ON THE NUMBER OF POINTS INCLUDED IN A PLANE FIGURE WITH LARGE PAIRWISE DISTANCES

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Abstract. Using the method of compression we show that the number of points that can be placed in a plane figure with mutual distances at least $d > 0$ satisfies the lower bound

$$\gg_2 d^{d-1+\epsilon}$$

for some small $\epsilon > 0$.

1. Introduction

Let $d > 0$, then the following question appears in [1]

Question 1.1. What is the maximum number of points included in a plane figure (generally: in a space body) such that the distance between any two points is greater than or equal to d?

Though it belongs to the class of discrete geometry problems involving certain configurations of points and lines in the plane (resp. Euclidean space), the problem 1.1 is relatively unknown and unsolved. Depending on the dimension of the space in which the points dwell, the problem demands a precise arrangement of points so that their mutual distances are not small and are totally covered by a planar figure (resp. space body). In theory, the problem might be investigated by selecting a planar (resp. space curve) that contains all of these points in the correct configuration, as this curve can be embedded in a planar shape (resp. space body) or its slightly expanded and translated equivalents. This is the main concept we will use to get the major result in this paper. By using the method of compression [2], we show that the maximum number of points that can be included in a planar figure with mutual distances at least $d > 0$ is at least $d^{d-1+\epsilon}$. In particular, we obtain the following lower bound

Theorem 1.2. *Let $\Delta_2(d)$ denotes the maximum number of points that can be placed inside a geometric figure in $\mathbb{R}^2$ such that their mutual distances is at least $d > 0$ satisfies the lower bound*

$$\Delta_2(d) \gg_2 d^{d-1+\epsilon}$$

for some small $\epsilon > 0$.

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2. Preliminaries and background

Definition 2.1. By the compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. Compression is a term that refers to the process of re-scaling points in $\mathbb{R}^n$ for $n \geq 2$. It is vital to note that a compression, roughly speaking, pushes points very close to the origin away from the origin by a given scale, while also drawing points away from the origin close to the origin. A compression of scale $1 \geq m > 0$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map. To see this, suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective.

2.1. **The mass of compression.**

In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m\sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) - k} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m\sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}$$

and it is easy to notice that this inequality is absurd. By extension, one may try to equalize the sub-sequence by assigning the supremum and infimum and getting an estimate, however this would contradict the mass of compression inequality after a minor reassignment. As a result, it is critical for the estimate to make sense in order to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i, j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i, j \leq n$.

Lemma 2.4. *The estimate holds*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$ *is the Euler-Macheroni constant.*

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m>0$.

Proposition 2.1. *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for each $1\leq i\leq n$ and $x_i\neq x_j$ for $i\neq j$, then the estimates holds*

$$m\log\left(1-\frac{n-1}{\sup(x_j)}\right)^{-1}\ll\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)])\ll m\log\left(1+\frac{n-1}{\operatorname{Inf}(x_j)}\right)$$

for $n\geq 2$.

Proof. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ for $n\geq 2$ with $x_j\neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)])&=m\sum_{j=1}^{n}\frac{1}{x_j}\\&\leq m\sum_{k=0}^{n-1}\frac{1}{\operatorname{Inf}(x_j)+k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)])&=m\sum_{j=1}^{n}\frac{1}{x_j}\\&\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.\end{aligned}$$

□

Definition 2.6. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for all $i=1,2\ldots,n$. Then by the gap of compression of scale $m>0$, denoted $\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]$, we mean the expression

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\left|\left|\left(x_1-\frac{m}{x_1},x_2-\frac{m}{x_2},\ldots,x_n-\frac{m}{x_n}\right)\right|\right|$$

Definition 2.7. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for all $1\leq i\leq n$. Then by the ball induced by $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ under compression of scale $m>0$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ we mean the inequality

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},x_2+\frac{m}{x_2},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|<\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)].$$

A point $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n=2$.

Remark 2.8. In the geometry of balls under compression of scale $m>0$, we will assume implicitly that $1\geq m>0$. The circle induced by points under compression is the ball induced on points when we take $n=2$.

Proposition 2.2. *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ for $n\geq 2$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]+m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]-2mn.$$

In particular, we have the estimate

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]-2mn+O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\right)$$

for $\vec{x}\in\mathbb{N}^n$, *where* $m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]$ *is the error term in this case.*

Lemma 2.9 (Compression estimate). *Let* $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ *for* $n\geq 2$ *and* $x_i\neq x_j$ *for* $i\neq j$, *then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\ll n\mathrm{sup}(x_j^2)+m^2\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)^2}\right)-2mn$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\gg n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1}-2mn.$$

Theorem 2.10. *Let* $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ *with* $z_i\neq z_j$ *for all* $1\leq i<j\leq n$. *Then* $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ *if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$, then it follows that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||\leq||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.2 that $||\vec{z}||<||\vec{y}||$. It follows that

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|&<\left|\left|\vec{y}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|\\&=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 2.11. *Let* $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ *with* $x_i\neq x_j$ *for all* $1\leq i<j\leq n$. *If* $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ *then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 2.10 that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\vec{y}] &> \mathcal{G}\circ\mathbb{V}_m[\vec{z}]\\ &\geq \mathcal{G}\circ\mathbb{V}_m[\vec{x}]\\ &> \mathcal{G}\circ\mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 2.12. Theorem 2.11 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

2.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.13. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 2.14. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball.

Theorem 2.15. *The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.10, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.2 that $||\vec{x}||<||\vec{y}||$ or $||\vec{y}||<||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y}$ is an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The latter equality follows from assertion that two balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the indistinguishable balls and must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\\ &=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that
$$\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=\bigg|\bigg|\vec{y}-\frac{1}{2}\bigg(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\bigg)\bigg|\bigg|$$
and $\vec{y}$ is indeed admissible, thereby ending the proof. $\square$

Next we obtain an equivalent notion of the circumference of the circle induced by points under compression in the plane $\mathbb{R}^2$ in the following result.

Proposition 2.3. *Let $\vec{x}\in\mathbb{R}^2$ with $x_i\neq 0$ for each $1\leq i\leq 2$. Then the circumference of the circle induced by point $\vec{x}$ under compression of scale m, denoted $\mathbb{V}_m[\vec{x}]$, is given by*
$$\delta(\mathbb{V}_m[\vec{x}])=\pi\times(\mathcal{G}\circ\mathbb{V}_m[\vec{x}]).$$

Proof. This follows from the mere definition of the circumference of a circle and noting that the radius r of the circle induced by the point $\vec{x}\in\mathbb{R}^2$ under compression is given by
$$r=\frac{\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}{2}.$$
$\square$

Remark 2.16. We note that we can replace the set $\mathbb{N}^n$ used in our construction with $\mathbb{R}^n$ at the compromise of imposing the restrictions $\vec{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$ such that $x_i>1$ for all $1\leq i\leq n$ and $x_i\neq x_j$ for $i\neq j$. The following construction in our next result in the sequel employs this flexibility.

3. Lower bound

Theorem 3.1. *Let $\Delta_2(d)$ denotes the maximum number of points that can be placed inside a geometric figure in $\mathbb{R}^2$ such that their mutual distances is at least $d>0$ satisfies the lower bound*
$$\Delta_2(d)\gg_2 d^{d-1+\epsilon}$$
for some small $\epsilon>0$.

Proof. Pick arbitrarily a point $(x_1,x_2)=\vec{x}\in\mathbb{R}^2$ for $1\leq j\leq 2$ such that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\geq d^d$. Next we apply the compression of scale $m>0$, given by $\mathbb{V}_m[\vec{x}]$ and construct the circle induced by the compression given by
$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$
with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2}\geq\frac{d^d}{2}$ by choosing
$$\inf(x_i)_{1\leq i\leq 2}=d^{d+\epsilon}$$
for some small $\epsilon>0$. On this circle locate admissible points so that the chord joining each pair of adjacent admissible points is at least $d>0$. Invoking Proposition 2.3, the circumference of the circle induced under compression is given by
$$\delta(\mathbb{V}_m[\vec{x}])=\pi\times\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$
We join all pairs of adjacent admissible points considered by a chord. We note that we can use the length of the arc induced by any two adjacent admissible points on the circle to determine the number of pairwise admissible points with mutual

distances at least $d > 0$. It follows that the number of admissible points on the circle with mutual distances at least $d > 0$ satisfies the lower bound

$$\begin{aligned}\Delta_2(d) := &\frac{\pi \times (\mathcal{G} \circ \mathbb{V}_m[\vec{x}])}{2d \sin \theta}\\ &\gg_2 \frac{\inf(x_i)_{1 \leq i \leq 2}}{d}\\ &= \frac{d^{d+\epsilon}}{d}.\end{aligned}$$

If the circle of compression constructed lives in the plane figure then the lower bound follows. On the other hand, if it pokes outside then we can enlarge the plane figure and apply translation under suitable translation vectors so that it covers the circle of compression $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and the lower bound also holds in this case. This completes the construction. □

4. **Data availability statement**

The manuscript has no associated data.

5. **Conflict of interest**

The authors declare no conflict of interest regarding this manuscript. [1].

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