ANALYTIC NUMBER THEORY
Sum of powers of integers

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The purpose of this study was to rewrite the formulas for the sum of powers of integers in a subsequent general mathematical formula independent of Bernoulli polynomials and numbers, starting from the formula of Faulhaber.

Abstract:
The domain of functions throughout the work are the exponents "r" natural numbers N+.

The history:

The sum of powers of integers is defined: $\sum_{k=1}^{n} k^r$

$$\sum_{k=1}^{n} k^1 = \frac{1}{2} n (1+n)$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n (1+n) (1+2n)$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (1+n)^2$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{30} n (1+n) (1+2n) \left(-1+3n+3n^2\right)$$

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In 1631 Johann Faulhaber published in the journal "Algebra Academiae" a general formula which was later proved by Carl Jacobi in 1834 where we used the Bernoulli polynomials and numbers.

$$\sum_{k=1}^{n} k^{r} == \frac{1}{r+1} * \left(\sum_{k=0}^{r} \binom{r+1}{k} * (n+1)^{(r+1-k)} * B_k \right)$$

PROCESSING

I ° Objective: To replace the binomial formula and the Bernoulli numbers, respectively, with mathematical formulas containing the Gamma and the Zeta function.

The Bernoulli numbers can be written as a function of ζ(k) and extrapolate from Euler's formula to find the integer values of ζ(2k)

$$\zeta(2k) = \frac{2^{2k-1} * \pi^{2k} * \text{Abs}[B_{2k}]}{(2k)!}$$

In this formula, Euler considered the absolute value of the Bernoulli numbers as in Faulhaber’s formula are used for each:

$$k \in N \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66} \; ....\right))$$

For each k > = 2 the formula is the same:

$$B_k = \left(-2^{1-k}\right) \pi^{-k} \, \Gamma(k+1) \, \mathrm{Cos}\left[\frac{3 k \pi}{2}\right] \zeta(k) \quad \forall\, k \in \mathbb{N} \bigvee k \geq 2$$

The addition operator $\mathrm{Cos}\left[\frac{3 k \pi}{2}\right]$ or $\mathrm{Re}\left[e^{\frac{3 k \pi i}{2}}\right]$ have the function take the value of the Bernoulli numbers used in Faulhaber's formula.

Processing the formula:

$\mathbb{N}+$	B_k	$\left(-2^{1-k}\right) \pi^{-k} \, \Gamma(k+1) \, \mathrm{Cos}\left[\frac{3 k \pi}{2}\right] \zeta(k)$
2	$\frac{1}{6}$	$\frac{1}{6}$
3	0	0
4	$-\frac{1}{30}$	$-\frac{1}{30}$
5	0	0
6	$\frac{1}{42}$	$\frac{1}{42}$
7	0	0

Binomial formula can be rewritten in terms of Gamma function:

$$\binom{r+1}{k} = \frac{\Gamma(r+2)}{\Gamma(k+1)\,\Gamma(-k+r+2)}$$

Substituting these two functions in Faulhaber's formula

$$\sum_{k=1}^{n} k^{r} = \frac{1}{r+1} * \left(\sum_{k=0}^{r} \binom{r+1}{k} * (n+1)^{(r+1-k)} * B_k\right) = \frac{1}{r+1} * \sum_{k=0}^{r} \left(\frac{\Gamma(r+2)}{(\Gamma(k+1)\,\Gamma(-k+r+2))} * (n+1)^{(r+1-k)} * \left(\left(-2^{1-k}\right) \pi^{-k}\, \Gamma(k+1) \operatorname{Cos}\left[\frac{3 k \pi}{2}\right] \zeta(k)\right)\right)$$

The mathematical formula so constructed will not work because the function ζ (k) does not converge for k = 1 (Series Harmonica -> ∞) then we can decompose the sum in:

$$\frac{1}{r+1} * \left(\sum_{k=0}^{1} \binom{r+1}{k} * (n+1)^{(r+1-k)} * B_k\right) + \frac{1}{r+1} * \left(\sum_{k=2}^{r} \left(-\frac{2^{1-k}\, (1+n)^{1-k+r}\, \pi^{-k}\, \Gamma(r+2) \operatorname{Cos}\left[\frac{3 k \pi}{2}\right] \zeta(k)}{\Gamma(-k+r+2)}\right)\right)$$

The Bernoulli numbers assume the value 1 when k = 0 and (- 1/ 2) when k = 1

$$\frac{1}{r+1} * \left(\sum_{k=0}^{0} \binom{r+1}{k} * (n+1)^{(r+1-k)} * 1\right) + \frac{1}{r+1} * \left(\sum_{k=1}^{1} \binom{r+1}{k} * (n+1)^{(r+1-k)} * \left(-\frac{1}{2}\right)\right) + \frac{1}{r+1} * \left(\sum_{k=2}^{r} \left(-\frac{2^{1-k}\, (1+n)^{1-k+r}\, \pi^{-k}\, \Gamma(r+2) \operatorname{Cos}\left[\frac{3 k \pi}{2}\right] \zeta(k)}{\Gamma(-k+r+2)}\right)\right)$$

$$= \; -\frac{(1+n)^{r}\, (-1-2n+r)}{2\,(1+r)} + \frac{1}{r+1} * \left(\sum_{k=2}^{r} \left(-\frac{2^{1-k}\, (1+n)^{1-k+r}\, \pi^{-k}\, \Gamma(r+2) \operatorname{Cos}\left[\frac{3 k \pi}{2}\right] \zeta(k)}{\Gamma(-k+r+2)}\right)\right)$$

$$= \; -\frac{(1+n)^{r}\, (-1-2n+r)}{2\,(1+r)} + \frac{(1+n)^{1+r}\, (\Gamma(r+2))}{r+1} * \left(\sum_{k=2}^{r} \left(-\frac{2^{1-k}\, (1+n)^{-k}\, \pi^{-k} \operatorname{Cos}\left[\frac{3 k \pi}{2}\right] \zeta(k)}{\Gamma(-k+r+2)}\right)\right)$$

Processing functions: in the second column Faulhaber's formula in the third column, the last formula:

for r=1	$\frac{1}{2} n (1+n)$	$\frac{1}{2} n (1+n)$
for r=2	$\frac{1}{6} n (1+n) (1+2n)$	$\frac{1}{6} n (1+n) (1+2n)$
for r=3	$\frac{1}{4} n^2 (1+n)^2$	$\frac{1}{4} n^2 (1+n)^2$
for r=4	$\frac{1}{30} n (1+n) (1+2n) (-1+3n(1+n))$	$\frac{1}{30} n (1+n) (1+2n) (-1+3n(1+n))$
for r=5	$\frac{1}{12} n^2 (1+n)^2 (-1+2n(1+n))$	$\frac{1}{12} n^2 (1+n)^2 (-1+2n(1+n))$
for r=6	$\frac{1}{42} (n - 7n^3 + 21n^5 + 21n^6 + 6n^7)$	$\frac{1}{42} (n - 7n^3 + 21n^5 + 21n^6 + 6n^7)$
for r=7	$\frac{1}{24} (2n^2 - 7n^4 + 14n^6 + 12n^7 + 3n^8)$	$\frac{1}{24} (2n^2 - 7n^4 + 14n^6 + 12n^7 + 3n^8)$
for r=8	$\frac{1}{90} n (-3 + n^2 (20 + n^2 (-42 + 5n^2 (12 + n(9+2n)))))$	$\frac{1}{90} n (-3 + n^2 (20 + n^2 (-42 + 5n^2 (12 + n(9+2n)))))$
for r=9	$\frac{1}{20} n^2 (-3 + n^2 (10 + n^2 (-14 + n^2 (15 + 2n(5+n)))))$	$\frac{1}{20} n^2 (-3 + n^2 (10 + n^2 (-14 + n^2 (15 + 2n(5+n)))))$

....

2nd Objective: Replacing the function ζ (k) with the Riemman integral and exchange integral with the summation.

$$-\frac{(1+n)^r (-1-2n+r)}{2(1+r)} + \frac{(1+n)^{1+r} (\Gamma(r+2))}{r+1} * \left(\sum_{k=2}^{r} \left(-\frac{2^{1-k} (1+n)^{-k} \pi^{-k} \operatorname{Cos}\left[\frac{3k\pi}{2}\right]}{\Gamma(-k+r+2)} * \frac{\int_0^\infty \left(\frac{x^{k-1}}{e^x-1} \right) dx}{\Gamma(k)} \right) \right)$$

$$-\frac{(1+n)^r (-1-2n+r)}{2(1+r)} + \frac{(1+n)^{1+r} (\Gamma(r+2))}{r+1} * \sum_{k=2}^{r} \int_0^\infty \left(-\frac{2^{1-k} (1+n)^{-k} \pi^{-k} \operatorname{Cos}\left[\frac{3k\pi}{2}\right]}{\Gamma(-k+r+2)} * \frac{\left(\frac{x^{k-1}}{e^x-1} \right)}{\Gamma(k)} \right) dx$$

Exchange integral with the summation:

$$\int_0^\infty \sum_{k=2}^{r} \left(- \frac{2^{1-k} (1+n)^{-k} \pi^{-k} \operatorname{Cos}\left[\frac{3 k \pi}{2}\right]}{\Gamma(-k+r+2)} * \frac{\left(\frac{x^{k-1}}{e^x - 1}\right)}{\Gamma(k)} \right) dx$$

$$\int_0^\infty \left(\sum_{k=2}^{r} \left(- \frac{2^{1-k} (1+n)^{-k} \pi^{-k} x^{-1+k} \operatorname{Cos}\left[\frac{3 k \pi}{2}\right]}{(-1+e^x)\, \Gamma(k)\, \Gamma(-k+r+2)} \right) \right) dx = \int_0^\infty \frac{(1+n)^{-1-r} (2\pi)^{-1-r} \left(i (1+n)^r \left(\left(2\pi - \frac{i x}{1+n}\right)^r - \left(2\pi + \frac{i x}{1+n}\right)^r \right) + 2 x^r \operatorname{Sin}\left[\frac{3\pi r}{2}\right] \right)}{(-1+e^x)\, \Gamma(r+1)} dx$$

$$= \frac{(1+n)^{-1-r}}{\Gamma(r+1)} \int_0^\infty \frac{(2\pi)^{-1-r} \left(i (1+n)^r \left(\left(2\pi - \frac{i x}{1+n}\right)^r - \left(2\pi + \frac{i x}{1+n}\right)^r \right) + 2 x^r \operatorname{Sin}\left[\frac{3\pi r}{2}\right] \right)}{(-1+e^x)} dx$$

Adding functions omit the first integral:

$$- \frac{(1+n)^r (-1-2n+r)}{2(1+r)} + \left(\frac{(1+n)^{1+r} (\Gamma(r+2))}{r+1} * \frac{(1+n)^{-1-r}}{\Gamma(r+1)} \right) \int_0^\infty \frac{(2\pi)^{-1-r} \left(i \left((2\pi * (n+1) - i x)^r - (2\pi * (n+1) + i x)^r \right) + 2 x^r \operatorname{Sin}\left[\frac{3\pi r}{2}\right] \right)}{(-1+e^x)} dx$$

On arrive at the final mathematical formula:

$$\sum_{k=1}^{n} k^r = - \frac{(1+n)^r (-1-2n+r)}{2(1+r)} + \frac{(n+1)^r\, i\, \pi^{-r-1}}{2^{r+1}} \left(\int_0^\infty \frac{\left(\left(2\pi - \frac{i x}{1+n}\right)^r - \left(2\pi + \frac{i x}{1+n}\right)^r \right)}{(-1+e^x)} dx \right) + 2^{-r} \pi^{-1-r} \operatorname{Sin}\left[\frac{3\pi r}{2}\right] \int_0^\infty \left(\frac{x^r}{-1+e^x} \right) dx$$

$$\boxed{\sum_{k=1}^{n} k^r = - \frac{(1+n)^r (-1-2n+r)}{2(1+r)} + i (1+n)^r (2\pi)^{-1-r} \left(\int_0^\infty \frac{\left(\left(2\pi - \frac{i x}{1+n}\right)^r - \left(2\pi + \frac{i x}{1+n}\right)^r \right)}{(-1+e^x)} dx \right) + 2^{-r} \pi^{-1-r} \operatorname{Sin}\left[\frac{3\pi r}{2}\right] \Gamma(r+1)\, \zeta(r+1)}$$

Processing functions: in the second column Faulhaber's formula in the third column, the last formula highlighted:

for r=1	$\frac{1}{2} n (1+n)$	$\frac{1}{2} n (1+n)$
for r=2	$\frac{1}{6} n (1+n)(1+2n)$	$\frac{1}{6} n (1+n)(1+2n)$
for r=3	$\frac{1}{4} n^2 (1+n)^2$	$\frac{1}{4} n^2 (1+n)^2$

for r=4 $\frac{1}{30} n (1+n) (1+2n) (-1+3n(1+n))$ $\frac{1}{30} n (1+n) (1+2n) (-1+3n(1+n))$

for r=5 $\frac{1}{12} n^2 (1+n)^2 (-1+2n(1+n))$ $\frac{1}{12} n^2 (1+n)^2 (-1+2n(1+n))$

for r=6 $\frac{1}{42} \left(n-7n^3+21n^5+21n^6+6n^7\right)$ $\frac{1}{42} \left(n-7n^3+21n^5+21n^6+6n^7\right)$

for r=7 $\frac{1}{24} \left(2n^2-7n^4+14n^6+12n^7+3n^8\right)$ $\frac{1}{24} \left(2n^2-7n^4+14n^6+12n^7+3n^8\right)$

for r=8 $\frac{1}{90} n \left(-3+n^2 \left(20+n^2 \left(-42+5n^2 (12+n(9+2n))\right)\right)\right)$ $\frac{1}{90} n \left(-3+n^2 \left(20+n^2 \left(-42+5n^2 (12+n(9+2n))\right)\right)\right)$

for r=9 $\frac{1}{20} n^2 \left(-3+n^2 \left(10+n^2 \left(-14+n^2 (15+2n(5+n))\right)\right)\right)$ $\frac{1}{20} n^2 \left(-3+n^2 \left(10+n^2 \left(-14+n^2 (15+2n(5+n))\right)\right)\right)$

Text Reference

[1] Rademacher H. (Springer 1973). “Topics in Analytic Number Theory

https://www.skuola.net/matematica/analytic-number-theory-sum-of-powers-of-integers.html

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