choice is that the gauged symmetry H_0 is automatically anomaly-free. However, there can still be anomalies in global currents. In particular, we can have mixed anomalies among global and gauged currents as well as pure global anomalies. Thanks to the proper local counter terms added (i.e. choice of UV-regulator), the mixed anomalies come entirely from the global currents. Under a global transformation $\hat{\gamma} \in \mathbf{k}_1$, the CFT is anomalous and the anomaly is captured by the phase factor in eq. (219). We emphasize that the anomaly factor depends on both the classical source B^a and the gauged dynamical field B^i. In detail, this single anomaly phase contains both pure global anomalies as well as mixed global-gauge anomalies. This of course is equivalent to the statement that the WZW action contains local operators that match chiral anomalies of the UV phase [**?**]. For example, in QCD, the gauged WZW Lagrangian contains $\mathcal{L}_{\text{WZW}} \supset n\frac{e^2}{24\pi^2 F_\pi}\pi^0 F\tilde{F}$, which on one hand explains the $\pi^0 \to \gamma\gamma$ decay, and on the other hand, reproduces the ABJ anomaly when the quantization condition $n = N_c = 3$ is chosen. The QCD WZW Lagrangian also contains $\mathcal{L}_{\text{WZW}} \supset -\frac{2}{3}ie\frac{n}{\pi^2 F_\pi^3}\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu\pi^+\partial_\rho\pi^-\partial_\sigma\pi^0$, which in turn reproduces the QCD VAAA anomaly for $n = N_c = 3$.

Once we specify G, H_1, and H_0, our formalism allows explicit computations of all of these using the results contained.

We consider the most general possibility in which $H_0 \subset G$ is gauged and G is spontaneously broken to H_1. The transformation of the partition function is obtained through a series of similar steps. The final result has the same form as eq. (217) but the transformation parameter $\hat{\gamma}$ is not constrained to be just $\hat{\gamma} \in \mathbf{k}_1$ and instead takes any value $\hat{\alpha} \in \mathbf{g}$. There are three different kinds of transformations we need to consider separately. First, under $\hat{\beta} \in \mathbf{h}_1$, the partition function is invariant. As a result, there is no UV-brane surface terms associated with these transformations, hence no "remedy" is needed. Second, we can perform a pure global transformation $\hat{\gamma}$ which is not part of $\mathbf{h}_1$ ($G/(H_0 \cup H_1)$ transformations). For these transformations, the partition function changes in exactly the same manner and in the 4D dual description we get anomalies in the global currents. Finally, we can consider $\hat{\delta} \in \mathbf{h}_0$ which is not part of $\mathbf{h}_1$ ($H_0 \setminus H_0 \cap H_1$ transformations). For these transformations, we get UV-localized anomaly factors, and given that these generators are not broken on the UV brane, we need to cure this problem. This is done by inserting

$$e^{i\Gamma_{\mathrm{UV}}\left[B^i\right]} \equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\; e^{iS_{\mathrm{UV}}\left[B^i,\psi,\bar{\psi}\right]} \tag{220}$$

into the partition function. As usual, we require it to transform so as to cancel the variance of the third kind (i.e. $\hat{\delta}$ transformations). Specifically, $\Gamma_{\mathrm{UV}}\left[B^i\right]$ is invariant under both H_1 and $G/(H_0 \cup H_1)$ transformations, whilst it changes under a $H_0 \setminus H_0 \cap H_1$ transformation in an opposite way to the way the bulk action does. The final form of the consistent partition function is then written as

$$Z^{\text{g.f.}}\left[B^A\right] = \int \mathcal{D}B^i\; e^{i\Gamma_{\mathrm{UV}}\left[B^i\right]} \int \mathcal{D}\Sigma_1 \mathcal{D}A_\mu\Big|_{\hat{A}}^{\hat{A}=B^{\Sigma_1^{-1}}}\; e^{iS_0[A]+i\tilde{S}_{\mathrm{CS}}\left[A^\Lambda\right]}. \tag{221}$$

This is invariant under any $H_0 \cup H_1$ transformations, although under an infinitesimal transformation $g \approx 1 - \hat{\gamma} \in G/(H_0 \cup H_1)$ it changes as

$$Z^{\text{g.f.}}\left[(B^a)^{\hat{\gamma}}\right] = \int \mathcal{D}B^i\; e^{i\Gamma_{\mathrm{UV}}\left[B^i\right]}\; e^{-ic\int_{\mathrm{UV}}\tilde{\omega}_4^{(1)}(\hat{\gamma},B^i,B^a)} \int \mathcal{D}\Sigma_1 \mathcal{D}A_\mu\Big|_{\hat{A}}^{\hat{A}=B^{\Sigma_1^{-1}}}\; e^{iS_0[A]+i\tilde{S}_{\mathrm{CS}}\left[A^\Lambda\right]}. \tag{222}$$

We are now ready to discuss the dual 4D CFT description. The UV theory consists of the CFT sector and an external sector of Weyl fermions. The subgroup H_0 of G is weakly gauged. In the IR, the CFT sector confines and this breaks G down to a subgroup H_1. Unlike what happens in QCD, this unbroken subgroup H_1 does not need to be aligned with the gauged H_0, a typical situation occurring in dynamical symmetry breaking. We added proper local counter terms to the theory so that the UV-regulator preserves the H_1-symmetry. For this reason, the external fermions in the UV theory form an anomaly-free set under the $H_0 \cap H_1$ gauge forces.

As already mentioned in the previous section, the NGBs associated with the broken *gauged* generators are a gauge artifact and can be eliminated in the unitary gauge. In this section, we confirm that this expectation is fulfilled in our holographic description of anomaly inflow. In fact, we show that this is achieved via field redefinitions, or change of integration variables, which is an allowed operation for path-integrated, i.e. dynamical, fields. Recall that the boundary value B for the Dirichlet UV-BC corresponds to a classical source in the 4D dual theory. On the other hand, the boundary value B for the Neumann UV-BC is dual to a dynamical gauge field and is path-integrated. Therefore, we see that for the latter case, the integration field variable can be redefined so as to eliminate the corresponding Σ_1-dependence. Such freedom, however, is absent for the purely global symmetry. In order to illustrate the point in a clean setup, instead of dealing with the most general case, in this section we consider the pure Neumann UV-BC.

Since all of the G generators are gauged, none of Σ_1 corresponds to physical degrees of freedom. Therefore, we should be able to entirely remove the Σ_1-dependence. We first note that for small NGB field ξ, we can expand $\Sigma_1 \approx 1-\xi$ and get

$$\begin{aligned}\tilde{S}_{\rm CS}\left[A^\Lambda\right] &= c\int_{5\rm D}\tilde{\omega}_5^{(0)}\left(\left(A^\Lambda\right)_h, A^\Lambda\right)\\ &\approx c\int_{5\rm D}\tilde{\omega}_5^{(0)}(A_h, A) + c\int_{5\rm D} d\tilde{\omega}_4^{(1)}(\xi, A_h, A).\end{aligned} \tag{223}$$

Using this, the partition function can be expressed as

$$Z^{\rm g.f.} = \int \mathcal{D}\Sigma_1\mathcal{D}B\; e^{i\Gamma_{\rm UV}[B]}\; e^{-ic\int_{\rm UV}\tilde{\omega}_4^{(1)}\left(\hat{\xi}, B^{\Sigma_1^{-1}}\right)}\; \mathcal{D}A_\mu|_{\hat{A}}^{\hat{A}=B^{\Sigma_1^{-1}}}\; e^{iS_0[A]+i\tilde{S}_{\rm CS}[A]}. \tag{224}$$

The Σ_1-dependence can be eliminated by a change of variable, $B\to B^{\Sigma_1}$. Under this we get

$$\begin{aligned}Z^{\rm g.f.} &= \int \mathcal{D}\Sigma_1\mathcal{D}B^{\Sigma_1}\; e^{i\Gamma_{\rm UV}[B^{\Sigma_1}]}\; e^{-ic\int_{\rm UV}\tilde{\omega}_4^{(1)}\left(\hat{\xi}, B\right)}\; \mathcal{D}A_\mu|_{\hat{A}}^{\hat{A}=B}\; e^{iS_0[A]+i\tilde{S}_{\rm CS}[A]}\\ &= \int \mathcal{D}\Sigma_1\mathcal{D}B\; \left(e^{iG_{\rm UV}\left[\hat{\xi},B\right]}\; e^{i\Gamma_{\rm UV}[B]}\right)\; e^{-ic\int_{\rm UV}\tilde{\omega}_4^{1}\left(\hat{\xi}, B\right)}\; \mathcal{D}A_\mu|_{\hat{A}}^{\hat{A}=B}\; e^{iS_0[A]+i\tilde{S}_{\rm CS}[A]},\end{aligned} \tag{225}$$

where $G_{\rm UV}\left[\hat{\xi}, B\right]$ is the anomaly functional of the external sector. Hence, we observe that the "would-be" NGB dependence is completely encoded in the two anomaly factors associated with the CFT and external degrees of freedom. In particular, we see that, provided the gauge anomaly cancels, the entire Σ_1-dependence disappears from the whole integrand, and the overall immaterial Σ_1-integral can be dropped. In other words, provided the gauged group is free of anomalies (hence suitable for gauging to begin with), the NGBs are unphysical and removable.

We review a standard fact about QFT: the symmetry property of the theory can be probed by checking how the partition function transforms as a functional of the source field. We first recall that even for a *global* symmetry G, the Ward identity is derived by performing a *local* version of the G transformation. Since we are interested in studying gauge theories, we would like to consider

$$Z[B_\mu] = \int \mathcal{D}\phi_i\; e^{iS[\phi_i]+i\int {\rm Tr}\left[B_\mu^a J^{a\mu}\right]} = \left\langle e^{i\int {\rm Tr}\left[B_\mu^a J^{a\mu}\right]}\right\rangle. \tag{226}$$

The ϕ_i's are fundamental fields of the underlying theory, and J_μ is the current they make up. Let's now look at the local G transformation of B_μ. Using

$$g = e^{-\omega^a T^a} = e^{-\omega}, \qquad A_\mu = A_\mu^a T^a dx^\mu \to g\,(A+d)\,g^\dagger \;\;\Rightarrow\;\; \delta A = d\omega + [A,\omega] \tag{227}$$

we get

$$\begin{aligned}Z\left[B^{g^{-1}}\right] &= \int \mathcal{D}\phi_i\; e^{iS[\phi_i]+i\int {\rm Tr}\left[B^{g^{-1}}\cdot J\right]}\\ &= \int \mathcal{D}\phi_i\; e^{iS[\phi_i]+i\int {\rm Tr}[B\cdot J^g+\omega dJ]}.\end{aligned} \tag{228}$$

Now, let us make a change of variable $\phi\to\phi^g$. Allowing for a possibly non-trivial (i.e. anomalous) Jacobian factor but assuming that the action is invariant under this transformation, we get

$$\begin{aligned}Z\left[B^{g^{-1}}\right] &= \int \mathcal{D}\phi_i^g\; J\left[\frac{\partial\phi}{\partial\phi^g}\right]\; e^{iS\left[\phi_i^g\right]+i\int {\rm Tr}[B\cdot J^g+\omega dJ]}\\ &= \int \mathcal{D}\phi_i\; e^{-i\int {\rm Tr}[\omega\mathcal{A}(B)]}\; e^{iS[\phi_i]+i\int {\rm Tr}[B\cdot J]}\; \left(1+i\int {\rm Tr}[\omega dJ]\right),\end{aligned} \tag{229}$$

where we have written the anomalous Jacobian factor as an anomaly phase $e^{-i\int {\rm Tr}[\omega\mathcal{A}(B)]}$. Therefore, we see that the condition that the partition function is invariant under a formal local transformation

$B \to B^g = g(B+d)g^{-1}$ is equivalent to the statement that the global symmetry G satisfies a complete set of (anomalous) Ward identities: $\langle dJ \rangle = \mathcal{A}$. Furthermore, we can use this in a slightly different way. Namely, we instead perform a homogeneous transformation $B \to gBg^{-1}$. We see that under this, the partition function will change by $Z[B] \to Z[B^g] = e^{-i\int \omega \mathcal{A}(B)} Z[B]$. In other words, if the theory satisfies a non-anomalous Ward identity for G, we will not see this anomaly phase, while the anomaly phase will show up whenever the theory is actually anomalous under a G transformation.

9 Path Integral Optimization in CFTs and Holography

We start by briefly reviewing the path integral optimization in CFTs [148, 149] and its holographic interpretation. Most of this material is described pedagogically in original works so readers should consult them for further details and clarifications.

The goal of the *path integral optimization* [148, 149] is to sharpen the intuitions behind the emergence of co-dimension-one slices of holographic geometries from TN in CFTs. The main object of interest, for a CFT defined in d-dimensional flat Euclidean spacetime $\mathbb{R}^d$, is the Euclidean path integral that prepares a ground state

$$\begin{aligned}\Psi_0[\tilde{\varphi}(\vec{x})] &:= \lim_{\beta\to\infty} \left\langle \tilde{\varphi}(\vec{x}) | e^{-\beta \hat{H}_{\mathrm{CFT}}} | \Psi_0 \, |{=}| \int \right\rangle [\mathcal{D}\varphi]\, e^{-I_{\mathrm{CFT}}[\varphi]}\, \delta\left(\varphi|_{\partial(\mathbb{R}^d)} - \tilde{\varphi}\right) \\ &= \int \left(\prod_{\vec{x}} \prod_{\epsilon \le z < \infty} \mathcal{D}\varphi(z,\vec{x}) \right) e^{-I_{\mathrm{CFT}}[\varphi]} \times \prod_{\vec{x}} \delta\left(\varphi(\epsilon,\vec{x}) - \tilde{\varphi}(\vec{x})\right) \ ,\end{aligned} \tag{230}$$

where ϵ is a UV cut-off identified with a lattice spacing in a discretized setting, $(z,\vec{x})$ are local coordinates in $\mathbb{R}^d$, $\tau := -z$ is the Euclidean time and $\vec{x} = (x^1,\ldots,x^{d-1})$ are local coordinates in $(d-1)$-dimensional Euclidean space $\mathbb{R}^{d-1}$. I_{CFT} is the CFT action given in terms of the fields $\varphi(z,\vec{x})$, whose boundary condition at $z=\epsilon$ is $\tilde{\varphi}(\vec{x})$.

We then perform an "optimization" of (230), which can be intuitively visualised in the following way: we first discretize $\mathbb{R}^d$ into a square, evenly-spaced "unoptimized" lattice, as shown in the left panel of Fig. **??**. Next, we optimize this lattice by effectively removing the unnecessary lattice sites on which the path integral is computed. This can be interpreted as a "coarse-graining" procedure where only low-energy modes $|\vec{k}| \ll -1/z = 1/\tau$ remain in the path integral for a given time τ. This implies that a number of lattice sites of order $\mathcal{O}(\tau/\epsilon)$ can be combined into one without losing much accuracy in the evaluation of (230). This optimization procedure of the path integral is represented in the middle panel of Fig. **??**. The optimized lattice can be interpreted in the continuum as hyperbolic metric (TN) over which the Euclidean path integral computes the CFT ground state $|\Psi_0\rangle$.

Moreover, it is well known [**?**] that while Weyl rescaling is a symmetry of the CFT action, it leads to anomalous transformation of the path integral measure such that

$$\Psi[\tilde{\varphi}(\vec{x})]\Big|_{g_{ab}=e^{2\phi}\delta_{ab}} = e^{I_L[\phi]-I_L[0]} \cdot \Psi[\tilde{\varphi}(\vec{x})]\Big|_{g_{ab}=\delta_{ab}} \ , \tag{231}$$

where $I_L[\phi]$ is the famous *Liouville action*

$$I_L[\phi] = \frac{c}{24\pi} \int_{-\infty}^{+\infty} \mathrm{d}x \int_{\epsilon}^{+\infty} \mathrm{d}z \ \left((\partial_x\phi)^2 + (\partial_z\phi)^2 + \mu\, e^{2\phi} \right) \ , \tag{232}$$

where c is the central charge of the CFT and where μ is an $\mathcal{O}(1)$ constant identified with $1/\epsilon^2$ in a discretized setting. The kinetic term in the Liouville action (232) is proportional to the Ricci scalar and describes the conformal anomaly in two dimensions, while the potential term $\mu e^{2\phi}$ arises from the UV regularization. As such, the potential term should dominate over the kinetic term as the UV cut-off $\mu \sim 1/\epsilon^2$ is taken to infinity, which is realized when

$$(\partial_i\phi)^2 \ll e^{2\phi} \ , \ (i = z, x) \ . \tag{233}$$

Given this observation, it was proposed in [148] that the optimization of the path integral should be done by choosing the background metric that minimizes the Liouville action subjected to boundary conditions. In other words, optimal metrics should solve the *Liouville equation* which is in fact equivalent to the constraint that the Ricci scalar R of the 2-dimensional metric is constant

$$(\partial_x^2 + \partial_z^2)\phi = \mu e^{2\phi} \qquad \Leftrightarrow \qquad R = -2\mu. \tag{234}$$

A solution to this equation which satisfies the boundary condition given by the Weyl factor and metric of the hyperbolic plane

$$e^{2\phi} = \frac{1}{\mu z^2}, \qquad \mathrm{d}s^2 = \frac{1}{\mu z^2}(\mathrm{d}z^2 + \mathrm{d}x^2)\ . \tag{235}$$

This hyperbolic metric with $\mu = 1$ corresponds in fact to the minimum of the Liouville action (232) satisfying the boundary condition (**??**) as can be seen by rewriting the former as

$$I_L = \frac{c}{24\pi}\int \mathrm{d}x\,\mathrm{d}z\,\left[(\partial_x\phi)^2 + (\partial_z\phi + e^{\phi})^2\right] - \frac{c}{12\pi}\int \mathrm{d}x\,\left[e^{\phi}\right]_{z=\epsilon}^{z=\infty} \geq \frac{c\,L_x}{12\pi\epsilon}\ , \tag{236}$$

where $L_x = \int \mathrm{d}x$ is the infinite volume (length in this case) of the spatial x direction. However, as we will discuss momentarily, one can view the metrics arising from (235) as corresponding to different degrees of optimization for different values of $0 < \mu \leq 1$ with $\mu = 1$ corresponding to the maximally optimized geometry.

The appearance of the hyperbolic space from the optimization was interpreted as an explicit realization of the AdS/TN correspondence in which such TN could be thought of as a slice of the holographic AdS$_3$. In [148, 149] it was also shown that the geometries obtained via the optimization of Euclidean path integrals for other states in 2-dimensional CFTs such as excited states (given by primaries) or thermal states lead consistently to time-slices of AdS$_3$ and the proposal for general spacetimes was described in [150]. However, a subtle issue arises when taking a closer look at the hyperbolic solution (235). In this case, $(\partial_i\phi)^2$ and $e^{2\phi}$ are found to be of the same order, which is at odds with the expectation (233) obtained in the limit where the UV cut-off ϵ is taken to infinity. This observation suggests that the path integral optimization via the Liouville action is in fact qualitative and therefore there should be finite cut-off corrections to this procedure. For example in the explicit Heat-Kernel derivation of (231) for free theories, one neglects higher curvature terms that are suppressed with powers of the UV cut-off. The main open question is how such terms should be included and under what assumptions (e.g. holographic CFTs) this can be done universally.

9.1 Path Integral Complexity

Intuitively, the optimization of the path integral that prepares a wavefunction corresponds to a minimization of the number of operations that need to be performed in the discretized description. This discrete Euclidean path-integration can be then mapped into a TN, whose optimization can be carried out by *tensor network renormalization* (TNR). In this sense, the optimization of Euclidean path integrals is a natural counterpart of TNR. This implies an interesting connection between the optimization and a notion of *complexity*, as measured by the number of tensors that are needed to construct the TN. Indeed, one can intuitively associate a notion of complexity to a state represented by a TN by counting the number of tensors (volume of the optimal TN) that are needed to accurately represent it: the more tensors are needed, the more "complex" is the state.

This naturally led to a notion of *path integral complexity* as described in [149], where the complexity $\mathcal{C}_\Psi$ of a CFT state $|\Psi\rangle$ is obtained by minimizing the functional $I_\Psi[g_{ab}(z,\vec{x})]$ defined by the ratio of the two wavefunctions

$$I_\Psi[g_{ab}(z,\vec{x})] \equiv \log\left(\frac{\Psi_{g_{ab}}}{\Psi_{\delta_{ab}}}\right), \tag{237}$$

and the actual complexity of $|\Psi\rangle$ is given by the on-shell value

$$\mathcal{C}_\Psi := \min_{g_{ab}(z,\vec{x})} \left[I_\Psi[g_{ab}(z,\vec{x})]\right] . \tag{238}$$

That is, the functional $I_\Psi[g_{ab}(z,\vec{x})]$ estimates the complexity of the TN corresponding to the path integral computed for a specific metric g_{ab} relatively to $g_{ab} = \delta_{ab}$.

This path integral complexity (238) acquires a precise realization in the case of 2-dimensional CFTs given the identification of the functional which determines the path integral optimization with the Liouville action (232). In particular, since the hyperbolic geometry (235) saturates the bound (236), this means that the path integral complexity for the ground state of 2-dimensional CFTs is given by the Liouville action on the hyperbolic geometry and is also proportional to the spatial volume

$$\mathcal{C}_{\Psi_0} = \min_{\phi}[I_L[\phi]] = \frac{c\,L_x}{12\pi\epsilon} , \tag{239}$$

a result which agrees with the expected leading UV behaviour of the ground state of a CFT.

This connection between the Liouville action and a notion of complexity in 2-dimensional CFTs through path integral optimization has been further generalized to various CFTs and QFTs, and has also been connected with more direct approaches to circuit complexity. Moreover, in connection with the TN interpretation of complexity, it was proposed that the terms appearing in the Liouville action (232) correspond to tensors in MERA. Qualitatively, the kinetic terms $(\partial_x\phi)^2 + (\partial_z\phi)^2$ corresponding to isometries and the potential term $e^{2\phi}$ to unitaries. Similarly, the authors discussed a relation between the path integral complexity measured by the Liouville action (232) and a notion of circuit complexity arising from non-unitary circuits built from components of the stress tensor in 2-dimensional CFTs. In particular, they observed that one way of extending the Liouville action to finite cut-off corrections could be done by considering a complexity functional (cost function) resembling the well known *Dirac–Born–Infeld* (DBI) action

$$I_{\mathrm{DBI}} \propto -\tilde{T}\int \mathrm{d}^2\chi(z,x)\sqrt{-\det\left(g_{ab} + \epsilon^2\partial_a\chi(z,x)\partial_b\chi(z,x)\right)}, \tag{240}$$

where $\tilde{T}$ is known as the *brane tension*, which is proportional to $\mathcal{O}\left((G_N^{(3)})^{-1}\right) \propto c$, and where $\chi(x,z) = (\chi_1(x,z), \chi_2(x,z))$ represents a coordinate transformation from the original flat coordinates (z,x) to curvilinear coordinates (χ_1,χ_2). Even though this guess was not derived in any systematic way from CFTs in [**?**], we will see below that complexity actions arising from gravity optimization indeed hint on similar structures.

9.2 Holographic Path Integral Optimization

As mentioned above, a recent proposal provides a dual description of the path integral optimization procedure from the gravitational perspective within the AdS/CFT correspondence in terms of the Hartle–Hawking wavefunctional taken to evolve from the boundary of AdS up to a certain slice of the bulk. This corresponds to an evaluation of the gravitational action in the blue shaded region in Fig. 5, computed for an Euclidean AdS$_{d+1}$ geometry written in Poincaré coordinates (z,τ,x^i)

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(\mathrm{d}z^2 + \mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2\right) . \tag{241}$$

More precisely, the idea is to consider the Hartle–Hawking (HH) wavefunctional $\Psi_{\mathrm{HH}}[g_{ab}]$ in an asymptotically AdS$_{d+1}$ spacetime which evaluates the path integral of Euclidean gravity from a cut-off surface Σ near the asymptotic boundary given by $z = \epsilon$ and $\tau < 0$ to the surface B, given by $z = f(\tau)$, which is located in the bulk and stems from $z = \epsilon$ and $\tau = 0$. See Fig. 5. The HH wavefunctional is defined as

$$\Psi_{\mathrm{HH}}[g_{ab}] := \int [\mathcal{D}g_{ab}]e^{-I_{\mathrm{G}}[g_{ab}]}\delta(g_{ab}|_B - e^{2\phi}\delta_{ab}) , \tag{242}$$

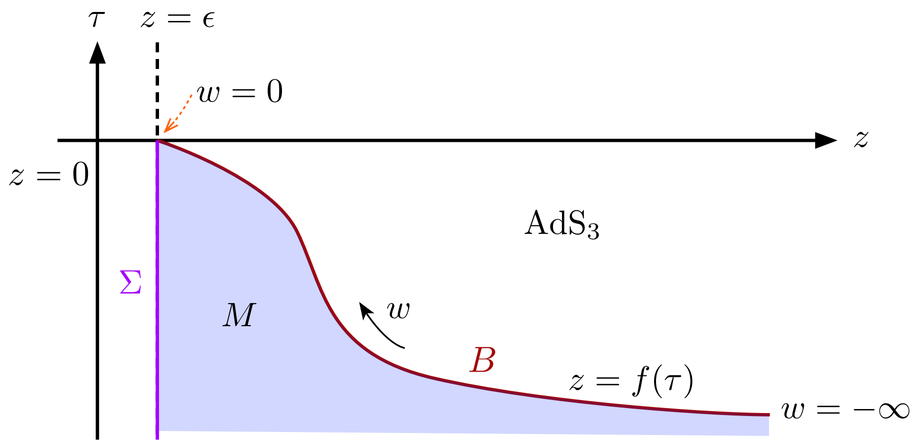

Figure 5: Diagram of the geometric region M over which an evaluation of the gravitational action I_G yields the computation of the Hartle–Hawking wavefunctional in AdS_3.

where the metric on the surface B is assumed to have the translational invariant form

$$\mathrm{d}s^2 = e^{2\phi}(\mathrm{d}w^2 + \mathrm{d}\vec{x}^2) \ , \tag{243}$$

where the Weyl factor $\phi(w, \vec{x})$ contains all the relevant information about the metric (243). One can also take more general metric on B but it would require starting from a more complicated solution of Einstein's equations. One should note that this procedure contemplates a semiclassical computation of the path integral (242). Another remark is that there is an implicit dependence of the coordinate w which characterizes the surface B and the Euclidean time τ defined on the AdS space: $w = w(\tau)$.

The gravitational action I_G on the $(d+1)$-dimensional AdS spacetime which contains a bulk and Gibbons–Hawking–York (GHY) boundary contributions is given by

$$I_\text{G} = -\frac{1}{16\pi G_N^{(d+1)}} \int_M \mathrm{d}^{d+1}x\sqrt{g}\,(R - 2\Lambda) - \frac{1}{8\pi G_N^{(d+1)}} \int_{B\cup\Sigma} \mathrm{d}^d x \sqrt{h}\, K \ , \tag{244}$$

where Λ is the cosmological constant, R is the Ricci scalar of the $(d+1)$-dimensional AdS spacetime, g is the determinant of the metric (241), h is the determinant of the induced metric on $B \cup \Sigma$ and K is the trace of the extrinsic curvature also on $B \cup \Sigma$.

Another crucial ingredient to this interpretation is that the surface B in the bulk should be looked at as a *probe brane* which extends from the boundary Σ and into the bulk, according the AdS/BCFT prescription. That is, one adds a tension term on B to (244) given by

$$I_\text{T} = \frac{T}{8\pi G_N^{(d+1)}} \int_B \mathrm{d}^d x\, \sqrt{h} \ , \tag{245}$$

which is proportional to the volume of the surface B and whose contribution to the gravitational action (244) is controlled by the sign of the *tension* T. In such a way, one obtains a one-parameter family of deformed HH wavefunctionals given by

$$\Psi_\text{HH}^{(T)}[\phi] := \int [\mathcal{D}g_{ab}] e^{-I_\text{G}[\phi] - I_\text{T}[e^{2\phi}]} \delta(g_{ab}|_B - e^{2\phi}\epsilon_{ab}) \ , \tag{246}$$

from which the standard HH wavefunctional (242) is obtained by setting $T = 0$. Note that it is also important that the brane B does not back-react on the AdS geometry.

These deformed HH wavefunctionals can be evaluated semi-classically using the saddle-point approximation. In particular, the actions $I_\text{G} + I_\text{T}$ can be evaluated directly and, for example in 2 dimensions,

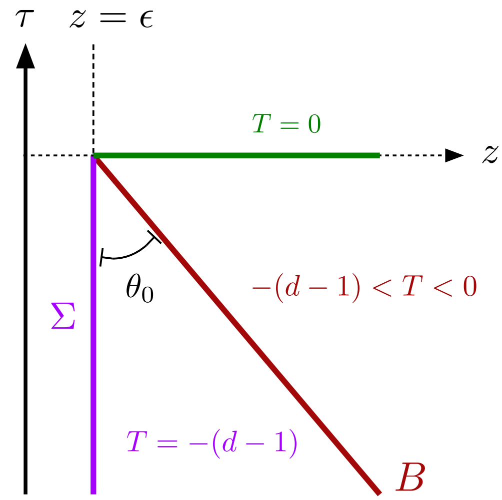

Figure 6: The brane B interpolates between the boundary Σ at $T = -(d-1)$ and the $\tau = 0$ time slice at $T = 0$. The angle θ_0 between B and Σ is given by $\theta_0 = \arcsin\left(1 - T^2/(d-1)^2\right)^{1/2}$.

neglecting finite cut-off corrections and assuming $(\partial_i\phi)^2 \ll e^{2\phi}$ one reproduces the Liouville action together with the optimal geometries derived for various universal classes of CFT states. For example, surfaces B for the vacuum state are given by half-planes (see Fig. 6)

$$z = \epsilon + \tau\frac{\sqrt{1-T^2}}{T} \ , \tag{247}$$

parametrized by $-1 < T < 0$ and their induced metric matches the 2d surface from the Liouville optimization for the vacuum. In particular, the coefficient μ in the Liouville action translates into the tension parameter

$$\mu = 1 - T^2 \ . \tag{248}$$

As we saw previously, the parameter μ can be thought of as measuring how optimized the background metric (TN) is within the path integral optimization scheme. As a consequence, from the gravitational perspective this corresponds to changing the tension T from -1 to 0, where $T = 0$ corresponds to fully-optimized solution. Geometrically, this variation of the tension positions the boundary-anchored brane B moving from the boundary Σ at $T = -1$ to a time slice $\tau = 0$, as can be seen in Fig. 6.

In general dimensions d, varying the on-shell action $I_{\mathrm{G}} + I_{\mathrm{T}}$ is equivalent to imposing the Neumann boundary condition on B, consistent with the AdS/BCFT construction, given by

$$K_{ab} - Kh_{ab} = -Th_{ab} \ , \tag{249}$$

where K_{ab}, K and h_{ab} are respectively the extrinsic curvature, its trace and the induced metric on B. Note that by the Hamiltonian constraint, which is always satisfied for on-shell solutions, this implies

$$K^2 - K^{ab}K_{ab} = \frac{d}{d-1}T^2 = R - 2\Lambda \ , \tag{250}$$

where R is the Ricci scalar on B and Λ is the cosmological constant of AdS_{d+1}, and where we substituted $K|_B = T\,d/(d-1)$ that is just the trace of (249). This is another confirmation of the holographic path integral optimization since, after inserting (248), this constraint becomes precisely the CFT optimization (234). While the maximization of the HH wavefunctional can be performed unambiguously for any dimension d, and gives a clear prediction for the CFT path integral complexity action in the UV limit, there are still important questions regarding the precise optimization procedure in higher-dimensional CFTs.

9.3 Higher-Dimensional CFTs

A natural question in the context of path integral optimization is whether an explicit form of the functional $I_\Psi[g_{ab}]$ (238) whose minimization leads to the optimization of the Euclidean path integral $\Psi_0[\tilde{\varphi}(\vec{x})]$ can be found in higher dimensions. This is also necessary to determine the path integral complexity $\mathcal{C}_\Psi$ in higher-dimensional CFTs. On this matter, there exists a proposal for "effective" path integral complexity action [148, 149] $I_\Psi[g_{ab}]$ constructed in the following way: Starting from a metric g_{ab} of the form

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = e^{2\phi(x)}\hat{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b \ , \tag{251}$$

the following action should be minimized for a vacuum state of a d-dimensional CFT (as well as some small excitations around the ground state)

$$I_\Psi[\phi,\hat{g}] := \frac{d-1}{16\pi G_N^{(d)}}\int \mathrm{d}^dx\sqrt{\hat{g}}\left(e^{(d-2)\phi}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \frac{e^{(d-2)\phi}R_{\hat{g}}}{(d-1)(d-2)} + \mu e^{d\phi}\right) \ , \tag{252}$$

where $R_{\hat{g}}$ is the Ricci scalar of the metric $\hat{g}_{ab}$. Among various other features which led to this identification is the fact that such a functional satisfies the so-called co-cycle conditions [148, 149]. Interestingly, (252) can be re-written as the Einstein–Hilbert action in d-dimensions with negative cosmological constant $\Lambda^{(d)} = -(d-1)(d-2)/2$. This generalized the optimization equation obtained by variation with respect to $\phi(x)$ that implies taking the trace of vacuum Einstein's equations, i.e., the condition that the Ricci scalar of (251) should be a negative constant. Last but not the least, the action (252) was also reproduced in the UV limit of the holographic path integral complexity action explained in the previous section.

Despite these non-trivial consistency checks and observations, there are still some puzzles when identifying the functional (252) as a higher-dimensional generalization of the Liouville action. Firstly, from the perspective of the action itself it is not clear why it should be restricted to having quadratic derivatives of the Weyl field ϕ. Generally, it is quite natural in AdS/CFT that ("sub-leading") higher-derivative terms will also contribute in higher dimensions. This is similar to the problem of the gravitational action in spacetime dimensions higher than $d+1=4$ in which one generically views the Einstein–Hilbert action as a low energy effective theory containing only terms that are quadratic in the derivatives of the metric. This is best seen by considering Lovelock's theorem which is used to construct natural higher-dimensional generalizations of Einstein gravity which include higher-curvature corrections. These so-called *Lovelock theories* are metric theories of gravity which lead to conserved second order equations of motion that naturally take into account higher-curvature terms in the action which become topological in lower-dimensional theories.

This is even more pronounced once we consider even-dimensional CFTs and intend to define the complexity functional from the ratio of wave functions (237). This would naturally lead to the so-called anomaly actions of the Riegert type that are also referred to as Q-curvature actions. For example, in 4d holographic CFTs with central charges $a=c$ the Weyl anomaly reads

$$\langle T^\mu{}_\mu\rangle = \frac{c}{2\pi}\mathcal{Q}_4, \tag{253}$$

and is responsible for the transformation of partition functions (see e.g. [?])

$$Z_{\mathrm{CFT}}(e^{2\phi}\hat{g}) = e^{\frac{c}{4\pi}\int\sqrt{\hat{g}}(\phi\mathcal{P}_4\phi+2\mathcal{Q}_4\phi)}Z_{\mathrm{CFT}}(\hat{g}), \tag{254}$$

where the Q-curvature $\mathcal{Q}_4$ and $\mathcal{P}_4$ will be discussed below. There is a similar expectation in 6d holographic CFTs, where the six-dimensional Q-curvature $\mathcal{Q}_6$ captures the type-A anomaly directly related to the six-dimensional Euler density E_6.

Similarly as in 2d, we may expect that the action (254) will play a similar role to Liouville in the optimization of the holographic (at least those with holographic Weyl anomalies CFT wavefunctions. In the following sections, we will follow this CFT prediction, and discuss similarities and differences between higher-dimensional path-integral optimization done with the Q-curvature actions as in (254) and (252) proposed in [148, 149]. Last but not the least, from the gravitational perspective it is an interesting question how other geometrical or physical (*e.g.* matter) properties of the surface B could be incorporated in the holographic path integral proposal. In a precise sense, the surface B can be understood as a time-dependent cut-off and *e.g.* adding counter-terms-like higher-derivative on B may be a natural step. Finally, similarly to 2D, it would be interesting to give a clear interpretation (*e.g.* counting gates) of different terms in higher-dimensional complexity action, as well as have a set of purely quantum computation arguments (*e.g.* penalty factors for certain gates) for discarding some of the possible contributions. We will discuss and propose resolutions to some of these issues in what follows.

10 Uniformization and the Q-curvature Action

In this section we discuss a systematic and geometric way of interpreting the path integral optimization and the functional $I_\Psi[\phi, \hat{g}]$ in *even-dimensional* CFTs using the Q-curvature action. We introduce the basic objects used in later discussions with a special focus on the Q-curvature, which is the higher-dimensional analogue of the Gauss curvature. We will see that higher-dimensional path integral complexity actions obtained from so-called uniformization problem, which we will also discuss, have a natural interpretation in terms of Q-curvature actions. Furthermore, we verify an essential property, namely the co-cycle condition, that must be satisfied in order for the Q-curvature action to be a valid path-integral complexity action. We also provide an intuitive explanation of the path integral optimization and connect it with the tensor network picture.

10.1 Q-curvature

Consider a compact even-dimensional manifold $(\mathcal{M}, \hat{g}_{ab})$ and a Weyl transformation of the metric: $\hat{g}_{ab} \to g_{ab} = e^{2\phi(x)}\hat{g}_{ab}$, where $\phi(x)$ is a scalar function capturing the effect of the transformation. Under this transformation the Ricci scalar transforms as

$$e^{2\phi(x)} R(e^{2\phi(x)}\hat{g}) = R(\hat{g}) - 2(d-1)\Box_{\hat{g}}\phi(x) - (d-1)(d-2)|\nabla_{\hat{g}}\phi(x)|^2, \tag{255}$$

where the subscript $\hat{g}$ indicates that the respective operators are evaluated on that metric, d is the dimension of the manifold $\mathcal{M}$, and where $\Box_{\hat{g}}$ and $\nabla_{\hat{g}}$ are respectively the Laplace–Beltrami operator the covariant derivative with respect to $\hat{g}$. The notation $R(\hat{g})$ and $R(g) = R(e^{2\phi(x)}\hat{g})$ means that the Ricci scalar has to be evaluated on the metrics $\hat{g}$ and $g = e^{2\phi(x)}\hat{g}$ respectively. We also define a scalar $\mathcal{J}(g)$ by

$$\mathcal{J}(g) = \frac{R(g)}{2(d-1)}, \tag{256}$$

whose interpretation will be clear later on. The introduction of $\mathcal{J}(g)$ allows us to rewrite the transformation (255) as

$$e^{2\phi(x)}\mathcal{J}(e^{2\phi(x)}\hat{g}) = \mathcal{J}(\hat{g}) - \Box_{\hat{g}}\phi(x) - \left(\frac{d}{2} - 1\right)|\nabla_{\hat{g}}\phi(x)|^2. \tag{257}$$

Specializing to $d = 2$, the above transformation simplifies to

$$e^{2\phi(x)}\mathcal{J}(e^{2\phi(x)}\hat{g}) = \mathcal{J}(\hat{g}) - \Box_{\hat{g}}\phi(x). \tag{258}$$

This relation is exactly equivalent to the *Gauss-curvature prescription* [**?**]

$$e^{2\phi(x)}\mathcal{K}(e^{2\phi(x)}\hat{g}) = \mathcal{K}(\hat{g}) - \Box_{\hat{g}}\phi(x), \tag{259}$$

which shows how the Gauss curvature $\mathcal{K}(\hat{g})$ for the metric $\hat{g}$ changes under a Weyl transformation. Hence, in $d = 2$, we identify $\mathcal{J} = \mathcal{K}$. One immediate question one can ask is whether there is an analogous generalized version of Eq.(258) in terms of higher-curvature invariants. To answer this, one defines the *Schouten tensor* for $d > 2$ as

$$S_{ab}(\hat{g}) = \frac{1}{d-2}\Big(R_{ab}(\hat{g}) - \mathcal{J}(\hat{g})\,\hat{g}_{ab}\Big). \tag{260}$$

We are now in a position to define the Q-curvature. For a given metric $\hat{g}_{ab}$, the *Branson Q-curvature* of order four in general dimensions $d > 4$ is defined as

$$\mathcal{Q}_{4,d}(\hat{g}) = -\frac{d}{2}\mathcal{J}(\hat{g})^2 + 2\,S_{ab}(\hat{g})S^{ab}(\hat{g}) + \Box_{\hat{g}}\mathcal{J}(\hat{g}). \tag{261}$$

Note that we have two indices in $\mathcal{Q}_{4,d}$. The first index denotes the order of the curvature and the second index represents the dimension. It is easy to see that $\Box_{\hat{g}}\mathcal{J}(\hat{g})$ contains fourth-order derivatives of the given metric and hence $\mathcal{Q}_{4,d}$ also contains them. From now onwards, we often suppress the dependence of the metric for convenience. Using Eq.(256) and Eq.(260), we write the $\mathcal{Q}_{4,d}$ in a more convenient form

$$\mathcal{Q}_{4,d} = \frac{1}{2(d-1)}\Box_{\hat{g}}R + \frac{2}{(d-2)^2}R_{ab}R^{ab} - \frac{d^2(d-4)+16(d-1)}{8(d-1)^2(d-2)^2}R^2. \tag{262}$$

Our interest is $\mathcal{Q}_4 = \mathcal{Q}_{4,4}$, i.e., the $\mathcal{Q}_{4,d}$ in 4-dimensions. Hence, from now onwards when we refer to the Q-curvature in 4-dimensions, we mean $\mathcal{Q}_4 \equiv \mathcal{Q}_{4,4}$. Setting $d = 4$ in Eq.(262), we obtain the expression of $\mathcal{Q}_4$ as

$$\mathcal{Q}_4 = \frac{1}{6}\Big(\Box_{\hat{g}}R + 3R_{ab}R^{ab} - R^2\Big). \tag{263}$$

Now, we come back to the question whether there is a generalization of Eq. (258). The answer is affirmative and we can directly generalise the Gauss-curvature prescription to the $\mathcal{Q}_4$-curvature prescription by the following theorem.
Theorem 1: For a four-dimensional manifold equipped with a metric $\hat{g}_{ab}$, the $\mathcal{Q}_4$-curvature prescription states that the $\mathcal{Q}_4$ curvatures of conformally-related metrics satisfy

$$e^{4\phi(x)}\mathcal{Q}_4(e^{2\phi(x)}\hat{g}) = \mathcal{Q}_4(\hat{g}) - \mathcal{P}_4(\hat{g})(\phi(x)), \tag{264}$$

where $\mathcal{P}_4(\hat{g})$ is a differential operator given by

$$\mathcal{P}_4(\hat{g}) = \Box^2_{\hat{g}} + \nabla_a\big(2\mathcal{J}g^{ab} - 4S^{ab}\big)\nabla_b. \tag{265}$$

Here $\mathcal{J}$ and S_{ab} are defined by Eq.(256) and Eq.(260) respectively. Note that this is the generalization of Eq.(258) or Eq.(259) to 4-dimensions, where the Gauss curvature and the Laplace-Beltrami operator are replaced by the $\mathcal{Q}_4$ and $\mathcal{P}_4$ respectively. This result suggests that the Q-curvature is the generalization of Gauss curvature in higher dimensions.

Similar to 4-dimensions, for a 2-dimensional manifold equipped with a metric $\hat{g}_{ab}$, the $\mathcal{Q}_2$-curvature prescription states that

$$e^{2\phi(x)}\mathcal{Q}_2(e^{2\phi(x)}\hat{g}) = \mathcal{Q}_2(\hat{g}) - \mathcal{P}_2(\hat{g})(\phi(x)). \tag{266}$$

This equation is nothing but Eq.(258) if one identifies $\mathcal{P}_2(\hat{g}) \equiv \Box_{\hat{g}}$ and $\mathcal{Q}_{2,d}$ as

$$\mathcal{Q}_{2,d}(\hat{g}) = \frac{R(\hat{g})}{2(d-1)} = \mathcal{J}(\hat{g}). \tag{267}$$

for $d \geq 2$. In particular the second order Q-curvature in 2-dimensions satisfies $\mathcal{Q}_2 \equiv \mathcal{Q}_{2,2} = R/2$, which immediately leads back to Eq.(255) in terms of $R(\hat{g})$. Hence $\mathcal{Q}_{2,d} \equiv \mathcal{J}$ for all d.

With these operational definitions, we have encountered two differential operators namely $\mathcal{P}_2$ and $\mathcal{P}_4$, which are conformally covariant. The theorem below gives the transformation law of $\mathcal{P}_4$.
Theorem 2: Under the Weyl transformation $g_{ab} = e^{2\phi(x)}\hat{g}$, the operator $\mathcal{P}_4(\hat{g})$ transforms according to

$$e^{4\phi(x)}\,\mathcal{P}_4(e^{2\phi(x)}\hat{g}) = \mathcal{P}_4(\hat{g}), \tag{268}$$

i.e., $\mathcal{P}_4(\hat{g})$ is conformally covariant.
Proof: Using Theorem 1 (264), we write the LHS of the above equation as

$$\begin{aligned} e^{4\phi}\,\mathcal{P}_4(e^{2\phi}\hat{g})(\psi) &= -e^{4(\phi+\psi)}\mathcal{Q}_4(e^{2(\phi+\psi)}\hat{g}) + e^{4\phi}\mathcal{Q}_4(e^{2\phi}\hat{g}), \\ &= \Big(-\mathcal{Q}_4(\hat{g}) + \mathcal{P}_4(\hat{g})(\phi+\psi)\Big) - \Big(-\mathcal{Q}_4(\hat{g}) + \mathcal{P}_4(\hat{g})(\phi)\Big), \\ &= \mathcal{P}_4(\hat{g})(\psi), \end{aligned} \tag{269}$$

which is the RHS of (268). Here the second line follows from (264) and in the third line, we have used the fact that $\mathcal{P}_4$ is a linear operator, and hence $\mathcal{P}_4(\hat{g})(\phi+\psi) = \mathcal{P}_4(\hat{g})(\phi) + \mathcal{P}_4(\hat{g})(\psi)$, completing the proof.

By a similar argument, one can prove that $\mathcal{P}_2(\hat{g})$ is also conformally covariant. i.e., under the conformal transformation $g = e^{2\phi(x)}\hat{g}$, the operator $\mathcal{P}_2(\hat{g})$ transforms according to

$$e^{2\phi(x)}\,\mathcal{P}_2(e^{2\phi(x)}\hat{g}) = \mathcal{P}_2(\hat{g}). \tag{270}$$

In general, one could define a general form of the operator $\mathcal{P}_{2,d}$ in d-dimensions which is known as the *Yamabe operator*

$$\mathcal{P}_{2,d}(\hat{g}) = \Box_{\hat{g}} - \left(\frac{d}{2} - 1\right)\mathcal{Q}_{2,d}(\hat{g}) = \Box_{\hat{g}} - \frac{d-2}{4(d-1)}R(\hat{g}). \tag{271}$$

Similar to the Q-curvature, here the first index denotes the order of the curvature while the second one indicates the dimension. One can define $\mathcal{P}_{4,d}$ for $d \geq 2$, known as the *Paneitz operator* [**? ? ?**]. It is defined as

$$\mathcal{P}_{4,d}(\hat{g}) = \Box_g^2 + \nabla_a\big((d-2)Jg^{ab} - 4S^{ab}\big)\nabla_b - \left(\frac{d}{2} - 2\right)\mathcal{Q}_{4,d}(\hat{g}). \tag{272}$$

Note that for $d = 4$ this reduces to Eq.(265).

The important aspect of the Yamabe and Paneitz operators is that they are conformally covariant

$$e^{(\frac{d}{2}+1)\phi}\,\mathcal{P}_{2,d}(e^{2\phi}\hat{g})(\psi) = \mathcal{P}_{2,d}(\hat{g})(e^{(\frac{d}{2}-1)\phi}\psi), \tag{273}$$

$$e^{(\frac{d}{2}+2)\phi}\,\mathcal{P}_{4,d}(e^{2\phi}\hat{g})(\psi) = \mathcal{P}_{4,d}(\hat{g})(e^{(\frac{d}{2}-2)\phi}\psi), \tag{274}$$

Note that the above covariance property reduces to Eq.(270) and Eq.(268) for $d = 2$ and $d = 4$ respectively.

The generalization of Theorem 1 (264) to general dimensions is straightforward. For an even d-dimensional manifold equipped with a metric $\hat{g}_{ab}$ the following identity holds

$$e^{d\phi(x)}\mathcal{Q}_d(e^{2\phi(x)}\hat{g}) = \mathcal{Q}_d(\hat{g}) - \mathcal{P}_d(\hat{g})(\phi(x)), \tag{275}$$

where $\mathcal{Q}_d \equiv \mathcal{Q}_{d,d}$ and $\mathcal{P}_d \equiv \mathcal{P}_{d,d}$ are higher-dimensional generalizations of the $\mathcal{Q}_{2,d}$ and $\mathcal{Q}_{4,d}$ Q-curvatures and the Yamabe $\mathcal{P}_{2,d}$ and Paneitz $\mathcal{P}_{4,d}$ operators and are respectively known as Branson's Q-curvature and the Graham–Jenne–Mason–Sparling (GJMS) operator. We will return to these objects in the following section. A proof of (275) for even-dimensional Riemannian manifolds, known as the fundamental identity

theorem, is given in [?]. This identity leads to the following theorem.
Theorem 3: For an even d-dimensional manifold, the following functional

$$\mathcal{T}_d(g) = \int_{\mathcal{M}^d} \mathcal{Q}_d(\hat{g})\,\mathrm{vol}(\hat{g}), \tag{276}$$

is invariant under conformal transformations.
Proof: First, we write

$$\mathcal{T}_d(e^{2\phi}\hat{g}) = \int_{\mathcal{M}^d} \mathcal{Q}_d(e^{2\phi}\hat{g})\,\mathrm{vol}(e^{2\phi}\hat{g}) = \int_{\mathcal{M}^d} e^{d\phi}\,\mathcal{Q}_d(e^{2\phi}\hat{g})\,\mathrm{vol}(\hat{g}), \tag{277}$$

where $\mathrm{vol}(\hat{g})$ is a convenient notation for $\sqrt{\hat{g}}$ and where we have neglected the overall multiplicative factor. The second equality follows from the fact that, under rescaling, $\mathrm{vol}(e^{\hat{2\phi}}\hat{g}) = e^{d\phi}\,\mathrm{vol}(\hat{g})$ for d-dimensions. Now, using Eq.(275), we can write this as

$$\mathcal{T}_d(e^{2\phi}\hat{g}) = \int_{\mathcal{M}^d} \big[\mathcal{Q}_d(\hat{g}) - \mathcal{P}_d(\hat{g})(\phi(x))\big]\mathrm{vol}(\hat{g}). \tag{278}$$

It has been shown that the integral over $\mathcal{P}_d$ vanishes. This implies

$$\mathcal{T}_d(e^{2\phi}\hat{g}) = \int_{\mathcal{M}^d} \mathcal{Q}_d(\hat{g})\,\mathrm{vol}(\hat{g}) = \mathcal{T}_d(\hat{g}) \tag{279}$$

completing the proof. Equipped with these definitions, we now state the *Yamabe problem*.

Yamabe problem (in $2d$ and $4d$): Consider a 2- and 4-dimensional manifold equipped with a metric $\hat{g}_{ab}$, and a Weyl transformation $g = e^{2\phi(x)}\hat{g}$, which defines an equivalence class of conformally-equivalent metrics $[g]$. Can we find a class of metrics which have a constant Q-curvature $\mathcal{Q}_2$ and $\mathcal{Q}_4$ in $d = 2$ and $d = 4$ respectively?

To state this problem more clearly, consider Eqs.(266) and (264). The Yamabe problem demands that the Q-curvatures of the Weyl-rescaled metric should be constant, i.e., $\mathcal{Q}_2(e^{2\phi}\hat{g}) = \Lambda_2$ and $\mathcal{Q}_4(e^{2\phi}\hat{g}) = \Lambda_4$. Here, we look for constants Λ_2, Λ_4 which are negative *i.e.*, we want to find a class of conformal transformation for which $\Lambda_2, \Lambda_4 < 0$. In these cases, Eqs.(266) and (264) are simplified to

$$\mathcal{Q}_2(\hat{g}) - \mathcal{P}_2(\hat{g})(\phi) = \Lambda_2\, e^{2\phi}\ , \tag{280}$$
$$\mathcal{Q}_4(\hat{g}) - \mathcal{P}_4(\hat{g})(\phi) = \Lambda_4\, e^{4\phi}\ . \tag{281}$$

The above equations can be recast as a variational problem, *i.e.* one can view them as the Euler-Lagrange equations obtained by the variation of the action

$$I_d[\phi,\hat{g}] = k\int_{\mathcal{M}^d} \mathrm{d}^d x\Big(\phi\,\mathcal{P}_d(\hat{g})\,\phi - 2\mathcal{Q}_d(\hat{g})\,\phi + \frac{2}{d}\Lambda_d\, e^{d\phi}\Big)\,\mathrm{vol}(\hat{g}), \tag{282}$$

where $\mathcal{Q}_d \in \{\mathcal{Q}_2, \mathcal{Q}_4\}$ and $\mathcal{P}_d \in \{\mathcal{P}_2, \mathcal{P}_4\}$ are the Q-curvature and Yamabe/Paneitz operators in $d = 2$ and $d = 4$ respectively, k is a proportionality constant, and $\Lambda_d \in \{\Lambda_2, \Lambda_4\}$ are the (negative) constants Q-curvature of the Weyl-rescaled metric.

Note that (280) can be re-written in terms of the Weyl-rescaled metric $e^{2\phi}\hat{g}_{ab}$ as

$$R(e^{2\phi}\hat{g}) = 2\mathcal{Q}_2(e^{2\phi}\hat{g}) = 2\Lambda_2 < 0\ , \tag{283}$$

which is nothing else than the Liouville equation discussed previously and cast as in terms of the Ricci scalar as in (234). This implies that (281), written in terms of the Weyl rescaled metric as $\mathcal{Q}_4(e^{2\phi}\hat{g}) = \Lambda_4$, can be regarded as a natural generalization of the Liouville equation in $d = 4$.

It is illustrative to find interesting solutions of Eq.(280) and (281). For convenience, we choose the reference metric $\hat{g}$ as Euclidean flat $\hat{g}_{ab} = \delta_{ab}$. This implies that $\mathcal{Q}_2$ and $\mathcal{Q}_4$ vanish identically, and $\mathcal{P}_2 = \Box = \partial^2$ and $\mathcal{P}_4 = \Box^2 = \partial^4$. Hence in this case the equations (280) (281) simplify to

$$\partial^2\phi = \Theta_2\, e^{2\phi}\,, \tag{284}$$

$$\partial^4\phi = \Theta_4 e^{4\phi}\,, \tag{285}$$

where $\Theta_2 = -\Lambda_2$ and $\Theta_4 = -\Lambda_4$ respectively. Along with the boundary condition $e^{2\phi(-\tau=\epsilon,x)} = 1/\epsilon^2$, where $(-\tau)$ is the Euclidean time, the solution to both equations is given by

$$e^{2\phi(\tau,x)} = \frac{1}{\tau^2} \tag{286}$$

for $d = 2, 4$ with $\{\Theta_2, \Theta_4\} = \{1, 6\}$ respectively. We will again come across this fact later on. The above discussion directly leads to the *uniformization problem* of conformally-equivalent metrics which we will discuss in the following section.

10.2 Path Integral Optimization as a Uniformization Problem

In conformal geometry, one can formulate the following *uniformization* problem: given a reference metric $\hat{g}_{ab}$, can one find a metric g_{ab} with a constant (negative) Q-curvature $\Lambda_d < 0$ that is conformally equivalent to $\hat{g}_{ab}$? The answer turns out to be affirmative and the required metric can be found by extremizing the *Q-curvature action* in d even dimensions, which is given by

$$I_d[\phi, \hat{g}] = \frac{d}{2\Omega_d(d-1)!}\int_{\mathcal{M}^d} \mathrm{d}^d x\Big(\phi\,\mathcal{P}_d(\hat{g})\,\phi - 2\mathcal{Q}_d(\hat{g})\,\phi + \frac{2}{d}\Lambda_d\, e^{d\phi}\Big)\,\mathrm{vol}(\hat{g}), \tag{287}$$

where $\Omega_d = 2\pi^{(d+1)/2}/\Gamma[(d+1)/2]$ is the d-dimensional volume of sphere S^d, the conformally covariant differential operator $\mathcal{P}_d$ is the aforementioned GJMS operator and $\mathcal{Q}_d$ is the Q-curvature scalar defined for the reference metric $\hat{g}_{ab}$. Note that $\mathcal{Q}_{\hat{g}}$ is the Q-curvature of the reference metric whereas Λ is the Q-curvature of the "uniformised" metric. These objects are generalizations of the Laplace-Beltrami operator and Gauss curvature respectively from 2-dimensions to higher even-dimensions and transform in a similar way as their 2-dimensional counterparts, as we have seen in previous sections. In particular, $\mathcal{P}_d(\hat{g})$ is the generalization of the Yamabe and Paneitz operators in even d-dimensions. The form of their leading structure is given by

$$\mathcal{P}_d = \Box^{d/2} + \text{lower order}, \quad \mathcal{Q}_d = \frac{1}{2(d-1)}\Box^{\frac{d}{2}-1}R + \cdots \tag{288}$$

where d is any even dimension and where the ellipsis denotes higher-curvature invariants constructed from the Ricci scalar and tensor as well as from their derivatives. Their explicit forms in 2- and 4-dimensions are given in the previous section.

For a hyperbolic metric (241) in $d = 2$ and $d = 4$, one obtains $\mathcal{Q}_2 = -1$ and $\mathcal{Q}_4 = -6$ respectively, which we will use later (they will be denoted by Λ since they are the Q-curvatures of the optimized metric). The transformation properties of the GJMS operator $\mathcal{P}_d$ allows us to construct conformal invariant quantities, which we have encountered in the previous section. For example, for an even-dimensional conformally flat manifold $\mathcal{M}$, the integral of the Q-curvature yields

$$\mathcal{T}_d(\hat{g}) = \int_{\mathcal{M}^d} \mathcal{Q}_d(\hat{g})\,\mathrm{vol}(\hat{g}) = \frac{1}{2}\Omega_d(d-1)!\,\chi(\mathcal{M}), \tag{289}$$

where $\Omega_d = 2\pi^{(d+1)/2}/\Gamma[(d+1)/2]$ is the d-dimensional volume of sphere S^d and $\chi(\mathcal{M})$ is known as the *Euler characteristic* of $\mathcal{M}$. It is easy to see that in $d = 4$, the invariant is $8\pi^2\chi(\mathcal{M})$. In general, Eq.(289) will be supplemented by an integral over the Weyl tensor, which is conformally invariant for any dimensions $d > 2$, as we showed in (276). We will be particularly interested in conformally-flat spacetimes, in which case this expression vanishes identically.

Moreover, one can regard (287) as the higher even-dimensional version of the Liouville action (232). The equation of motion obtained from the variation of the Q-curvature action (287) is given by

$$\mathcal{P}_d(\hat{g})\,\phi - \mathcal{Q}_d(\hat{g}) = \Theta_d\, e^{d\phi}, \tag{290}$$

where $\Theta_d = -\Lambda_d > 0$ is the cosmological constant. This is the higher-dimensional version of Liouville equation (234) for even-dimensional manifolds.

For convenience and along the lines of path integral optimization we take our reference metric to be the Euclidean flat metric, as we did at the end of the last section. This gives $\mathcal{Q}_d = 0$ and the GJMS operator reduces to $\mathcal{P}_d = \Box^{d/2} = \partial^d$, resulting in the equation of motion

$$\partial^d \phi = \Theta_d\, e^{d\phi}. \tag{291}$$

Along with the boundary condition

$$e^{2\phi(-\tau=\epsilon,x)} = \frac{1}{\epsilon^2}\ , \tag{292}$$

where $(-\tau)$ is the Euclidean time, the solution is given by

$$e^{2\phi(\tau,x)} = \left[\frac{(d-1)!}{\Theta_d}\right]^{2/d} \frac{1}{\tau^2}, \tag{293}$$

confirming the optimal geometry as hyperbolic. This hyperbolic solution was shown to be a minimum of the Liouville action in [149]. In higher dimensions we have not been able to prove that this is the lower bound of the Q-curvature action. Still, the Q-curvature action may still be a meaningful measure of complexity since the optimization procedure should be stoped when $e^{\phi} \sim O(1)$ such that we cannot coarse-grain more than the original lattice. This solution can be rewritten in terms of a parameter $\mu = [\Theta_d\,/(d-1)!]^{2/d}$ simply as

$$e^{2\phi(\tau,x)} = \frac{1}{\mu\tau^2}\ . \tag{294}$$

This result can be directly linked to the discussion of path integral optimization and the optimization of the hyperbolic metrics (235). As the optimized metric corresponds to $\Theta_d = (d-1)!$, the optimization condition is given by $\mu = 1$. For example, in 2-dimensions we readily obtain $\Theta_2 = \mu = 1$ for the optimized metric and the optimized geometry is the Poincaré half-plane. Alternatively, this implies $\Lambda_2 = -1$, corresponding to the Gaussian curvature of the Poincaré half-plane. In 4-dimensions, the optimized geometry $\mu = 1$ corresponds to $\Theta_4 = 6$. This again implies $\Lambda_2 = -6$ which is the Q-curvature of the optimized hyperbolic geometry in 4-dimensions.

Note that the value of the cosmological constant Θ_d is *automatically* fixed according to the spacetime dimension and physically corresponds to the (negative) Q-curvature of the optimal geometry. This also explains why we need to set $\mu = 1$ for the optimized geometry. This result intuitively suggests that the amount of Q-curvature of the the optimal (hyperbolic) geometry sets the scale of the optimization and the boundary geometry automatically picks up the optimal way of performing the path integration. It is natural to follow the analogy and propose that the corresponding path integral complexity is then given by the on-shell value of the Q-curvature action, which for d-dimensions behaves as $\sim V_{d-1}/\epsilon^{d-1}$, where V_{d-1} is the $(d-1)$-dimensional spatial volume, consistent with the holographic "complexity=volume" proposal.

It is also instructive to verify the optimization constraint in the context of the Q-curvature. In 2-dimensions, the optimization constrain reads $R^{(2)} = -2\mu$. From Eq.(256) (note that $\mathcal{J} = \mathcal{Q}_2$), we obtain $\mathcal{Q}_2 = R^{(2)}/2$, which implies that the optimization constraint $\mu = 1$ corresponds to the optimized Q-curvature $\mathcal{Q}_2 = -1$, which is the result for the hyperbolic geometry. In higher dimensions the optimization constraint is instead given by (290). In 4-dimensions, this optimization corresponds to $R^{(4)} = -12\mu$, and $\mathcal{Q}_4 = -6$, which is the optimized Q-curvature for the geometry in 4-dimensions, that was discussed previously. This holds for all even-dimensions. As a consequence, the optimization constraint is naturally incorporated within the uniformization formulation via the Q-curvature action.

10.3 Improved Q-curvature Action and the Co-cycle Condition

The Liouville action has a number of interesting properties. For example, an improved version of the Liouville action has been defined in [149] by subtracting the potential term proportional to the volume in (252), which satisfies the so-called *co-cycle condition*

$$I_\Psi[g_1, g_2] + I_\Psi[g_2, g_3] = I_\Psi[g_1, g_3], \tag{295}$$

where $I_\Psi[g_1, g_2]$ computes the complexity between two TNs described by metrics g_1 and g_2. It has been argued that a legitimate path integral complexity action (in any dimension) should obey this co-cycle condition. Hence, it is important to verify whether the Q-curvature action defined in Eq.(287) also satisfies this condition.

Claim: The improved Q-curvature action

$$I_d^{\rm im}[\phi, \hat{g}] = I_d[\phi, \hat{g}] - I_d[0, \hat{g}]. \tag{296}$$

obeys the cocycle relation (295) where $I_d[\phi, \hat{g}]$ is given by Eq.(287).

Proof: First, we separate $I_d^{\rm im}[\phi, \hat{g}]$ into two parts

$$I_d^K[\phi, \hat{g}] = \int_{\mathcal{M}^d} \big(\phi \mathcal{P}_d(\hat{g})\, \phi - 2\mathcal{Q}_d(\hat{g})\, \phi\big)\, \mathrm{vol}(\hat{g}), \tag{297}$$

$$I_d^V[e^{2\phi}\hat{g}, \hat{g}] = \int_{\mathcal{M}^d} (e^{d\phi} - 1)\, \mathrm{vol}(\hat{g}), \tag{298}$$

and ignore the overall constant $d/2\Omega_d(d-1)!$ that will not play any role in this proof. We then separately show that each of the above terms satisfy the co-cycle condition. For convenience, we show the proof for the first term (297) in $d = 4$, but it can be generalized to any even dimensions.

Let us then consider the following action

$$I_4^K[\phi, \hat{g}] = \int_{\mathcal{M}^4} \big(\phi \mathcal{P}_4(\hat{g})\, \phi - 2\mathcal{Q}_4(\hat{g})\, \phi\big)\, \mathrm{vol}(\hat{g}). \tag{299}$$

From Theorem 1 (264), we obtain

$$e^{4\phi}\mathcal{Q}_4(e^{2\phi}\hat{g}) = \mathcal{Q}_4(\hat{g}) - \mathcal{P}_4(\hat{g})(\phi), \tag{300}$$

where we suppress the coordinate dependence of ϕ. Adding a term $\mathcal{Q}_4(\hat{g})$ to both sides and multiplying them by ϕ and $\mathrm{vol}(\hat{g})$, the above equation can be written as

$$\phi\big[2\mathcal{Q}_4(\hat{g}) - \mathcal{P}_4(\hat{g})(\phi)\big]\mathrm{vol}(\hat{g}) = \phi\big[\mathcal{Q}_4(\hat{g})\, \mathrm{vol}(\hat{g}) + \mathcal{Q}_4(e^{2\phi}\hat{g})\, \mathrm{vol}(e^{2\phi}\hat{g})\big], \tag{301}$$

where we have used the fact that $\mathrm{vol}(e^{2\hat{\phi}}\hat{g}) = e^{4\phi}\, \mathrm{vol}(\hat{g})$. Hence Eq.(299) can be re-written as the integral

$$\mathcal{S}[e^{2\phi}\hat{g}, \hat{g}] = \int_{\mathcal{M}^4} \phi\big[\mathcal{Q}_4(\hat{g})\, \mathrm{vol}(\hat{g}) + \mathcal{Q}_4(e^{2\phi}\hat{g})\, \mathrm{vol}(e^{2\phi}\hat{g})\big], \tag{302}$$

and we define its integrand as

$$\mathcal{L}[e^{2\phi}\hat{g},\hat{g}] = \phi\big[\mathcal{Q}_4(\hat{g})\,\mathrm{vol}(\hat{g}) + \mathcal{Q}_4(e^{2\phi}\hat{g})\,\mathrm{vol}(e^{2\phi}\hat{g})\big]. \tag{303}$$

From the transformation rules of its elements, it can be shown that the integrand satisfies the following identity

$$\begin{aligned}\mathcal{L}[e^{2(\phi+\psi)}\hat{g},e^{2\phi}\hat{g}] &= \psi\big[\mathcal{Q}_4(e^{2\phi}\hat{g})\,\mathrm{vol}(e^{2\phi}\hat{g}) + \mathcal{Q}_4(e^{2(\phi+\psi)}\hat{g})\,\mathrm{vol}(e^{2(\phi+\psi)}\hat{g})\big].\\ &= \psi\big[2\mathcal{Q}_4(e^{2\phi}\hat{g}) - \mathcal{P}_4(e^{2\phi}\hat{g})(\psi)\big]\mathrm{vol}(e^{2\phi}\hat{g}),\end{aligned} \tag{304}$$

where in the second line we have used Eq.(301). Now using $\mathrm{vol}(e^{2\hat{\phi}}\hat{g}) = e^{4\phi}\,\mathrm{vol}(\hat{g})$, we write

$$\begin{aligned}\mathcal{L}[e^{2(\phi+\psi)}\hat{g},e^{2\phi}\hat{g}] &= \psi\big[2e^{4\phi}\mathcal{Q}_4(e^{2\phi}\hat{g}) - e^{4\phi}\mathcal{P}_4(e^{2\phi}\hat{g})(\psi)\big]\mathrm{vol}(\hat{g})\\ &= \psi\big[2\mathcal{Q}_4(\hat{g}) - 2\mathcal{P}_4(\hat{g})(\phi) - \mathcal{P}_4(\hat{g})(\psi)\big]\mathrm{vol}(\hat{g}),\end{aligned} \tag{305}$$

where in the last line we have used Eq. (264) and Eq.(268). Similarly one can write

$$\mathcal{L}[e^{2\phi}\hat{g},\hat{g}] = \phi\big[2\mathcal{Q}_4(\hat{g}) - \mathcal{P}_4(\hat{g})(\phi)\big]\mathrm{vol}(\hat{g}), \tag{306}$$

Hence, by adding Eq.(305) and Eq.(306), we obtain

$$\begin{aligned}\mathcal{L}[e^{2(\phi+\psi)}\hat{g},e^{2\phi}\hat{g}] + \mathcal{L}[e^{2\phi}\hat{g},\hat{g}] &= (\phi+\psi)\big[2\mathcal{Q}_4(\hat{g}) - \mathcal{P}_4(\hat{g})(\phi+\psi)\big]\mathrm{vol}(\hat{g})\\ &\quad + [\phi\mathcal{P}_4(\hat{g})\psi - \psi\mathcal{P}_4(\hat{g})\phi]\mathrm{vol}(\hat{g}).\end{aligned} \tag{307}$$

The first term is $\mathcal{L}[e^{2(\phi+\psi)}\hat{g},\hat{g}]$, and the second term can be written as a total derivative term, which can be neglected. Hence, the integral (302) yields,

$$\mathcal{S}[e^{2(\phi+\psi)}\hat{g},e^{2\phi}\hat{g}] + \mathcal{S}[e^{2\phi}\hat{g},\hat{g}] = \mathcal{S}[e^{2(\phi+\psi)}\hat{g},\hat{g}], \tag{308}$$

which further implies

$$I_4^K[e^{2(\phi+\psi)}\hat{g},e^{2\phi}\hat{g}] + I_4^K[e^{2\phi}\hat{g},\hat{g}] = I_4^K[e^{2(\phi+\psi)}\hat{g},\hat{g}]. \tag{309}$$

Choosing $g_1 = e^{2(\phi+\psi)}\hat{g},\ g_2 = e^{2\phi}\hat{g}$ and $g_3 = \hat{g}$, this proves our claim.

The proof for the potential term can be done in general even dimensions. From the definition (298), we readily verify the identity

$$I_d^V[e^{2(\phi+\psi)}\hat{g},e^{2\phi}\hat{g}] + I_d^V[e^{2\phi}\hat{g},\hat{g}] = I_d^V[e^{2(\phi+\psi)}\hat{g},\hat{g}]. \tag{310}$$

i.e., the action I_d^V obeys the co-cycle condition. Hence, as we argued before, the improved action defined by

$$\begin{aligned}I_d^{\mathrm{im}}[\phi,\hat{g}] &= \frac{d}{2\Omega_d(d-1)!}\,(I_d^K + I_d^V)\\ &= \frac{d}{2\Omega_d(d-1)!}\int_{\mathcal{M}^d}\big(\phi\mathcal{P}_d(\hat{g})\,\phi - 2\mathcal{Q}_d(\hat{g})\,\phi + (e^{d\phi}-1)\big)\,\mathrm{vol}(\hat{g}),\end{aligned} \tag{311}$$

will also satisfy the co-cycle condition. Therefore the full improved Q-curvature action $I_d^{\mathrm{im}}[\phi,\hat{g}]$ obeys the co-cycle condition and is a legitimate candidate for a path-integral complexity action in even-dimensional spacetimes.

11 Holographic Path Integral Optimization and Higher Curvature on B

In this section we focus on the second question i.e., of higher curvature in the holographic path integral optimization and discuss yet another way that such corrections may enter or be tuned in the path integral optimization. Namely, we perform the optimization be adding by hand (with arbitrary coefficients) higher curvature terms in the induced metric on B. At first, including such terms may seem arbitrary and it is not clear at which curvature order one should terminate such procedure. On the other hand, finding B from extremizing an on-shell gravity action with counter-terms computed up to a finite-cutoff region of the bulk is natural in the $T\bar{T}$ context. We discuss this procedure below and point the main difference with the TN ideas based on $T\bar{T}$ deformations.

11.1 Higher Curvature and Hartle-Hawking Wavefunction

We first compute a family of Hartle-Hawking wavefunctions discussed before. However, not only with tension T but now with a more general counterterm-like action added on the surface B with arbitrary coefficients. More precisely, we evaluate the classical wavefunction as

$$\Psi_{\rm HH}[\phi] = e^{-I_{\rm HH}[\phi]}, \qquad I_{\rm HH}[\phi] = I_G + I_{\rm B}, \tag{312}$$

where

$$I_{\rm G} = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{g}\,(R-2\Lambda) - \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{h}K, \tag{313}$$

where $\mathcal{M}$ is the region bounded by B and Σ, and $\partial\mathcal{M} = B\cup\Sigma$, as in Fig. 6. R is the Ricci scalar on region $\mathcal{M}$ and K is the extrinsic curvature on $\partial\mathcal{M}$. Moreover, we take the counterterm-like action on B written in terms of the higher curvature terms as

$$I_{\rm B} = \frac{1}{\kappa^2}\int_B \mathrm{d}^d x\sqrt{h}\left[T + \alpha\mathcal{R} + \beta\mathcal{R}_{ab}\mathcal{R}^{ab} + \gamma\mathcal{R}^2 + \cdots\right], \tag{314}$$

where $\mathcal{R}_{ab}$ and $\mathcal{R}$ denote the Ricci tensor and Ricci scalar of the induced metric on B. Note that this is not the exact counterterm action in AdS/CFT, as the coefficients are arbitrary and should be fixed by the optimization.

For simplicity, we analyze the vacuum case in Poincaré AdS$_{d+1}$ coordinates

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + \mathrm{d}x_i^2 + \mathrm{d}\tau^2}{z^2}\ , \tag{315}$$

and consider the region $\mathcal{M}$ contained between the surfaces $z=\epsilon$, denoted as Σ, and $z=f(\tau)$, denoted as B. The induced metric on B is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_i^2 + (1+f'^2)\mathrm{d}\tau^2}{f^2} = e^{2\phi(w)}\left(\mathrm{d}w^2 + \mathrm{d}x_i^2\right)\ , \tag{316}$$

where we introduced a field $\phi(w)$ and coordinate w as

$$e^{2\phi(w)} = \frac{1}{f^2(w)}, \qquad w'(\tau) = \sqrt{1+f'(\tau)^2}\ . \tag{317}$$

The trace of the extrinsic curvature on B is given by

$$K_B = -\frac{ff'' + d(1-f'^2)}{\sqrt{1-f'^2}} = \frac{e^{-2\phi}\left(\ddot{\phi} + (d-1)\dot{\phi}^2\right) - d}{\sqrt{1-e^{-2\phi}\dot{\phi}^2}}\ , \tag{318}$$

in which case the gravity action (313) can be directly evaluated yielding [? ?]

$$I_{\rm G}[\phi] = \frac{V_x(d-1)}{\kappa^2}\int {\rm d}w e^{d\phi}\left[\sqrt{1-\dot\phi^2 e^{-2\phi}}+\dot\phi e^{-\phi}\arcsin(\dot\phi e^{-\phi})\right] - \frac{(d-1)}{\kappa^2}\frac{V_x L_\tau}{\epsilon^d} - \frac{V_x}{\kappa^2}\left[e^{(d-1)\phi}\arcsin\left(\dot\phi e^{-\phi}\right)\right]_{-\infty}^{0} . \tag{319}$$

Before we go to a more general case, let us first just consider the example where in addition to the tension, we also add the curvature $\mathcal{R}$ with coefficient α. With only these two contributions, the action (314) becomes

$$I_{\rm B} = \frac{V_x}{\kappa^2}\int {\rm d}w\,\left[Te^{d\phi} - \alpha(d-1)e^{(d-2)\phi}\left(2\ddot\phi + (d-2)\dot\phi^2\right)\right] , \tag{320}$$

which can be further integrated by parts

$$I_{\rm B} = \frac{V_x}{\kappa^2}\int {\rm d}w\, e^{d\phi}\left[T + \alpha(d-1)(d-2)e^{-2\phi}\dot\phi^2\right] - \frac{2\alpha(d-1)V_x}{\kappa^2}\left[e^{(d-2)\phi}\dot\phi\right]_{-\infty}^{0} . \tag{321}$$

Interestingly, this new term not only modifies the bulk equations of motion but also the corner (Hayward) term. The equations of motion arising from the extremisation are given by

$$K_B - \frac{d}{d-1}T = -\alpha(d-2)e^{-2\phi}\left(2\ddot\phi + (d-2)\dot\phi^2\right) , \tag{322}$$

which can be written as

$$K_B = \frac{d}{d-1}T + \alpha\frac{d-2}{d-1}\mathcal{R}. \tag{323}$$

This is nothing more than the trace of the general Neumann condition

$$K_{ij} - K\,h_{ij} = -T\,h_{ij} + 2\alpha\mathcal{G}_{ij}, \tag{324}$$

where $\mathcal{G}_{ij} = \mathcal{R}_{ij} - \frac{1}{2}\mathcal{R}\,h_{ij}$ is the Einstein tensor written in terms of the brane curvature and the brane metric. If we again look for the solutions of the form

$$e^{2\phi(w)} = \frac{1}{\mu(w+b)^2}, \tag{325}$$

we obtain the condition between parameters

$$\frac{T}{d-1} + \sqrt{1-\mu} = \alpha(d-2)\mu. \tag{326}$$

It is interesting to note that for $d=2$ the new contribution vanishes and surface B is independent on α i.e., the optimized metric always corresponds to $T=0$.

Let us then consider $d>2$. Note that $T=-(d-1)$ implies $\mu=0$, corresponding to a fully un-optimized geometry. Suppose now that we want to keep the condition $-(d-1)\leq T\leq 0$. Then, from Eq. (326) and considering the optimized metric for $\mu=1$ we can solve for the coefficient α

$$\alpha = \frac{T}{(d-1)(d-2)} . \tag{327}$$

In order to keep the condition $-(d-1)\leq T\leq 0$, we further require that

$$-\frac{1}{d-2} \leq \alpha \leq 0. \tag{328}$$

Hence, e.g. the brane action

$$I_{\rm B} = \frac{1}{\kappa^2}\int_B \mathrm{d}^d x\sqrt{h}\left[T + \frac{T}{(d-1)(d-2)}\mathcal{R}\right]. \tag{329}$$

will lead to an optimized geometry with $\mu = 1$. The lesson from this result is that it is possible to judiciously add a curvature term in the brane action and recover the fully-optimized metric.

Continuing with this procedure, one can add higher curvature terms on the brane according to (314). After postulating a solution of the form (325), we then get the following constraint by varying the action with respect to ϕ

$$\frac{T}{d-1} + \sqrt{1-\mu} = (d-2)\alpha\mu - (d-4)(d-1)\mu^2(\beta + d\gamma). \tag{330}$$

Again, the higher-dimensional contributions identically vanish in $d = 4$. If we are interested in $d > 4$, then we can solve this constraint by e.g. taking α, β, γ as

$$\alpha = \frac{T}{(d-1)(d-2)}, \qquad \beta = -d\gamma, \qquad \gamma = T, \tag{331}$$

however, there are many other choices that will equivalently lead to the optimized geometry with $\mu = 1$. At the moment we do not have a strong argument to resolve this ambiguity and we hope that better understanding of the role of Q-curvature action in the CFT optimization may help in this task.

More generally, we can think about the above procedure as follows. When we vary the on-shell action with general series of counter-terms with respect to the induced metric on B, we compute

$$\delta I_{HH} \sim T_{ij}h^{ij}, \tag{332}$$

where T_{ij} (denoted in analogy with the holographic stress-tensor) is generally

$$T_{ij} \sim K_{ij} - Kh_{ij} + t_{ij}, \tag{333}$$

and t_{ij} comes from the variation of the additional terms on B. If we set this variation to 0, we impose the Neumann boundary condition on B, *i.e.*, $T_{ij} = 0$. Moreover, in our gauge $h_{ij} = e^{2\phi}\delta_{ij}$, the condition that variation of I_{HH} with respect to ϕ vanishes corresponds to

$$T^i_i = 0 \quad \Leftrightarrow \quad K = \frac{1}{d-1}t^i_i. \tag{334}$$

If we only work in pure gravity (as in [**?** **?**]) and add "geometric" terms on B, the Hamiltonian constraint gives the condition

$$\mathcal{R} = 2\Lambda + K^2 - K_{ij}K^{ij} = 2\Lambda + \frac{1}{d-1}(t^i_i)^2 - t_{ij}t^{ij}. \tag{335}$$

For example, for only the tension term on B, we have $t_{ij} = Th_{ij}$ and the Ricci scalar on B is constant negative. Similarly, with higher curvature terms, we can find our solution (325) that also has negative curvature $\mathcal{R} = -d(d-1)\mu$ where μ is expressed in terms of T, α and the other parameters via the maximization equation.

11.2 $T\bar{T}$ and Holographic Path Integral Optimization

Let us now discuss the connection to the so-called $T\bar{T}$-deformations. According to the proposal, we could interpret the bulk on-shell action with Dirichlet boundary condition on B as an effective holographic description of a CFT deformed by the higher-dimensional T^2 operator. Given the interpretation that the holographic path integral optimization should be thought of as the boundary action plus finite cut-off terms, it is tempting to speculate that, in holographic settings, $T\bar{T}$ deformations could be used as a tool to introduce such finite cut-off corrections. Making this precise is beyond the scope of this work however we discuss below how these two approaches may be mutually consistent.

More precisely, the effective gravity action that describes a T^2-deformed holographic CFT is given by

$$I_{T\bar{T}} = -\frac{1}{2\kappa^2}\int \sqrt{g}\,(R-2\Lambda) - \frac{1}{\kappa^2}\int \sqrt{h}K + S_{ct} \tag{336}$$

where the appropriate holographic counter-terms integrated up to the cut-off surface B are

$$S_{ct} = \frac{1}{\kappa^2}\int \sqrt{h}\left[(d-1) + \frac{\mathcal{R}}{2(d-2)} + \frac{(\mathcal{R}_{ij}\mathcal{R}^{ij} - \frac{d}{4(d-1)}\mathcal{R}^2)}{2(d-4)(d-2)^2}\right]. \tag{337}$$

Once we compute the holographic stress tensor from this action, solve it for K_{ij} and K (in terms of T_{ij} and T^i_i), the Hamiltonian constraint of gravity can be written as the anomaly equation

$$\langle \hat{T}^i_i\rangle = -\frac{1}{16\pi G_N}\mathcal{R} - 4\pi G_N\left(\hat{T}_{ij}\hat{T}^{ij} - \frac{1}{d-1}(\hat{T}^i_i)^2\right), \tag{338}$$

where $\hat{T}_{ij} = T_{ij} + a_d t_{ij}$ is the appropriately renormalised (by the counter-terms) holographic stress tensor. In $d=2$ this relation is simply the anomaly equation together with the $T\bar{T}$ operator but in higher (even) dimensions one can also separate holographic anomalies (e.g. in $d=4$ with central charges $a=c$) and the remaining part define the T^2 operators on curved background in the holographic large-N regime. If we would naively minimize this action with respect to the choice of the induced metric on B this would be equivalent to setting T_{ij} to zero. But this is not what is being done in the $T\bar{T}$ (or T^2 in higher-dimensional) TN. There, we would simply consider constant mean curvature slices B with a non-trivial stress tensor i.e., Dirichlet boundary condition on B. On the other hand, in the path integral optimization, we fix B by imposing the Neumann boundary condition on these slices. Still the two approaches can be consistent and give rise to the same slices of the bulk that have a constant Ricci scalar $\mathcal{R}$. A precise understanding of the relation between these two constructions may involve some version of the Legendre transform that has been discussed in the context of the $T\bar{T}$ deformation and we leave this as an exciting future problem.

12 The Holographic Complexity of Extremal Branes in Lower Dimensions

In the discussion section, we commented on a discrepancy in our analysis related to logarithmic divergences in the CV complexity and the induced gravity action on the brane. In particular, for odd d, the CV complexity in the bulk contains a logarithmic contribution but the latter is not generated by our generalized complexity proposal (41) applied to the corresponding brane action. Similarly for even d, applying our geometric formula to the logarithmically divergent terms in the induced action naively yields contributions that do not appear in the CV complexity. Further, this issue becomes immediately evident in lower dimensions, where the logarithmic divergences appear as the leading or first subleading contributions. Explicitly, one can see that our proposal to the generalized CV for a d-dimensional gravity theory

$$\mathcal{C}_{\rm V}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}}\left[\frac{W_{\rm gen}(\mathcal{B}) + W_K(\mathcal{B})}{G_{\rm N}\ell}\right], \tag{339}$$

is only valid for $d>3$ due to superficial divergences in the coefficients

$$\alpha_d = \frac{2(d-4)}{(d-2)(d-3)}, \qquad \gamma_d = \frac{2}{(d-2)(d-3)}, \qquad A_d = \frac{4(d-4)}{(d-2)^2(d-3)}, \tag{340}$$

when $d=2$ or $d=3$. (Recall that $\beta_d = 0$ for all dimensions.) We examine this issue by revisiting our analysis in section 2.2 for lower-dimensional gravity theories.

12.1 Three-Dimensional Extremal Brane

We begin here with the case of $d=3$. It is obvious that there is a problem for the subleading contributions in eq. (34) coming from integrating the volume of the extremal surface in the vicinity of the three-dimensional brane. The divergence in the corresponding coefficients is a signal of the appearance of logarithmic terms. Explicitly, performing the z-integral for $d=3$, we find that the subregion complexity for four-dimensional bulk gravity reads

$$\begin{aligned}
\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) &\equiv \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}}\left[\frac{V(\mathcal{B}_{\rm L})+V(\mathcal{B}_{\rm R})}{G_{\rm bulk}\ell}\right] \\
&\simeq \frac{2L^2}{G_{\rm bulk}\ell}\int_{\widetilde{\mathcal{B}}} d^2\sigma\int_{z_{\rm B}} dz\sqrt{\det \overset{(0)}{h}}\left(\frac{L}{z}\right)^3\left(1-\frac{z^2}{8}K^2+\frac{z^2}{2L^2}\overset{(0)}{h}{}^{ab}\overset{(1)}{h}_{ab}+\cdots\right) \\
&\simeq \frac{LV(\widetilde{\mathcal{B}})}{G_{\rm bulk}\ell}+\log\left(\frac{\ell_{\rm IR}}{z_{\rm B}}\right)\frac{L^3}{G_{\rm bulk}\ell}\int_{\widetilde{\mathcal{B}}} d^2\sigma\sqrt{\det\tilde{h}}\left(\frac{\tilde{K}^2}{4}-\frac{1}{2}\tilde{R}-\tilde{R}_{ij}\tilde{n}^i\tilde{n}^j\right)+\mathcal{O}(z_{\rm B}^0)\,,
\end{aligned} \tag{341}$$

where $\ell_{\rm IR}$ is some scale from deep in the bulk which makes the argument of the logarithmic term dimensionless. Hence the leading term in $\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R})$ still yields the expected volume contribution for the brane gravity, *i.e.*, $V(\widetilde{\mathcal{B}})/(G_{\rm eff}\ell')$ with $\ell'=2\ell$ and $G_{\rm eff}=G_{\rm bulk}/(2L)$ as before. However, the proposed functional for the generalized CV proposal must be modified at higher orders to match the logarithmic divergence

$$\mathcal{C}_{{\rm V},d=3}^{\log}(\widetilde{\mathcal{B}})=\log\left(\frac{\ell_{\rm B}^2}{L^2}\right)\frac{L^2}{4G_{\rm eff}\,\ell'}\int_{\widetilde{\mathcal{B}}} d^2\sigma\sqrt{\det\tilde{h}}\left(\frac{\tilde{K}^2}{2}-\tilde{R}-2\tilde{R}_{ij}\tilde{n}^i\tilde{n}^j\right)\,, \tag{342}$$

where we have substituted $\ell_{\rm B}=L^2/z_{\rm B}$ and made the simple choice $\ell_{\rm IR}=L$. Recall that $\ell_{\rm B}$ and L correspond to the AdS curvature and the UV cutoff scales, respectively, in the effective theory on the brane [1, 2]. Then, we arrive at the generalized CV expression for the induced gravity on the three-dimensional brane,

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R})\simeq\mathcal{C}_{\rm V,d=3}^{\rm Island}\equiv\max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}}\left[\frac{V(\widetilde{\mathcal{B}})}{G_d\ell'}+\mathcal{C}_{{\rm V},d=3}^{\log}(\widetilde{\mathcal{B}})\right] \tag{343}$$

where the logarithmic term is explicitly shown in eq. (342) and denotes the contributions from curvature-squared terms in the gravitational action (16).

Following the approach in the main text, it is straightforward to extend eqs. (39) and (40) to the present case if we allow for logarithmic coefficients. Explicitly, we obtain

$$\begin{aligned}
\mathcal{C}_{\rm V,d=3}^{\rm Island}=\frac{1}{G_{\rm eff}\ell'}\int_{\widetilde{\mathcal{B}}} d^2\sigma\sqrt{\tilde{h}}\Bigg[&\left(1+\log\left(\frac{\ell_{\rm B}^2}{L^2}\right)-\log\left(\frac{\ell_{\rm B}^2}{L^2}\right)\frac{\partial\mathbf{L}_{\rm eff}}{\partial\tilde{R}_{ijkl}}\tilde{n}_i\tilde{h}_{ik}\tilde{n}_l\right)\,, \\
&-2\log\left(\frac{\ell_{\rm B}^2}{L^2}\right)\frac{\partial^2\mathbf{L}_{\rm eff}}{\partial\tilde{R}_{ijkl}\partial\tilde{R}^{mnop}}\tilde{K}_{jl}\tilde{h}_{ik}\tilde{K}^{np}\tilde{h}^{mo}\Bigg]\,.
\end{aligned} \tag{344}$$

That is, we are using the same functional $\widetilde{W}_{\rm gen}+\widetilde{W}_K$ as before but with new coefficients

$$\alpha_3=-\log\left(\frac{\ell_{\rm B}^2}{L^2}\right)\,,\quad \gamma_3=1+\log\left(\frac{\ell_{\rm B}^2}{L^2}\right)\,,\qquad A_3=-2\log\left(\frac{\ell_{\rm B}^2}{L^2}\right)\,, \tag{345}$$

for a general curvature-squared gravity theories in three dimensions.

We emphasize that we included the first subleading contributions in eq. (34) and so the issue of the logarithmic divergence in the holographic complexity became manifest for $d=3$. However, the same issue will arise for any odd d, *i.e.*, with an even dimension in the bulk. Carrying out the same calculations to a sufficiently high order will reveal an extra logarithm in the holographic complexity. In particular, with $d = 2n+1$, one should only apply eqs. (6) and (7) for the generalized CV proposal for higher curvature interactions up to R^{2n-1}. It will be possible to include the R^{2n} interactions if one adds an extra contribution with a logarithmic coefficient, as in eq. (343). It would be interesting to examine this issue in greater detail in higher dimensions.

12.2 Two-Dimensional Extremal Brane

Now turning to the case of $d = 2$, we expect to find a logarithmic divergence in the induced action which is not reflected in the holographic complexity. Furthermore, we should stress that the generalized CV for $d = 2$ is more subtle because the usual relations $\ell' = \frac{d-1}{d-2}\ell$ and $G_{\text{eff}} = (d-2)G_{\text{bulk}}/(2L)$ break down for this dimension.

First of all, we recall the FG expansion for the metric with a three-dimensional bulk becomes

$$g_{ij}(z, x^i) = \overset{(0)}{g}_{ij}\left(x^i\right) + \frac{z^2}{\delta^2}\left(\overset{(1)}{g}_{ij}(x^i) + f_{ij}(x^i)\log\left(\frac{z}{L}\right)\right) + \cdots . \tag{346}$$

where the subleading term $\overset{(1)}{g}_{ij}(x^i)$ is not completely fixed and $f_{ij}(x^i)$ depends on the stress tensor on the boundary [124]. Similarly, the embedding function for the extremal surface $\mathcal{B}$ in the bulk is given by

$$x^i\left(z, \sigma^a\right) = \overset{(0)}{x^i}\left(\sigma^a\right) + \frac{z^2}{L^2}\left(\overset{(1)}{x^i}\left(\sigma^a\right) + \overset{(1)}{y^i}\left(\sigma^a\right)\log\left(\frac{z}{L}\right)\right) + \mathcal{O}\left(\frac{z^4}{L^4}\right) . \tag{347}$$

From this expansion, we see that the subleading terms are not fully geometric anymore and depend on the details of the boundary state.

Explicitly, performing the CV integral in the vicinity of the brane with $d = 2$ yields

$$\mathcal{C}_{\text{V}}^{\text{sub}}\left(\mathbf{R}\right) = \max_{\partial\mathcal{B}=\mathbf{R}\cup\Sigma_{\mathbf{R}}}\left[\frac{V\left(\mathcal{B}_{\text{L}}\right) + V\left(\mathcal{B}_{\text{R}}\right)}{G_{\text{bulk}}\ell}\right] \approx \frac{2LV(\widetilde{\mathcal{B}})}{G_{\text{bulk}}\ell} + \mathcal{O}(z_{\text{B}}^0) . \tag{348}$$

Hence the leading term is still the volume of the island and the subleading contributions are dominated by the upper bound in the radial z-integral, *i.e.,* these should be included as quantum contributions to the brane complexity. As a result, we will only need to consider the leading contribution, *i.e.,* the volume term.

Now the expression for the effective action given in eq. (16) does not apply for $d = 2$. Rather after a careful examination of the FG expansion and integration over the radial direction (see section 2.3 in [1] for more details), the induced action for the $d = 2$ brane can be written as

$$I_{\text{induced}} = \frac{1}{16\pi G_{\text{eff}}}\int d^2x\sqrt{-\tilde{g}}\Big[\frac{2}{\ell_{\text{eff}}^2} - \tilde{R}\,\log\left(-\frac{L^2}{2}\tilde{R}\right) + \tilde{R} + \frac{L^2}{8}\,\tilde{R}^2 + \cdots\Big] . \tag{349}$$

where the two effective scales are

$$\left(\frac{L}{\ell_{\text{eff}}}\right)^2 = 2\left(1 - 4\pi G_{\text{bulk}} L T_o\right) , \qquad\qquad G_{\text{eff}} = G_{\text{bulk}}/L . \tag{350}$$

The unusual logarithmic term can be understood as arising from the nonlocal Polyakov action induced by the two-dimensional boundary CFT supported by the brane.

There is a certain degree of ambiguity in how to proceed at this point, but examining our ansatz (39) for the generalized volume $\widetilde{W}_{\text{gen}}(\widetilde{\mathcal{B}})$ (with undetermined α_2, β_2, γ_2), we obtain

$$\mathcal{C}_{\text{V,d=2}}^{\text{Island}} = \frac{\widetilde{W}_{\text{gen}}(\widetilde{\mathcal{B}})}{\hat{G}_{\text{eff}}\ell} \simeq \frac{1}{G_{\text{eff}}\ell'}\int_{\widetilde{\mathcal{B}}} d\sigma\,\sqrt{h}\left[-\frac{\alpha_2}{2}\,\log\left(-\frac{L^2}{2}\tilde{R}\right) + (0)\,\beta_2 + \gamma_2\right] . \tag{351}$$

Here we have ignored any contributions from the $\tilde{R}^2$ and higher terms (denoted by the ellipsis) in eq. (349). We note that these contributions do not contain any UV divergences in the limit $L/\ell_{\rm B} \to 0$, and so they can be included as part of the quantum contribution to the complexity. Further, note that tensor contraction multiplying the coefficient β_2 vanishes for $d = 2$. Now the following simple choice of the coefficients,

$$\alpha_2 = 0\,, \qquad \gamma_2 = 2\,, \qquad \ell' = \ell\,, \tag{352}$$

yields the desired identification for the two-dimensional complexity

$$\mathcal{C}_{\rm V}^{\rm sub}(\mathbf{R}) \simeq \mathcal{C}_{\rm V,d=2}^{\rm Island} = \frac{\widetilde{W}_{\rm gen}(\tilde{\mathcal{B}})}{\hat{G}_{\rm eff}\,\ell'} = \frac{2\,V(\tilde{\mathcal{B}})}{G_{\rm eff}\,\ell'}\,. \tag{353}$$

We again note that a similar mismatch from logarithmic divergences in the induced action will appear for any $d = 2n$. In this case, no corresponding divergence appears in the holographic complexity in the bulk, which is odd-dimensional. Hence one should only apply eqs. (6) and (7) for the generalized CV proposal for higher curvature interactions up to R^{2n-2}. A logarithmic divergence will appear at the next order, *i.e.,* R^{2n-1}, and the corresponding contribution to the complexity will have to be treated separately. Again, it would be interesting to explicitly examine this question in greater detail for higher dimensions.

13 The Higher Derivative Actions of Extremal Branes

In exceptional order to study the details near the transition point we should consider all the replica-nonsymmetric configurations, re-sum them into an effective action which we then compute semi-classically as a whole. By now we have understood what these configurations correspond to holographically, let us proceed with the re-summation in the context of AdS/CFT.

We leave the subsystem energies unfixed, but still impose the boundary condition for the total energy $E_A + E_{\bar{A}} = E$. Holographically this means that we do not fix the black hole geometries involved for the $\mathcal{M}_i$'s. Denote the portion of the black hole geometry extending along A by $\mathcal{B}_A(E')$, and the corresponding semi-classical contribution to the Euclidean action by

$$I_A(E') = \int_{\mathcal{B}_A(E')} (\mathcal{L}_{\rm E.H.} + \mathcal{L}_{\rm matter}) + \int_A \mathcal{L}_{\rm H.G.} \tag{354}$$

where the bulk Lagrangian densities $\{\mathcal{L}_{\rm E.H.}, \mathcal{L}_{\rm matter}, \mathcal{L}_{\rm H.G}\}$ are evaluated at the saddle point geometry $\mathcal{B}_A(E')$, as well as matter field configurations not specified here. Consequently, let us also denote the portion of black hole geometry and semi-classical Euclidean action along $\bar{A}$ by $\mathcal{B}_{\bar{A}}\left(\frac{E-fE'}{1-f}\right)$ and $I_{\bar{A}}\left(\frac{E-fE'}{1-f}\right)$ respectively, this is obtained after imposing the total energy conditions.

Using these ingredients, the total semi-classical Euclidean action for each of the $\mathcal{M}_i$'s takes the form:

$$I^n_{\mathcal{M}_i}(E) = \text{Min}\left\{ mI_A(E') + (n+1-m)I_{\bar{A}}\left(\frac{E-fE'}{1-f}\right) : E' \right\} \tag{355}$$

This corresponds to $\mathcal{M}_i$ obtained from gluing m copies of black hole portion $\mathcal{B}_A(E')$ with $(n+1-m)$ copies of $\mathcal{B}_{\bar{A}}\left(\frac{E-fE'}{1-f}\right)$. We have not specified the boundaries of $\mathcal{B}_A(E')$ or $\mathcal{B}_{\bar{A}}\left(\frac{E-fE'}{1-f}\right)$ in the bulk, and in general directly gluing them would result in discontinuous junctions across these boundaries. As commented before, these issues do not enter in the limit we are interested. We label the boundary and bulk manifolds by $\{\mathcal{M}_m\}$ and $\{\mathcal{B}_m\}$ respectively. This is not a one-to-one correspondence between the label m and the original label for the contractions i, because there are in general multiple ways to junction m and $(n+1-m)$ portions of black hole geometries by gluing the original EoW branes. For this reason we include a curly bracket to indicate that each $\{\mathcal{M}_m\}$ and $\{\mathcal{B}_m\}$ represent the class of all contraction or gluing choices resulting in the same m.

For each class $\{\mathcal{M}_m\}$, the bulk saddles in $\{\mathcal{B}_m\}$ can be constructed in a way analogous to the cosmic extremal brane prescription. We describe the construction as follows. For each $\{\mathcal{B}_m\}$, the saddle consists of n identical wedges $\tilde{\mathcal{B}}_m$,

$$\tilde{\mathcal{B}}_m = \mathcal{B}_m/\mathbb{Z}_n \tag{356}$$

Each wedge $\tilde{\mathcal{B}}_m$ is then constructed by inserting in the original black hole state $|E\rangle_{\text{bulk}}$ a pair of defects consisting of two cosmic extremal branes Σ_A and $\Sigma_{\bar{A}}$, homologous to A and $\bar{A}$ respectively.

Let us make a few comments regarding the double-defect construction. It may appear that we have proposed a procedure based on the replica-symmetry among the n wedges $\tilde{\mathcal{B}}_m$. However, we can do this only because there is an effective $U(1)$ rotation-symmetry along the bulk thermal circle that emerges in the limit of our interest: high energy $E\ell \gg 1$ and ignoring boundary effects near ∂A. It allows the division of the circle into n equal wedges that are therefore replica-symmetric under $\mathbb{Z}_n \subset U(1)$, justifying our construction. The origin for this emergent rotation-symmetry can be traced to two ingredients. Firstly, the m copies of black hole portions along A are all the same (and similarly for $\bar{A}$), we assume this to be the dynamically favored configuration. More importantly, we are approximating each of the m black hole portions as "featureless" and thus invariant under rotation along the bulk thermal circle. This is certainly not true in the exact solution. For example, solving the matching equations will in general give a set $\{\beta_1, ..., \beta_n\}$ that are not all equal. Once we include the details arising from gluing and smoothening across ∂A, they will produce "features" along the thermal circles that generically breaks the rotation-symmetry. Apart from this, these features also distinguish between the bulk saddles $\mathcal{B}_i$'s in the same class $\{\mathcal{B}_m\}$, because different contractions i will in general give different sets for $\{\beta_1, .., \beta_n\}$, and thus different gluing and smoothening effects. Since these features are localized near ∂A, in the high energy limit they only affect the radial-going portion of the bulk defects that we have ignored. It is interesting to consider the corrections they induce to the cosmic brane picture, we leave this for future investigations.

One way to represent the double-defect construction is to simply add the corresponding brane source terms into the bulk action. The total on-shell action $I_{\text{bulk}}(\mathcal{B}_m)$ can be divided into n quotient on-shell actions:

$$I_{\text{bulk}}(\mathcal{B}_m) = nI_{\text{bulk}}(\tilde{\mathcal{B}}_m) \tag{357}$$

with the quotient on-shell action $I_{\text{bulk}}(\tilde{\mathcal{B}}_m)$ given by a quotient path integral in the saddle point approximation:

$$e^{-I_{\text{bulk}}(\tilde{\mathcal{B}}_m)} = \int \mathcal{D}g\mathcal{D}\phi\mathcal{D}\Sigma_A\mathcal{D}\Sigma_{\bar{A}}\ e^{-I_{\text{bulk}}(g,\phi,E) - \frac{n-m}{4nG_N} I_{\text{brane}}(\Sigma_A) - \frac{m-1}{4nG_N} I_{\text{brane}}(\Sigma_{\bar{A}})} \tag{358}$$

where $I_{\text{bulk}}(g, \phi, E)$ is the bulk Euclidean action with boundary conditions specified by the original state $|E\rangle$, and $I_{\text{brane}}(\Sigma_A)$, $I_{\text{brane}}(\Sigma_{\bar{A}})$ are the Nambu-Goto actions for cosmic branes located on Σ_A, $\Sigma_{\bar{A}}$ respectively:

$$I_{\text{brane}}(\Sigma_A) = \int_{\Sigma_A} d^{d-1}y\sqrt{\gamma}, \quad I_{\text{brane}}(\Sigma_{\bar{A}}) = \int_{\Sigma_{\bar{A}}} d^{d-1}y\sqrt{\gamma}. \tag{359}$$

Here γ is the induced metric on the brane. Although not shown explicitly, I_{brane} depends implicitly on the spacetime metric g. In Eq. (358), $\mathcal{D}\Sigma_A$ denotes a path integral over the location of the brane.

Extremizing over $\{g, \phi, \Sigma_A, \Sigma_{\bar{A}}\}$ solves for the backreacted geometry for a particular saddle $\mathcal{B}_m$. As was alluded to, our strategy is to re-sum all the saddle configurations before extremizing. Before proceeding, we point out that while (358) computes the action of the $\mathbb{Z}_n$ quotient $\tilde{\mathcal{B}}_m$ of the "parent space" $\mathcal{B}_m$, it is the full action of the parent space $\mathcal{B}_m$ that we need to re-sum. In the semi-classical limit, we can simply write the latter as

$$e^{-I_{\text{bulk}}(\mathcal{B}_m)} = \int \mathcal{D}g\mathcal{D}\phi\mathcal{D}\Sigma_A\mathcal{D}\Sigma_{\bar{A}}\ e^{-nI_{\text{bulk}}(g,\phi,E) - \frac{n-m}{4G_N} I_{\text{brane}}(\Sigma_A) - \frac{m-1}{4G_N} I_{\text{brane}}(\Sigma_{\bar{A}})} \tag{360}$$

using Eqs. (357) and (358).

Comparing with the combinatorics factors worked out in Appendix **??**, we find that out of all the contractions i's, there are $N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$ number of "micro-configurations" for the class

$\{\mathcal{M}_m\}$. So we can re-sum all the contractions and obtain the full replicated partition function Z_n:

$$\begin{aligned} Z_n &= \sum_{m=1}^{n} N(n,m) \int \mathcal{D}g\mathcal{D}\phi\mathcal{D}\Sigma_A\mathcal{D}\Sigma_{\bar{A}} \; e^{-nI_{\text{bulk}}(g,\phi,E)-\frac{n-m}{4G_N}I_{\text{brane}}(\Sigma_A)-\frac{m-1}{4G_N}I_{\text{brane}}(\Sigma_{\bar{A}})} \\ &= \int \mathcal{D}g\mathcal{D}\phi\mathcal{D}\Sigma_A\mathcal{D}\Sigma_{\bar{A}} \; e^{-nI_{\text{bulk}}(g,\phi,E)} \sum_{m=1}^{n} N(n,m) \; e^{-\frac{n-m}{4G_N}I_{\text{brane}}(\Sigma_A)-\frac{m-1}{4G_N}I_{\text{brane}}(\Sigma_{\bar{A}})} \\ &= \int \mathcal{D}g\mathcal{D}\phi\mathcal{D}\Sigma_A\mathcal{D}\Sigma_{\bar{A}} \; e^{-nI_{\text{bulk}}(g,\phi,E)} G_n\left(\Sigma_A,\Sigma_{\bar{A}}\right) \\ &= \int \mathcal{D}g\mathcal{D}\phi\mathcal{D}\Sigma_A\mathcal{D}\Sigma_{\bar{A}} \; e^{-nI_{\text{bulk}}(g,\phi,E)-I_{\text{eff}}(\Sigma_A,\Sigma_{\bar{A}},n)} \end{aligned} \tag{361}$$

where we have defined

$$G_n\left(\Sigma_A,\Sigma_{\bar{A}}\right) = {}_2\mathrm{F}_1\left[1-n,-n;\,2;\,e^{-\frac{|\Delta I_{\text{brane}}|}{4G_N}}\right] \times \begin{cases} e^{-\frac{n-1}{4G_N}I_{\text{brane}}(\Sigma_A)}, & \Delta I_{\text{brane}}<0 \\ e^{-\frac{n-1}{4G_N}I_{\text{brane}}(\Sigma_{\bar{A}})}, & \Delta I_{\text{brane}}>0 \end{cases} \tag{362}$$

with

$$\Delta I_{\text{brane}} \equiv I_{\text{brane}}\left(\Sigma_A\right) - I_{\text{brane}}\left(\Sigma_{\bar{A}}\right), \tag{363}$$

and we have extracted a total effective action for the cosmic extremal branes

$$\begin{aligned} I_{\text{eff}} \quad & \left(\Sigma_A,\Sigma_{\bar{A}},n\right) = -\ln G_n\left(\Sigma_A,\Sigma_{\bar{A}}\right) \\ = & \begin{cases} \frac{n-1}{4G_N}I_{\text{brane}}(\Sigma_A) - \ln\left({}_2\mathrm{F}_1\left[1-n,-n;\,2;\,e^{\frac{\Delta I_{\text{brane}}}{4G_N}}\right]\right), & \Delta I_{\text{brane}}<0 \\ \frac{n-1}{4G_N}I_{\text{brane}}(\Sigma_{\bar{A}}) - \ln\left({}_2\mathrm{F}_1\left[1-n,-n;\,2;\,e^{-\frac{\Delta I_{\text{brane}}}{4G_N}}\right]\right), & \Delta I_{\text{brane}}>0 \end{cases} \end{aligned} \tag{364}$$

The first term in the effective action corresponds to that of a single cosmic extremal brane in the dominant configuration; the second term re-sums the (semi-classical) corrections from the subdominant configurations. They correspond to the decomposition into F_{dom} and F_{Δ}. The resulting I_{eff} has a modified dynamics in terms of extremal brane dynamics and backreactions on the bulk geometry. The bulk physics near the transition point is encoded in such modifications, which we turn to study next.

In this research article, assuming the existence of some isometry directions, we construct effective actions for various mixed-symmetry tensors that couple to exotic extremal branes. We consider the cases of the exotic extremal 5^2_2-brane, the 1^6_4-brane, and the p_{7-p}-brane, and argue that these exotic extremal branes are the magnetic sources of the non-geometric fluxes associated with polyvectors β^{ij}, $\beta^{i_1\cdots i_6}$, and $\gamma^{i_1\cdots i_{7-p}}$, respectively. As it is well-known, an exotic-brane background written in terms of the usual background fields is not single-valued and has a U-duality monodromy. However, with a suitable redefinition of the background fields, the U-duality monodromy of the exotic-brane background simply becomes a gauge transformation associated with a shift in a polyvector, which corresponds to a natural extension of the *β-transformation* known in the generalized geometry. Here we study the case of exotic super p-brane. The contribution of the boundary terms in the variation of S_p is given by

$$\delta S_p|_\Gamma = \oint ds_\mu \rho^\mu Y^\Lambda G_{\Lambda\Xi}\delta Y^\Xi, \tag{365}$$

where $ds^\nu = \frac{1}{p!}\varepsilon^{\nu\mu_1\mu_2\ldots\mu_p}dS_{\mu_1\mu_2\ldots\mu_p}$. Here, we consider the variational problem with the fix initial ($\tau=\tau_i$) and final ($\tau=\tau_f$) data, so the integral along the super p-brane profile for $\tau=(\tau_i,\tau_f)$ does not contribute to $\delta S_p|_\Gamma$

$$\int_{s_\tau} ds_\tau \rho^\tau Y^\Lambda G_{\Lambda\Xi}\delta Y^\Xi|_{\tau_i}^{\tau_f} = 0. \tag{366}$$

As a result, the variation $\delta S_p|_\Gamma$ is filled out by the integrals along the p-dimensional boundaries of the brane worldvolume containing the τ-direction

$$\delta S_p|_\Gamma = \Sigma_{i=1}^{i=p} \int_{s_i} ds_i \rho^i Y^\Lambda G_{\Lambda\Xi} \delta Y^\Xi|_{\sigma^i=0}^{\sigma^i=\pi} . \tag{367}$$

In the case of variational problem with free ends, when the field variations on the p-brane boundaries are arbitrary, the vanishing of these hypersurface terms in $\delta S_p|_\Gamma$ gives the super p-brane boundary conditions. Now, let us consider the dual action which additionally includes the Wess-Zumino term

$$S\big[\, \tilde{g}_{ij},\, \tilde{\phi},\, \beta_{ij}^{(8)} \,\big] = \frac{1}{2\kappa_{10}^2} \int \Big[-2\tilde{\phi} \, \big(\tilde{*}\, \tilde{R} + 4\, \tilde{\phi} \wedge \tilde{*}\, \tilde{\phi} \big) - \frac{1}{4}\, 2\tilde{\phi}\, \tilde{g}^{ik}\, \tilde{g}^{jl}\, {}^{(9)}_{ij} \wedge \tilde{*}\, {}^{(9)}_{kl} \Big] - \mu_{5^2_2} \int \frac{1}{2}\, \beta^{(8)} \wedge (x - X(\xi)) \, . \tag{368}$$

Taking a variation with respect to $\beta^{(8)}$, we obtain the following equation of motion:

$$\frac{1}{2\kappa_{10}^2}\, {}^{(1)} = \frac{\mu_{5^2_2}}{(2\pi R)(2\pi R)}\, n\, \delta^2(x - X(\xi))\, x^1 \wedge x^2 \, . \tag{369}$$

From (369), we conclude that the current for the $5^2_2(n_1 \cdots n_5, m_1 m_2)$-brane (in the absence of the Ramond-Ramond fields) is given by

$$\tilde{*}\, j_{5^2_2(n_1 \cdots n_5,\, m_1 m_2)} = \frac{(2\pi R_{m_1})(2\pi R_{m_2})}{2\kappa_{10}^2\, \mu_{5^2_2}}\, {}^{(1)\, m_1 m_2} \, . \tag{370}$$

According to the Wess-Zumino term of the $5^2_3(34567, 89)$-brane action (smeared in the isometry directions, x^8 and x^9) is written as

$$S^{5^2_3} = -\mu_{5^2_3} \int \gamma_{89}^{(8)} \wedge^{89} (x - X(\xi)) \, , \tag{371}$$

where the B-field, the Ramond-Ramond 0- and 4-forms, and the worldvolume gauge fields are turned off for simplicity, and ${}^{89}(x - X(\xi))$ is defined. As in the case of the 5^2_2-brane, if we consider the action

$$S\big[\, \tilde{g}_{ij},\, \tilde{\phi},\, \gamma_{ij}^{(8)} \,\big] = \frac{1}{2\kappa_{10}^2} \int \Big[-2\tilde{\phi} \, \big(\tilde{*}\tilde{R} + 4\, \tilde{\phi} \wedge \tilde{*}\, \tilde{\phi} \big) - \frac{1}{4}\, 4\tilde{\phi}\, \tilde{g}^{ik}\, \tilde{g}^{jl}\, {}^{(9)}_{ij} \wedge \tilde{*}\, {}^{(9)}_{kl} \Big] - \mu_{5^2_3} \int \frac{1}{2}\, \gamma^{(8)} \wedge (x - X(\xi)) \, , \tag{372}$$

and take a variation with respect to $\gamma^{(8)}$, we obtain the Bianchi identity for the P-flux with a source term:

$$\frac{1}{2\kappa_{10}^2}\, {}^{(1)} = \frac{\mu_{5^2_3}}{(2\pi R_)(2\pi R_)}\, n\, \delta^2(x - X(\xi))\, x^1 \wedge x^2 \, . \tag{373}$$

As in the case of the β-supergravity, we can further find a solution corresponding to the (Euclidean) background of an instanton that couples to γ^{ij} electrically. The explicit form of the background fields is presented.

We have presented various actions with the following form:

$$S\big[\tilde{g}_{ij},\, \tilde{\phi},\, {}^{i_1 \cdots i_{7-p}}\big] = \frac{1}{2\kappa_{10}^2} \int \Big[-2\phi \, \big(\tilde{*}\, \tilde{R} + 4\, \phi \wedge \tilde{*}\, \phi \big) - \frac{2\,(\alpha+1)\, \tilde{\phi}}{2\,(7-p)!}\, \tilde{g}_{i_1 j_1} \cdots \tilde{g}_{i_{7-p} j_{7-p}}\, {}^{(1)\, i_1 \cdots i_{7-p}} \wedge \tilde{*}\, {}^{(1)\, j_1 \cdots j_{7-p}} \Big] , \tag{374}$$

where ${}^{(1)\, i_1 \cdots i_{7-p}} \equiv^{i_1 \cdots i_{7-p}}$ is a non-geometric flux of which an exotic brane acts as the magnetic source, and α is an integer.

The equations of motion are given by

$$\tilde{R} + 4\big(\tilde{\nabla}^i \partial_i \tilde{\phi} - \tilde{g}^{ij}\, \partial_i \tilde{\phi}\, \partial_j \tilde{\phi}\big) + \tfrac{(\alpha+1)\, 2\, (\alpha+2)\, \tilde{\phi}}{2\, (7-p)!}\, {}_i{}^{j_1 \cdots j_{7-p}\, i}{}_{j_1 \cdots j_{7-p}} = 0 \, , \tag{375}$$

$$\tilde{R}_{ij} + 2\tilde{\nabla}_i \partial_j \tilde{\phi} - \tfrac{2\, (\alpha+2)\, \tilde{\phi}}{2\, (7-p)!} \Big({}_i{}^{k_1 \cdots k_{7-p}}{}_{j k_1 \cdots k_{7-p}} - (7-p)\, {}_{k_1}{}^{k_2 \cdots k_{7-p}\, k_1}{}_{j k_2 \cdots k_{7-p}} \tag{376}$$

$$- \tfrac{\alpha+2}{2}\, {}_k{}^{l_1 \cdots l_{7-p}\, k}{}_{l_1 \cdots l_{7-p}}\, \tilde{g}_{ij} \Big) = 0 \, , \tag{377}$$

$${}^{(9)}_{i_1 \cdots i_{7-p}} = 0 \, , \quad {}^{(9)}_{i_1 \cdots i_{7-p}} \equiv 2\, (\alpha+1)\, \tilde{\phi}\, \tilde{g}_{i_1 j_1} \cdots \tilde{g}_{i_{7-p} j_{7-p}}\, \tilde{*}\, {}^{(1)\, j_1 \cdots j_{7-p}} \equiv^{(8)}_{i_1 \cdots i_{7-p}} \, . \tag{378}$$

If we regard the dual potential $\overset{(8)}{}_{i_1\cdots i_{7-p}}$ as a fundamental field, the dual action is given by

$$
\begin{aligned}
S\big[\tilde{g}_{ij},\, \tilde{\phi},\, \overset{(8)}{}_{i_1\cdots i_{7-p}}\big] = \frac{1}{2\kappa_{10}^2} \int \Big[& -2\phi \left(\tilde{*}\, \tilde{R} + 4\, \phi \wedge \tilde{*}\, \phi \right) \\
& - \frac{2\,(\tilde{\alpha}+1)\, \tilde{\phi}}{2\,(7-p)!}\, \tilde{g}^{i_1 j_1} \cdots \tilde{g}^{i_{7-p} j_{7-p}}\, \overset{(8)}{}_{i_1\cdots i_{7-p}} \wedge \tilde{*}\, \overset{(8)}{}_{j_1\cdots j_{7-p}} \Big] ,
\end{aligned}
\tag{379}
$$

where we defined $\tilde{\alpha} \equiv -\alpha - 2$. We can add the Wess-Zumino term of the exotic $p^{7-p}_{-\alpha}$-brane extending in the $x^1, \cdots, x^p$-directions and smeared over the $x^1, \cdots, x^{7-p}$-directions:

$$
S = -\mu_{p^{7-p}_{-\alpha}} \sum_{1,\cdots,7-p} \int_{p+1 \times T^{7-p}_{1\cdots 7-p}} \frac{n^{1\cdots 7-p}}{(7-p)!}\, \overset{(8)}{}_{1\cdots 7-p} \wedge \frac{x^1 \wedge \cdots \wedge x^{7-p}}{(2\pi R_1)\cdots(2\pi R_{7-p})} = -\mu_{p^{7-p}_{-\alpha}} \int \frac{1}{(7-p)!}\, \overset{(8)}{}_{1\cdots 7-p} \wedge (x - X(\xi)) \,.
\tag{380}
$$

Then, taking variation, we obtain the following Bianchi identity as the equation of motion:

$$
{}_{2\,1\cdots 7-p} = 2\kappa_{10}^2\, \mu_{p^{7-p}_{-\alpha}} \frac{n^{1\cdots 7-p}}{(2\pi R_1)\cdots(2\pi R_{7-p})}\, \delta^2(x - X(\xi))\, x^1 \wedge x^2 \,.
\tag{381}
$$

It will be also important to investigate a reformulation of the effective worldvolume theory of exotic branes by using the newly introduced background fields $\big(\tilde{g}_{ij},\, \tilde{\phi},\, {}^{i_1\cdots i_{7-p}}\big)$. More generally, it will be important to find a manifestly U-duality covariant formulation for the effective worldvolume theory of exotic extremal branes.

We considered the general solutions of the equations of motion in the simple model of closed and open tensionless superstring and exotic p-branes. Using the $OSp(1,2M)$ invariant character of the differential one-form $Y^\Lambda G_{\Lambda\Xi} dY^\Xi$ and two-form $dY^\Lambda G_{\Lambda\Xi} dY^\Xi$ one can construct more general $OSp(1,2M)$ invariant super p-brane actions with enhanced supersymmetry. At first, we note that the closed $2n$-differential form $\Omega_{2n} = (G_{\Lambda\Xi} dY^\Lambda \wedge dY^\Xi)^n$

$$
\Omega_{2n} = d \wedge \Omega_{(2n-1)} \equiv G_{\Lambda_1\Xi_1} dY^{\Lambda_1} \wedge dY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} dY^{\Lambda_n} \wedge dY^{\Xi_n}
\tag{382}
$$

which is not equal to zero, because of the symplectic character of the supertwistor metric $G_{\Lambda\Xi}$, can be used to generate the Dirichlet boundary terms for the open super p-brane ($p = 2n-1$) described by the generalized action

$$
S = S_{2n-1} + \beta_{(2n-1)} \int_{M_{2n}} \Omega_{2n}.
\tag{383}
$$

Similarly to the open superstring case, the Wess-Zumino integral in (383) is transformed to the integral along the $(2n-1)$-dimensional boundary M_{2n-1} of the super $(2n-1)$-brane worldvolume

$$
\int_{M_{2n}} \Omega_{2n} = \oint_{M_{2n-1}} G_{\Lambda_1\Xi_1} Y^{\Lambda_1} \wedge dY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} dY^{\Lambda_n} \wedge dY^{\Xi_n}.
\tag{384}
$$

The sufficient conditions for the vanishing of the variations of the integral (384) with the fix initial and final data are the conditions

$$
\partial_\tau Y^\Lambda(\tau,\sigma)|_{\sigma^i=0,\pi} = 0, \quad (i = 1, 2, ..., 2n-1)
\tag{385}
$$

generalizing the Dirichlet boundary condition. Therefore, this open super p-brane is described by the pure static solution

$$
Y^\Lambda(\tau, \sigma \hat{\imath}) = Y^\Lambda_0(\sigma^i), \quad (i = 1, 2, ..., 2n-1)(386)
$$

generalizing the superstring static solution (**??**). On the other hand the integrals (384)

$$
S_{(2n-2)} = \beta_{(2n-2)} \int_{M_{2n-1}} \Omega_{2n-1} = \beta_{(2n-2)} \int_{M_{2n-1}} G_{\Lambda_1\Xi_1} Y^{\Lambda_1} dY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} dY^{\Lambda_n} \wedge dY^{\Xi_n}
\tag{387}
$$

can be considered as the $OSp(1,2M)$ invariant actions for the new models of super p-branes ($p=2n-2$) with enhanced supersymmetry. For $n=1$ we get the known action for superparticles, but for $n=2,3$ we find the new actions for the supermembrane

$$S_2 = \beta_2 \int_{M_3} \Omega_3 = \tilde{\beta}_2 \int d\tau d^2\sigma\, \varepsilon^{\mu\nu\rho} Y^\Lambda \partial_\mu Y_\Lambda \partial_\nu Y^\Xi \partial_\rho Y_\Xi, \tag{388}$$

or a domain wall in the symplectic superspace, and for the super four-brane

$$S_4 = \beta_4 \int_{M_5} \Omega_5 = \tilde{\beta}_4 \int d\tau d^4\sigma\, \varepsilon^{\mu\nu\rho\lambda\phi} Y^\Lambda \partial_\mu Y_\Lambda \partial_\nu Y^\Xi \partial_\rho Y_\Xi \partial_\lambda Y^\Sigma \partial_\phi Y_\Sigma. \tag{389}$$

When the Wess-Zumino terms are considered as the boundary terms generating the Dirichlet boundary conditions for the superstring and super p-branes (385) the breaking of the Weyl symmetry is localized at the boundaries. It shows that the spontaneous breaking of the $OSp(1,2M)$ symmetry on the boundaries is accompanied by the explicit breakdown of the Weyl gauge symmetry on the boundaries. Because the Dirichlet boundary conditions are associated with the Dp-branes attached on their boundaries, a question on the action of Dp-branes in the symplectic superspaces considered here appears. It implies the correspondent generalization of the proposed Wess-Zumino actions. One of the posssible generalizations is rather natural and is based on the observation that the Weyl invariance of the considered Wess-Zumino actions may be restored by the minimal lengthening of the differentials $d \to D = (d - A)$, where the worldvolume one-form A is the gauge field associated with the Weyl symmetry. The covariant differentials DY^Σ are homogeneously transformed under the Weyl symmetry transformations

$$(DY^\Sigma)' \equiv ((d-A)Y^\Sigma)' = e^\lambda DY^\Sigma, \quad A' = A + d\lambda. \tag{390}$$

Then the generalized $OSp(1,2M)$ invariant two and one-forms

$$\begin{aligned}(e^\phi DY^\Sigma G_{\Sigma\Xi} DY^\Xi)' &= e^\phi DY^\Sigma G_{\Sigma\Xi} DY^\Xi, \\ (e^\phi Y^\Sigma G_{\Sigma\Xi} DY^\Xi)' &= e^\phi Y^\Sigma G_{\Sigma\Xi} DY^\Xi\end{aligned} \tag{391}$$

become the invariants of the Weyl symmetry also, where the compensating scalar field ϕ, with the transformation low

$$\phi' = \phi - 2\lambda, \tag{392}$$

was introduced. Then the closed $2n$-differential form $\Omega_{2n} = (G_{\Lambda\Xi} dY^\Lambda \wedge dY^\Xi)^n$ may be changed by the Weyl invariant $2n$-differential form $\tilde{\Omega}_{2n} = (e^\phi G_{\Lambda\Xi} DY^\Lambda \wedge DY^\Xi)^n$

$$\tilde{\Omega}_{2n} \equiv e^{n\phi} G_{\Lambda_1\Xi_1} DY^{\Lambda_1} \wedge DY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} DY^{\Lambda_n} \wedge DY^{\Xi_n}, \tag{393}$$

and Ω_{2n-1} by $\tilde{\Omega}_{2n-1}$

$$\tilde{\Omega}_{2n-1} \equiv e^{n\phi} Y^{\Lambda_1} \wedge DY_{\Lambda_1} \wedge ... \wedge DY^{\Lambda_n} \wedge DY_{\Lambda_n}. \tag{394}$$

As a result, the actions (384) is transformed to the new super $(2n-1)$-brane action

$$\tilde{S}_{(2n-1)} = \beta_{(2n-1)} \int_{M_{2n}} \tilde{\Omega}_{2n} = \beta_{(2n-1)} \int e^{n\phi} G_{\Lambda_1\Xi_1} DY^{\Lambda_1} \wedge DY^{\Xi_1} \wedge ... \wedge G_{\Lambda_n\Xi_n} DY^{\Lambda_n} \wedge DY^{\Xi_n} \tag{395}$$

invariant under the $OSp(1,2M)$ and Weyl symmetries. Respectively, the action

$$\tilde{S}_{(2n-2)} = \beta_{(2n-2)} \int_{M_{2n-1}} \tilde{\Omega}_{2n-1} = \beta_{(2n-2)} \int e^{n\phi} Y^{\Lambda_1} \wedge DY_{\Lambda_1} \wedge ... \wedge DY^{\Lambda_n} \wedge DY_{\Lambda_n} \tag{396}$$

will describe a new $OSp(1,2M)$ and Weyl invariant super $(2n-2)$-brane.
These actions may be presented in the Dp-brane like form

$$\tilde{S}_p = \tilde{\beta}_p \int d\tau d^p\sigma\, e^{\frac{(p+1)}{2}\phi} \sqrt{|det[(\partial_\mu - A_\mu) Y^\Lambda G_{\Lambda\Xi} (\partial_\nu - A_\nu) Y^\Xi]|},\ (p = 2n-1), \tag{397}$$

where $\tilde{\beta}_p$ is the Dp-brane tension.

We generalized this model to the higher orders in the derivatives of the Goldstone fields and constructed the new Wess-Zumino like actions supposed to describe tensile exotic p-branes. It was shown in deep detail, that the bosonic couplings described above were consistent with all the linear couplings of closed superstring background fields with higher-dimensional supergravity theory including exceptional degrees of freedom of multiple D-branes. These couplings were originally computed in the current literature and then extended to Dp-branes with using T-duality symmetries. We will review the illustration of the general formalism with presentation of the Wess-Zumino term for multiple D-branes that is required to do such matching

$$
\begin{aligned}
S_{\mathcal{WZ}} \;=\; & \Xi_\Sigma \int \mathrm{Tr}\Bigg[P \wedge D^{(R)} + \Delta(\Xi_\Sigma)\big(D^{(V)} \wedge B\big) - \Delta(\Xi_\Sigma)\left(D^{(U)} \wedge B + \frac{1}{2} D^{(R)} \wedge B \wedge B\right) \\
& -\Delta(\Xi_\Sigma)\left(D^{(Z)} + D^{(T)} \wedge B + \frac{1}{2} D^{(V)} \wedge B \wedge B + \frac{1}{6} D^{(R)} \wedge B \wedge B \wedge B\right) \wedge B \wedge D^{(Z)} \\
& -\Delta(\Xi_\Sigma)\left(D^{(Z)} + \frac{1}{2} D^{(T)} \wedge B + \frac{1}{6} D^{(R)} \wedge B \wedge B + \frac{1}{24} D^{(Z)} \wedge B \wedge B \wedge B\right) \wedge B \wedge D^{(Z)} \\
& +\Delta(\Xi_\Sigma)\Big(D^{(U)} \wedge G + D^{(U)} \wedge G \wedge K^{(R)} - D^{(T)} \wedge K^{(R)} - D^{(T)} \wedge K^{(R)} \wedge G \wedge K^{(T)}\Big) \\
& +\Delta(\Xi_\Sigma)\big({}^{(T)} + B \wedge D^{(V)} - D^{(V)} \wedge K^{(R)} \wedge G - B \wedge G + D^{(V)} \wedge K^{(R)}\big) \wedge K^{(V)} \\
& -(D^{(W)} + D^{(S)} \wedge B) + B \wedge L^{(Z)} + D^{(S)} \wedge B - B \wedge L^{(R)} \wedge G + D^{(S)} \wedge G \wedge L^{(W)} \\
& -\Delta(\Xi_\Sigma)\big(B - D^{(Z)} \wedge D^{(S)} \wedge L^{(W)} \wedge L^{(V)}\big) + \left(B^{(X)} - \frac{1}{2} B \wedge D^{(Z)} \wedge D^{(S)} \wedge L^{(Z)}\right) \\
& + \big({}^{(W)} + D^{(S)} \wedge B\big) \wedge L^{(R)} + \big(D^{(W)} + D^{(S)} \wedge B \wedge L^{(W)} \wedge L^{(V)}\big) \wedge L^{(Z)} \wedge G \\
& + \big(D^{(W)} + D^{(S)} \wedge B \wedge L^{(R)} \wedge G +{}^{(W)} + D^{(S)} \wedge B \wedge L^{(Z)} \wedge L^{(R)} \wedge G\big) \wedge L^{(W)} \\
& -\Delta(\Xi_\Sigma)\big({}^{(W)} + D^{(S)} \wedge B - B \wedge L^{(R)} + B \wedge L^{(R)} \wedge G + D^{(S)} \wedge B \wedge G\big) \wedge L^{(R)} \\
& + \big(D^{(W)} + D^{(S)} \wedge B\big) \wedge L^{(R)} + \big(D^{(W)} + D^{(S)} \wedge B \wedge L^{(W)} \wedge L^{(V)}\big) \wedge L^{(Z)} \wedge G \\
& + \big(D^{(W)} + D^{(S)} \wedge B \wedge L^{(R)} \wedge G + D^{(W)} \wedge B \wedge L^{(Z)} \wedge G\big) \wedge L^{(W)} \wedge L^{(V)}\Bigg]
\end{aligned} \tag{398}
$$

The fundamental interaction between the hidden and dark sectors with branes implies that all the soft higher-dimensional terms acquire a supergravity dependent form and this has a drastic effect on the supergravity theory at low energy scales. We assumed that the dark energy hidden sector can be successfully incorporated into the theory of elementary particles and that the cosmological inclusion of membranes in the observable sector can somehow be solved in the hyperspace framework. There are two sectors, the observable sector and the hidden sector of extremal branes plus the assumption that the two sectors interact quantum gravitationally with the supersymmetric representations of quantum gravity in diverse dimensions. Then, the main issue is to understand how the extremal branes can be implemented into the theoretical framework. An appropriate restriction of the global symmetry group of the effective low energy limit supergravity theory to the integers is conjectured to be the U-duality symmetry of the relative superstring theory. Therefore, we claim these supergravity backgrounds are also transition duals. Supergravity equations of motion and the torsional constraint for heterotic superstrings pose severe restrictions on the allowed type of fluxes. We provide a theoretical example that is consistent with the IIB orientifold action, the IIB linearized supergravity equation of motion and the torsional relation in the U-dual heterotic supergravity background. These backgrounds come in pairs and we argued them to be related by a geometric transition, meaning that one of them contains branes, the other one only flux. A rigorous proof of this claim and the implications for weakstrong coupling dualities in the underlying field theories are left for future work. The supergeometrical origin of this supergravity theory is remarkable.

The higher-dimensional bulk and brane action for exceptional supergravity theory, can be organised and the solution is

$$
\begin{aligned}
S_T =& \int d^N x\, d^D Y \sqrt{|\mathcal{G}|}\Big(\mathcal{R}_{EXT}(\mathcal{G}) + \mathcal{L}_{KIN} + \mathcal{L}_{KR} + \mathcal{L}_{NS} + \mathcal{L}_{GFC} + \mathcal{L}_{INT} + \sqrt{|\mathcal{G}|}^{-1}\mathcal{L}_{CS}\Big)\\
&+ \mathcal{T}_{\mathcal{P}} \int e^{n\phi} G_{\Lambda_1\Xi_1} \mathcal{D}Y^{\Lambda_1} \wedge \mathcal{D}Y^{\Xi_1} \wedge \ldots \wedge G_{\Lambda_n\Xi_n} \mathcal{D}Y^{\Lambda_n} \wedge \mathcal{D}Y^{\Xi_n}\\
&= \int d^N x\, d^D Y \sqrt{|\mathcal{G}|}\Big(4\mathcal{H}^{MN}\mathcal{D}_M\mathcal{D}_N\Xi - \mathcal{D}_M\mathcal{D}_N\mathcal{H}^{MN} - 4\,\mathcal{H}^{MN}\mathcal{D}_M\Xi\,\mathcal{D}_N\Xi + 4\mathcal{D}_M\mathcal{H}^{MN}\,\mathcal{D}_N\Xi\\
&+ \frac{1}{8}\mathcal{H}^{MN}\mathcal{D}_M\mathcal{H}^{KL}\mathcal{D}_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\mathcal{D}_M\mathcal{H}^{KL}\,\mathcal{D}_K\mathcal{H}_{NL} - \frac{1}{2}\Gamma^M{}_{NK}\mathcal{H}^{NP}\mathcal{H}^{KQ}\mathcal{D}_P\mathcal{H}_{QM}\Big)\\
&+ \frac{1}{4}\mathcal{G}^{MN}\Big(\partial_\mu\mathcal{B}_{\nu M} - \partial_\nu\mathcal{B}_{\mu M} + 3\Lambda_{MNP}\mathcal{V}^N{}_\mu\mathcal{V}^P{}_\nu + 4\Delta^P_{MN}\mathcal{B}_{[\mu P}\mathcal{V}^N{}_{\nu]} + 4\Omega^I_{MN}\mathcal{A}^I{}_{[\mu}\mathcal{V}^N{}_{\nu]}\\
&- \mathcal{A}^I{}_M\mathcal{F}^I{}_{\mu\nu} - \mathcal{P}_{MP}\mathcal{V}^P{}_{\mu\nu}\Big)\Big(\partial_\mu\mathcal{B}_{\nu M} - \partial_\nu\mathcal{B}_{\mu M} + 3\Lambda_{MNP}\mathcal{V}^N{}_\mu\mathcal{V}^P{}_\nu + 4\Delta^P_{MN}\mathcal{B}_{[\mu P}\mathcal{V}^N{}_{\nu]}\\
&+ 4\Omega^I_{MN}\mathcal{A}^I{}_{[\mu}\mathcal{V}^N{}_{\nu]} - \mathcal{A}^I{}_N\mathcal{F}^{I\ \mu\nu} - \mathcal{P}_{NQ}\mathcal{V}^{Q\mu\nu}\Big) + \frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\mathcal{D}_{\mu MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\Big)\\
&+ \frac{1}{4}\mathcal{G}^{MN}\Big(\mathcal{B}_{\mu\nu M} - \mathcal{A}^I{}_M\mathcal{F}^I{}_{\mu\nu} - \mathcal{P}_{MP}\mathcal{V}^P{}_{\mu\nu}\Big)\Big(\mathcal{B}^{\mu\nu}{}_N - \mathcal{A}^I{}_N\mathcal{F}^{I\ \mu\nu} - \mathcal{P}_{NQ}\mathcal{V}^{Q\mu\nu}\Big)\\
&+ \frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\mathcal{D}_\mu\mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\Big)\Big(\mathcal{D}^\mu B_{PQ} + \mathcal{A}^J{}_{[P}\mathcal{D}^\mu\mathcal{A}^J{}_{Q]}\Big)\\
&+ \frac{3}{4}\mathcal{G}^{MQ}\mathcal{G}^{NR}\mathcal{G}^{PS}\Big(\Lambda_{MNP} + 2\mathcal{A}^I{}_{[M}\Omega^I_{NP]} - 2\mathcal{P}_{T[M}\Delta^G_{NP]}\\
&- \frac{1}{4}\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{G}_{MN}\mathcal{G}^{PQ}\mathcal{D}^\mu\mathcal{G}_{PQ} + \frac{1}{4}\mathcal{D}^\mu\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{G}_{MN} - \frac{1}{2}\mathcal{D}_\mu\Big(\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{G}_{MN}\Big)\\
&- \mathcal{G}_{MN}\mathcal{G}^{PQ}\mathcal{G}^{RS}\Delta^M_{PR}\Delta^N_{QS} - 2\mathcal{G}^{MN}\Delta^P_{MQ}\Delta^Q_{NP} + \mathcal{D}^\mu\Phi\mathcal{D}_\mu\Phi + \frac{1}{4}\Big(\mathcal{D}_\mu\mathcal{G}_{MN}\Big)\Big(\mathcal{D}^\mu\mathcal{G}^{MN}\Big)\\
&- \frac{1}{2}\mathcal{G}^{MN}\Big(\mathcal{D}_\mu\mathcal{A}^I{}_M\Big)\Big(\mathcal{D}^\mu\mathcal{A}^I{}_N\Big) - \frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\mathcal{D}_\mu\mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu\mathcal{A}^I{}_{N]}\Big)\Big(\mathcal{D}^\mu\mathcal{B}_{PQ}\\
&+ \mathcal{A}^J{}_{[P}\mathcal{D}^\mu\mathcal{A}^J{}_{Q]}\Big) + \frac{1}{4}\Big(\partial_\mu\mathcal{B}_{\nu\lambda} - \frac{1}{2}\mathcal{A}^I{}_\mu\mathcal{F}^I{}_{\nu\lambda} - \frac{1}{2}\mathcal{V}^M{}_\mu\mathcal{H}_{\nu\lambda M} - \frac{1}{2}\mathcal{B}_{\mu M}\mathcal{V}^M{}_{\nu\lambda}\\
&+ \frac{1}{2}\Lambda_{MNP}\mathcal{V}^M{}_\mu\mathcal{V}^N{}_\nu\mathcal{V}^P{}_\lambda - \Omega^I_{MN}A^I{}_\mu\mathcal{V}^M{}_\nu\mathcal{V}^N{}_\lambda - \Delta^M_{NP}\mathcal{B}_{\mu M}\mathcal{V}^N{}_\nu\mathcal{V}^P{}_\lambda\Big)\Big(\partial^\mu\mathcal{B}^{\nu\lambda}\\
&- \frac{1}{2}\mathcal{A}^I_\mu\mathcal{F}^I_{\nu\lambda} - \frac{1}{2}\mathcal{V}^{M\mu}\mathcal{H}^{\nu\lambda M} - \frac{1}{2}\mathcal{B}^{\mu M}\mathcal{V}^{M\nu\lambda} + \frac{1}{2}\Lambda_{MNP}\mathcal{V}^{M\mu}\mathcal{V}^{N\nu}\mathcal{V}^{P\lambda}\\
&- \Omega^I_{MN}A^{I\mu}\mathcal{V}^{M\nu}\mathcal{V}^{N\lambda} - \Delta^M_{NP}\mathcal{B}^{\mu M}\mathcal{V}^{N\nu}\mathcal{V}^{P\lambda}\Big) + \frac{1}{2}\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{A}^I{}_M\mathcal{D}^\mu\mathcal{A}^I{}_N\\
&+ \mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\Omega^I_{MN} + \mathcal{A}^I{}_R\Delta^R_{MN}\Big)\Big(\Omega^I_{PQ} + \mathcal{A}^I{}_S\Delta^S_{PQ}\Big) + \Big(\xi^K\mathcal{D}_K\mathcal{H}^I{}_J\\
&- \mathcal{D}^P\xi^I\,\mathcal{H}_{PJ} + \Big(\mathcal{D}_J\xi^P - \mathcal{D}^P\xi_I\Big)\mathcal{H}^I{}_P + \xi^K\mathcal{D}_k\mathcal{H}^I{}_J - \mathcal{D}_K\xi^I\,\mathcal{H}^K{}_J + \mathcal{D}_J\xi^K\,\mathcal{H}^I{}_K\\
&+ \mathcal{D}_J\tilde{\xi}_K\,\mathcal{H}^{IK} + \mathcal{D}_J\xi^B\,\mathcal{H}^I{}_B - \mathcal{D}_K\tilde{\xi}_J\,\mathcal{H}^{IK} + \hat{\mathcal{L}}_\xi\,\mathcal{H}^I{}_J + \Big(\mathcal{D}_J\tilde{\xi}_K - \mathcal{D}_K\tilde{\xi}_J\Big)\mathcal{H}^{IK}\\
&+ \mathcal{T}_{\mathcal{P}} \int e^{n\phi} G_{\Lambda_1\Xi_1} \mathcal{D}Y^{\Lambda_1} \wedge \mathcal{D}Y^{\Xi_1} \wedge \ldots \wedge G_{\Lambda_n\Xi_n} \mathcal{D}Y^{\Lambda_n} \wedge \mathcal{D}Y^{\Xi_n}
\end{aligned}
$$

We apply our universal recipe of the preceding section to write the supergravity corresponding action based on the moduli superspace construction with exceptional inclusion of exotic brane systems. For the general case of fundamental extremal brane system the appropriate higher-dimensional theory includes special bulk action $\widetilde{\mathcal{L}}_B(\Sigma_{\hat{\Delta}})$, the brane $\widetilde{\mathcal{L}}_{BR}(\Sigma_{\hat{\Delta}})$ and hidden brane lagrangian $\widetilde{\mathcal{L}}_{HBR}(\Sigma_{\hat{\Delta}})$, the brane fields coupling action $\widetilde{\mathcal{L}}_{BFC}(\Sigma_{\hat{\Delta}})$ and hidden brane couplings term $\widetilde{\mathcal{L}}_{HBC}(\Sigma_{\hat{\Delta}})$. The exclusive solution of the higher-dimensional corresponding action of exceptional supergravity for the fundamental extremal brane system is

$$
\begin{aligned}
S_{EXT} &= \hat{\Delta}\int_{\Sigma_{\hat{\Delta}}} d^Dx\sqrt{\mathcal{G}_{\hat{\Delta}}}\,\mathcal{R}_{\hat{\Delta}}\,\widetilde{\mathcal{L}}_{EXT}(\Sigma_{\hat{\Delta}}) + \sum_{\hat{\Delta}}\Big\{\int_{\Sigma_{\hat{\Delta}}} d^Dx\sqrt{-\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{B}(\Sigma_{\hat{\Delta}}) + \int_{\Sigma_{\hat{\Delta}}} d^Dx\sqrt{-\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{BR}(\Sigma_{\hat{\Delta}}) \\
&+ \int_{\Sigma_{\hat{\Delta}}} d^Dx\sqrt{-\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{HBR}(\Sigma_{\hat{\Delta}}) + \int_{\Sigma_{\hat{\Delta}}} d^Dx\sqrt{-\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{BFC}(\Sigma_{\hat{\Delta}}) + \int_{\Sigma_{\hat{\Delta}}} d^Dx\sqrt{-\mathcal{G}_{\hat{\Delta}}}\widetilde{\mathcal{L}}_{HBC}(\Sigma_{\hat{\Delta}})\Big\} \\
&+ \hat{\Delta}\sum_{\Sigma}\mathcal{T}_{\Xi}\prod_{\Sigma}\int_{\Sigma_{\hat{\Delta}}} d^Dx\sqrt{-\mathcal{G}_{\hat{\Delta}}}\Big\{\mathcal{T}_{\Sigma} + \frac{1}{2}X_{\Sigma}\,\mathcal{D}_M\Sigma\,\mathcal{D}^M\Sigma + \mathcal{D}_M X^M \Xi^A_M(X)\mathcal{D}_N X^N \Xi^C_N(X)\,B_{AC}(\mathcal{Z}) \\
&+ \frac{1}{2}U_{\Sigma}\,\mathcal{D}_M\Sigma\,\mathcal{D}^M\Sigma + \mathcal{D}_M U^M \Xi^A_M(U)\mathcal{D}_N U^N \Xi^C_N(U)\,D_{AC}(B) + \frac{1}{2}Z_{\Sigma}\,\mathcal{D}_M\Sigma\,\mathcal{D}^M\Sigma \\
&+ \mathcal{D}_M Z^M \Xi^A_M(Z)\mathcal{D}_N Z^N \Xi^C_N(Z)\,P_{AC}(Z) + \cdots\Big\}
\end{aligned} \tag{399}
$$

We construct a fully consistent and gauge invariant actions in higher-dimensional exceptional supergravity with presence of backgrounds, superstrings and membrane interpretations in D-dimensional spacetime supermanifolds realized in the theoretical framework. We discuss and surrendered the challenges involved in the advanced construction of the full higher-dimensional supergravities in modern and constructive fashion. Our main results are both of purely fundamental and mathematical interest and lead, from the physical point of view, to the construction of new realistic superstring theories in supergravity backgrounds. The future of modern theoretical and mathematical physics is dependent on the creation of higher-dimensional models in the theoretical framework used in theories such as supergravity, superstrings and supersymmetric membranes.

14 Discussion and Future Directions

As discussed in section 2.3, there is an interesting identification between the island rule (3) for the brane perspective and the RT prescription (4) for the bulk perspective in the doubly holographic model of [1, 2]. One feature is that the RT surfaces in eq. (4) are extremized in two stages: first, one finds surfaces that are extremal everywhere aware from the brane, and second, the intersection of the RT surfaces with the brane is extremized. The latter corresponds to finding the quantum extremal surface in the island rule (3). The on-shell bulk surfaces found in the first step describe the leading contributions to the entanglement entropy in the large N limit of the boundary CFT, for different candidate quantum extremal surfaces. These contributions may be divided into two classes: various geometric contributions corresponding to terms of the Wald-Dong entropy [116, 117, 118, 119] coming from the various gravitational interactions induced in the brane theory by the CFT, and the quantum contributions appearing as S_{QFT} in the island rule (3). Of course, the first set of contributions comes from integrating the bulk area of the RT surface near the brane, while the second set comes from the bulk region far from the brane.

As discussed in section 2.3, there seems to be a direct parallel between the above analysis of the holographic entanglement entropy and of the holographic complexity using the CV proposal. Hence beginning with the subregion complexity=volume proposal (18) in the bulk, we arrive at the following description of the complexity from the brane perspective:

$$
\mathcal{C}_{\text{V}}^{\text{sub}}(\mathbf{R}) = \max_{\partial\widetilde{\mathcal{B}}=\sigma_{\mathbf{R}}}\left[\frac{\widetilde{W}_{\text{gen}}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\text{eff}}\ell'} + \mathcal{C}_{\text{QFT}}(\mathbf{R}\cup\widetilde{\mathcal{B}})\right], \tag{400}
$$

where the geometric contribution is given by eqs. (49) and (55). Focusing on this geometric contribution, this result leads us to propose eqs. (6) and (7) as the extension of the CV proposal for holographic complexity in higher curvature theories. Our experience with the Wald-Dong entropy suggests that W_K provides an infinite series of corrections involving higher powers of the extrinsic curvature [119], and eq. (7) only presents the first K^2 term in this series. Further, in section 2.2, we noted that the K-term in eq. (40) was chosen for its simplicity and the similarity to the form of the K corrections in the Wald-Dong entropy, but we cannot rule out the possibility that it involves more complicated contractions than that in eq. (40). Perhaps, equally interesting in eq. (400) are the 'quantum' contributions coming from integrating the bulk volume of the extremal surface far from the brane. These contributions can play an important role in determining the geometry of $\widetilde{\mathcal{B}}$. Recall that the boundary of the extremal surface consists of $\partial\mathcal{B} = \mathbf{R} \cup \Sigma_{\mathbf{R}}$. Hence the profile of $\mathcal{B}$ depends on the full details of the geometry of the boundary subregion $\mathbf{R}$. Hence any two $\mathbf{R}$ and $\mathbf{R}'$ with $\partial\mathbf{R} = \partial\mathbf{R}'$ yield the same RT surface $\Sigma_{\mathbf{R}}$ and the same quantum extremal surface $\sigma_{\mathbf{R}}$ on the brane, but these different choices will produce different island surfaces $\widetilde{\mathcal{B}}$ – see figure 7. Of course, this reflects the fact that the holographic complexity is sensitive to the details of the state that are not captured by the corresponding entanglement entropy.

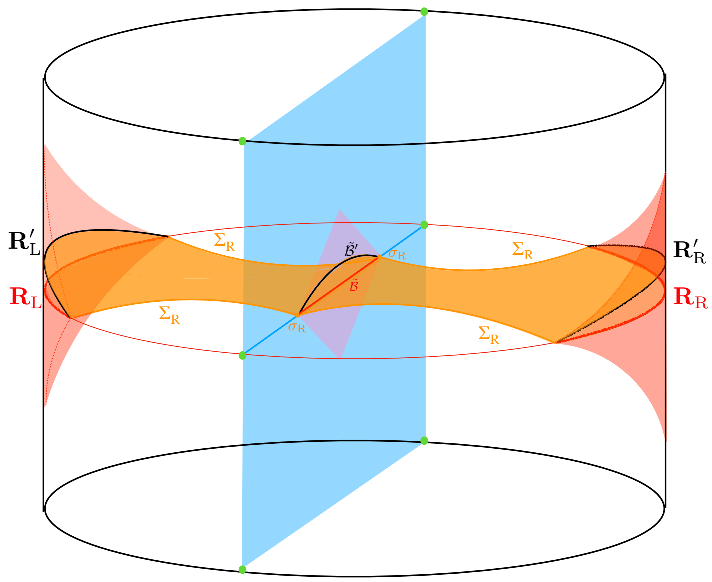

Figure 7: Different boundary subregions, $\mathbf{R} = \mathbf{R}_{\rm L} \cup \mathbf{R}_{\rm R}$ and $\mathbf{R}' = \mathbf{R}'_{\rm L} \cup \mathbf{R}'_{\rm R}$ with the same boundaries, *i.e.,* $\partial\mathbf{R} = \partial\mathbf{R}'$. The entanglement entropy and the RT surface remains the same for both subregions. However, the extremal surfaces $\mathcal{B}$ and $\mathcal{B}'$ (denoted by the orange regions) are different and hence they produce different islands $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}'$ on the brane (represented by the blue slice). The QES on the brane is unchanged and hence $\partial\widetilde{\mathcal{B}} = \partial\widetilde{\mathcal{B}}' = \sigma_{\mathbf{R}}$. The red shaded regions on the asymptotic boundary represent the causal domain of $\mathbf{R}$ ($\widetilde{\mathcal{B}}$). The subregion $\mathbf{R}'$ may be any spacelike surface in this causal domain. Similarly, $\widetilde{\mathcal{B}}'$ will always lie within the causal domain of the brane (denoted by the pink region).

A simple observation is that the holographic CV calculation picks out a special time slice on the brane (*i.e.,* $\widetilde{\mathcal{B}}$) for the island, in contrast to the corresponding entanglement entropy calculation which only fixes the boundary of the islands (*i.e.,* the quantum extremal surface). It would be interesting to explore how $\widetilde{\mathcal{B}}$ is deformed by making variations of the subregion $\mathbf{R}$ on the asymptotic AdS boundary, or perhaps by the insertion of extra operators in this subregion. While in principle these deformations could fill the causal domain of some canonical time slice with boundary $\sigma_{\mathbf{R}}$, our intuition is that generally, such variations will only produce perturbatively small deformations of $\widetilde{\mathcal{B}}$. If one examines the FG expansion (24) for embedding surface near the brane more closely, one finds

$$x^i\left(z,\sigma^a\right) = \overset{(0)}{x}{}^i\left(\sigma^a\right) + \overset{(1)}{x}{}^i\left(\sigma^a\right) + \cdots + \frac{z^d}{L^d}\left(\overset{(d)}{x}{}^i\left(\sigma^a\right) + \overset{(d)}{y}{}^i\left(\sigma^a\right)\log\left(\frac{z}{L}\right)\right) + \mathcal{O}\left(\frac{z^{d+1}}{L^{d+1}}\right). \tag{401}$$

The coefficients $\overset{(n)}{x}{}^i$ with $n < d$ are completely determined by the boundary profile $\overset{(0)}{x}{}^i$ and the boundary metric $\overset{(0)}{g}_{ij}$, *e.g.,* see [145]. The second independent coefficient in this expansion is $\overset{(d)}{x}{}^i$. This is precisely the coefficient that is determined by the infrared physics and the shape of $\mathbf{R}$ and so naively, its contributions on the brane are suppressed by the power $(z_{\rm B}/L)^d \ll 1$ in the regime of interest.

The above expansion also resolves a puzzle with eqs. (18) and (400). In the latter equation, the brane perspective seems to treat $\mathcal{C}_{\text{QFT}}$ as a higher-order term of the expansion in G_{eff}. However, both contributions arise at the same order in the G_{N} expansion in the bulk. There is no contradiction because the quantum corrections from the boundary CFT are enhanced by a power of the central charge $c_{\text{T}} \sim L^{d-1}/G_{\text{N}} \sim L^{d-2}/G_{\text{eff}}$ – where we use eq. (17) in the latter. However, the effect of the quantum contribution $\mathcal{C}_{\text{QFT}}$ can still be suppressed in the expansion on the brane in terms of powers of $z_{\text{B}}/L \sim L/\ell_{\text{B}}$.

A fascinating aspect of the second term in eq. (400) is that while this contribution has a geometric origin in our bulk calculations, it is interpreted as a quantum contribution from the brane perspective, *i.e.,* it is associated with the quantum fields on $\mathbf{R} \cup \tilde{\mathcal{B}}$. The interpretation follows the parallel with the holographic entanglement and the appearance of $S_{\text{QFT}}(\mathbf{R} \cup \text{islands})$ in the island rule (3). Of course, it points to an improved version of our generalized complexity=volume proposal (6) of the form

$$\mathcal{C}_{\text{V}}(\mathbf{R}) = \max_{\partial\mathcal{B}=\mathbf{R}} \left[\frac{W_{\text{gen}}(\mathcal{B}) + W_K(\mathcal{B})}{G_{\text{N}}\,\ell} + \mathcal{C}_{\text{bulk}} \right], \tag{402}$$

where $\mathcal{C}_{\text{bulk}}$ represents the contribution from the quantum field state in the bulk. This would be analogous to the appearance of quantum corrections in the holographic entanglement entropy [106, 107]. Of course, such additional contributions have long been expected because the CV proposal (1) has the form of a saddle point approximation of some more complete calculation. While eq. (400) is the first instance where these quantum corrections can be explicitly calculated, unfortunately, our doubly holographic model does not indicate what quantum calculation yields these contributions. Of course, it would be interesting to further investigate the properties of $\mathcal{C}_{\text{QFT}}$ in eq. (400) to gain further insight into this question.

In this vein, one immediate observation from examining eq. (400) is the tension between the maximization and the naive association of $\mathcal{C}_{\text{QFT}}$ with circuit complexity – or rather circuit depth. That is, if we associate $\mathcal{C}_{\text{QFT}}$ with the size of the quantum circuit needed to prepare the QFT state on the corresponding region (along the lines studied in, *e.g.,* [146, 147]) then the complexity follows from minimizing this quantity rather than maximizing. One simple resolution would be to consider our analysis with a Euclidean (rather than a Minkowski) signature. Then the CV conjecture (1) would correspond to minimizing the volume of the bulk surfaces and this minimization would naturally be inherited by the generalized proposal in eq. (400) or (402). This tension may suggest that $\mathcal{C}_{\text{QFT}}$ should instead be associated with an alternative interpretation of holographic complexity, *e.g.,* optimization of path integrals [148, 149], "quantum circuits" based on path-integrals [150] or using the equivalence of bulk and boundary symplectic forms [151, 152, 153]. Our doubly holographic model may also provide an interesting new forum to study these approaches.

General Higher Curvature Theories

While we are proposing that the generalized expressions for holographic complexity in eqs. (6) and (7) should apply for general theories of higher curvature gravity, we only applied our consistency tests in section 3 to two very specific theories. The feature that distinguished these theories was their boundary value problem. Namely, Gauss-Bonnet gravity can be solved with standard boundary conditions applied to the metric, while $f(\mathcal{R})$ gravity could be expressed in a form (*i.e.,* as a scalar-tensor theory) where the boundary conditions had a simple form. We note however that this limitation was because of issues in dealing with infinitely thin brane in higher curvature theories. Hence while this is a limitation of the doubly holographic model, we do not believe that it limits the applicability of our generalized proposal for holographic complexity. Certainly, the induced gravitational theories on the brane are outside this limited class of higher curvature theories.

However, we must admit that there are aspects of our consistency tests in section 3 that deserve further consideration. For example, one should better understand the appearance of the "effective Newton constant" in the generalized volume for $f(\mathcal{R})$ gravity. For the Gauss-Bonnet theory, it would be interesting to understand how to derive the expression for W_{bdy} in eq. (82) from the surface terms added to the gravitational action on either side of the brane.

We hope our generalized extension of the CV proposal will encourage further investigations of holographic complexity in higher curvature gravity models. Many studies of the CV proposal for higher derivative gravity (*e.g.,* [154, 155, 156, 157]) only consider the volume term. Therefore it will be interesting to explore the differences between the CV and our generalized CV approaches in various settings.

To close here, let us add that there is another interesting discrepancy in our approach which deserves further study. Setting aside the doubly holographic model and considering standard AdS holography for a moment, we observe that one finds logarithmic divergences in evaluating the boundary counterterms and the holographic entanglement entropy when the boundary dimension d is even. Of course, these divergences are related to the conformal anomaly of the boundary CFT. However, in evaluating the extremal volume for the holographic complexity, one finds that there are logarithmic divergences when d is odd. As a result, in the analysis of the doubly holographic model, one finds that one can account for the log divergences in the entanglement entropy (coming from the bulk region near the brane) by straightforwardly applying the Wald-Dong entropy to the induced gravitational action on the brane [1]. In contrast, there is no such match between the logarithmic divergences in the CV complexity in the bulk and the geometric contributions in our generalized complexity (402) on the brane (for odd d). Similarly, applying our geometric formula to the logarithmically divergent terms in the induced action naively yields contributions which do not appear in the CV complexity (for even d). In either case, one can adopt an approach where these logarithmic terms are treated separately. However, an alternative may be that the boundary between the geometric gravitational contributions and the quantum contributions is different for the generalized CV complexity in eq. (402), than say, for holographic entanglement entropy. This is certainly an issue that deserves further consideration.

While the above issue arises for all values of d when calculating corrections to sufficiently high orders, let us add that it is immediately apparent in our analysis in section 2.2 for lower dimensions, *e.g.,* the coefficients in eq. (48) diverge for $d = 2$ or 3. It arises there because the logarithmic divergence appears in the leading or first subleading contribution. We provide a detailed examination of these two cases in appendix 12. However, we emphasize again that the same issue arises in higher dimensions but only in higher-order contributions.

Mutual Complexity and Island Complexity

Much of our analysis focused on identifying the geometric terms in eq. (400) by looking at the contributions arising from the region near the brane, *i.e.,* the leading terms in the limit $z_{\rm B}/\tilde{L} \to 0$. However, we should recall that the quantum term $\mathcal{C}_{\rm QFT}(\mathbf{R}\cup\widetilde{\mathcal{B}})$ also includes the UV divergent terms associated with the cut-off surface near the asymptotic AdS boundary. These are less interesting for our purposes and so we point out that they can be eliminated by considering the mutual complexity, *e.g.,* [20, 36, 49, 158]

$$\Delta\mathcal{C}_{\rm V}^{\rm sub} = \mathcal{C}_{\rm V}^{\rm sub}\left(\mathbf{R}_{\rm L}\right) + \mathcal{C}_{\rm V}^{\rm sub}\left(\mathbf{R}_{\rm R}\right) - \mathcal{C}_{\rm V}^{\rm sub}\left(\mathbf{R}_{\rm L}\cup\mathbf{R}_{\rm R}\right) . \tag{403}$$

The UV divergent terms, which only depend on the boundary geometry of $\mathbf{R}_{\rm L}$ and $\mathbf{R}_{\rm R}$, cancel in this combination of complexities, leaving a UV finite quantity.

We also remark that the transition between the no-island phase to the island phase can also be diagnosed by the above mutual complexity. In particular, the latter vanishes in the no island phase, in which the bulk RT surfaces are disconnected phase – see figure 1. For the island phase, the mutual complexity jumps to a large negative value. In fact, we expect that this is dominated by the island contribution, *i.e.,*

$$\Delta\mathcal{C}_{\rm V}^{\rm sub} \simeq -\,\mathcal{C}_{\rm V}^{\rm Island} + \cdots = -\left.\frac{\widetilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) + \widetilde{W}_K(\widetilde{\mathcal{B}})}{G_{\rm eff}\,\ell'}\right|_{\widetilde{\mathcal{B}}_{\rm ext}} + \cdots . \tag{404}$$

Even though the entanglement entropy is continuous at the transition between these two phases, the complexity of the island state is much larger than that of the no island state. This reflects the fact that one is able to reconstruct the island on the brane from the asymptotic boundary state. Of course, similar discontinuities in the mutual complexity are seen in more conventional holographic settings, *e.g.,* [20, 47, 48, 49, 50, 51, 52, 98], but it would interesting to further understand the implications for quantum extremal islands.

Length Scale in Holographic Complexity

Both the holographic CV proposal (1) and our proposed generalization (6) involve an undetermined length scale ℓ. In most previous studies, *e.g.,* [21, 22, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52], this length scale is simply chosen to be the AdS curvature scale. However, our analysis was simplified by leaving ℓ undetermined, and in particular, we found a simple relation (38) between the scales associated with the holographic complexity in the bulk and on the brane. The AdS radius of the induced gravity on the brane, *i.e.,* $\ell_{\rm B} \approx L^2/z_{\rm B}$, is more or less independent of the bulk radius L, *i.e.,* the relation depends on the brane tension as shown in eq. (12). If one demands to identify the length scale in the complexity proposals to the AdS radius for the different gravity theories, the generalized CV for boundary subregion and the island are given by

$$\begin{aligned} C_{\rm V}^{\rm sub} &\equiv \max_{\mathcal{B}} \left[\frac{1}{G_{\rm bulk}\, L} \left(W_{\rm gen}(\mathcal{B}) + W_K(\mathcal{B}) \right) \right] , \\ C_{\rm V}^{\rm Island} &\equiv \max_{\widetilde{\mathcal{B}}} \left[\frac{1}{G_{\rm eff}\, \ell_{\rm B}} \left(\tilde{W}_{\rm gen}(\widetilde{\mathcal{B}}) + \tilde{W}_K(\widetilde{\mathcal{B}}) \right) \right] , \end{aligned} \tag{405}$$

and the two expressions do not agree, *i.e.,* $C_{\rm V}^{\rm sub}/C_{\rm V}^{\rm Island} \simeq \ell_{\rm B}/L$. Rather one would have to introduce an additional 'penalty factor' to produce the desired equivalence, *i.e.,*

$$C_{\rm V}^{\rm sub} \simeq \mathcal{P}\, C_{\rm V}^{\rm Island} + \cdots \qquad \text{with} \;\; \mathcal{P} = \frac{d-2}{d-1} \frac{\ell_{\rm B}}{L} . \tag{406}$$

In contrast to the simple relation in eq. (38), this additional factor has a complicated dependence on the physical parameters of the underlying theory.

Maximal Condition for Holographic Complexity

As we have stressed, the CV conjecture (1) and our generalized proposal (6) relies on maximizing the corresponding geometric functional on bulk hypersurfaces $\mathcal{B}$ with the appropriate boundary condition $\partial\mathcal{B} = \mathbf{R} \cup \Sigma_{\mathbf{R}}$. However, we only explicitly use the local equations, *i.e.,* $\frac{\delta \mathcal{C}_{\rm V}}{\delta X^\mu} = 0$ to find the extremum. For eq. (1), we are guaranteed that the extremal volume will be a maximum. However, with our generalization (6), we are no longer guaranteed that the corresponding geometric functional will reach a maximum in situations where the higher curvature contributions become important. That is, the solutions of the extremizing equation may be a maximum, a minimum, or a saddle point. Maximizing the holographic complexity further requires a necessary condition for the generalized CV functional to be a local maximum, *i.e.,*

$$\delta^2 \mathcal{C}_{\rm V}\ 0\,, \tag{407}$$

where the variation is defined with respect to perturbations of the extremal surface $\mathcal{B}$. Although, this condition is not necessary for the derivation of the results in this paper, it is still interesting to explore the meaning of this constraint on second variations of generalized complexity. From the viewpoint of holographic entanglement entropy $S_{\rm EE}$, its second variations (with respect to deformations of the entangling surface) are also constrained by strong stability, *i.e.,* $\delta^2 S_{\rm EE} \geq 0$, due to the fact the RT surface is a local minimum of its area. Similar strong stability should also be imposed on the generalized entropy $S_{\rm gen}$ – see [25, 159] for more discussion. It is remarked that strong stability is a nontrivial constraint independent of its extremality condition. As an important application, the second variation plays a crucial role in defining quantum null energy conditions [160, 161]. So we expect that there will be interesting applications of the stability condition (407) for holographic complexity.

Generalized First Law for Causal Diamonds

By applying Wald's Noether charge formalism [116, 117], the authors in [162, 163] derived an extended first law of causal diamond mechanics in Einstein gravity

$$\delta H_\zeta^{\text{matter}} = -\frac{\kappa}{8\pi G_{\text{N}}}\left[\delta A - k\delta V\right] , \tag{408}$$

where H_ζ^{matter} is the matter Hamiltonian associated with the flow generated by the conformal Killing vector ζ on the causal diamond, A is the area of the edge $\partial\Sigma$, and k denotes the extrinsic curvature of $\partial\Sigma$ embedded in the maximal slice. Connections to the first law of holographic complexity were also developed in [17, 19, 152]. Furthermore, it was extended to higher-curvature gravity in [120] as

$$\delta H_\zeta^{\text{matter}} = -\frac{\kappa}{2\pi G_{\text{N}}}\,\delta S_{\text{Wald}}\Big|_W + \int_{\partial\Sigma}\delta C_\zeta\,, \tag{409}$$

where $\delta C_\zeta = 0$ are the linearized equations of higher derivative theory and the Wald entropy evaluated on bifurcation surface $\partial\Sigma$ varies while keeping the generalized volume W fixed. Here the generalized volume W differs from ours in eq. (7) because the former has a constant term depending on the couplings of the higher derivative theory and it is normalized to be the regular volume for any higher-curvature gravity when evaluated in AdS background. The simplicity from our complexity formula is that the relevant coefficients only depend on the dimension of theory. Considering that our proposal suggests a new term W_K depending on the extrinsic curvature, it would be interesting to generalize the first law of causal diamond mechanics by connecting the Wald-Dong entropy and our generalized volume.

Generalizing Complexity=Action?

In the context of holographic complexity, the complexity=action (CA) conjecture [7, 8] and its subregion version [9] have also been widely studied. Generalizing our work to consider the CA proposal in the framework of our doubly holographic model is an obvious future direction. However, in contrast to the CV proposal, the CA approach already includes the corrections from higher-curvature terms due to the explicit dependence of action on these terms. So the real question to verify is whether the subregion-CA proposal in bulk theory produces the same complexity for the induced gravitational theory on the brane, *i.e.,* does one find

$$\mathcal{C}_{\text{A}}^{\text{sub}} \simeq \mathcal{C}_{\text{A}}^{\text{Island}} + \cdots \quad ? \tag{410}$$

If this is not the case, it may imply the need to consider a modified CA approach for higher-curvature gravity theory. Of course, subtlety is that surface and joint terms play a very important role in the CA approach [164], and determining the corresponding terms for higher curvature theories is quite demanding, *e.g.,* [165, 166, 167, 168]. Let us also note that the $\mathcal{C}_{\text{A}}^{\text{sub}}$ approach has already been studied in the literature, *e.g.,* [35, 51, 52], but the extension to the present context is not obvious from these results.

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