On the decomposition of a vector using matrix algebra

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June 17, 2022

Abstract

In this letter a theorem is stated on the decomposition of a vector with respect to two axes using matrix algebra. This theorem may serve to develop exercises for education in mathematics and physics. In most physics exercises the vectors are given by their polar coordinates. Motivated by practical applications, we denote the vectors by their Cartesian coordinates. Using Cartesian coordinates provide a didactic aid to develop exercises which can be solved without the use of a calculator. Several examples clarify the educational potential of the theorem.

1 Theorem

Let $\underline{F}$ be a given vector, $\underline{F_u}$ and $\underline{F_v}$ its components with respect to two given axes u and v, respectively. The directions of the u-axis and the v-axis are defined by the unit vectors $\underline{e_u}$ and $\underline{e_v}$, respectively. Then

$$\underline{F} = \underline{F_u} + \underline{F_v} = F_u \cdot \underline{e_u} + F_v \cdot \underline{e_v}$$

where F_u and F_v are the coordinates of the vector $\underline{F}$ with respect the axes u and v.
Define the unit vectors $\underline{e_u^\perp}$ and $\underline{e_v^\perp}$ as

$$\underline{e_u^\perp} := \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \cdot \underline{e_u}, \qquad \underline{e_v^\perp} := \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \cdot \underline{e_v}$$

Remark: note that

$$\left(\underline{e_u}\right)^T \cdot \left(\underline{e_u^\perp}\right) = \left(\underline{e_v}\right)^T \cdot \left(\underline{e_v^\perp}\right) = 0$$

and

$$\left(\underline{e_u}\right)^T \cdot \left(\underline{e_v^\perp}\right) = -\left(\underline{e_v}\right)^T \cdot \left(\underline{e_u^\perp}\right)$$

(1) The coordinates F_u and F_v are obtained as:

$$F_u := \frac{\left(\underline{F}\right)^T \cdot \left(\underline{e_v^\perp}\right)}{\left(\underline{e_u}\right)^T \cdot \left(\underline{e_v^\perp}\right)}, \qquad F_v := \frac{\left(\underline{F}\right)^T \cdot \left(\underline{e_u^\perp}\right)}{\left(\underline{e_v}\right)^T \cdot \left(\underline{e_u^\perp}\right)}$$

(2) The components $\underline{F_u}$ and $\underline{F_v}$ are obtained as:

$$\underline{F_u} := F_u \cdot \underline{e_u} = P_u \cdot \underline{F}, \qquad \underline{F_v} := F_v \cdot \underline{e_v} = P_v \cdot \underline{F}$$

where

$$P_u := \frac{\left(\underline{e_u}\right) \cdot \left(\underline{e_v^\perp}\right)^T}{\left(\underline{e_u}\right)^T \cdot \left(\underline{e_v^\perp}\right)}, \qquad P_v := \frac{\left(\underline{e_v}\right) \cdot \left(\underline{e_u^\perp}\right)^T}{\left(\underline{e_v}\right)^T \cdot \left(\underline{e_u^\perp}\right)}$$

Remark: for the 2×2 projection matrices P_u and P_v it holds that

$$P_u + P_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Proof

(1)

$$\left(\underline{F}\right)^T \cdot \left(\underline{e}_v^\perp\right) = \left(F_u \cdot \underline{e}_u + F_v \cdot \underline{e}_v\right)^T \cdot \left(\underline{e}_v^\perp\right) = F_u \cdot \left(\underline{e}_u\right)^T \cdot \left(\underline{e}_v^\perp\right) + F_v \cdot \left(\underline{e}_v\right)^T \cdot \left(\underline{e}_v^\perp\right) = F_u \cdot \left(\underline{e}_u\right)^T \cdot \left(\underline{e}_v^\perp\right) + F_v \cdot 0$$

$$= F_u \cdot \left(\underline{e}_u\right)^T \cdot \left(\underline{e}_v^\perp\right) \quad \Leftrightarrow \quad F_u = \frac{\left(\underline{F}\right)^T \cdot \left(\underline{e}_v^\perp\right)}{\left(\underline{e}_u\right)^T \cdot \left(\underline{e}_v^\perp\right)}$$

The result for the coordinate F_v follows by replacing the axes $u \to v$ and $v \to u$ in the expression for the coordinate F_u.

(2)

$$\underline{F}_u := F_u \cdot \underline{e}_u = \underline{e}_u \cdot \frac{\left(\underline{e}_v^\perp\right)^T \cdot \left(\underline{F}\right)}{\left(\underline{e}_u\right)^T \cdot \left(\underline{e}_v^\perp\right)} = \frac{\left(\underline{e}_u\right) \cdot \left(\underline{e}_v^\perp\right)^T \cdot \left(\underline{F}\right)}{\left(\underline{e}_u\right)^T \cdot \left(\underline{e}_v^\perp\right)} = \frac{\left(\underline{e}_u\right) \cdot \left(\underline{e}_v^\perp\right)^T}{\left(\underline{e}_u\right)^T \cdot \left(\underline{e}_v^\perp\right)} \cdot \underline{F} = P_u \cdot \underline{F}$$

The result for the component $\underline{F}_v$ follows by replacing the axes $u \to v$ and $v \to u$ in the expression for the component $\underline{F}_u$.

□

2 Example

Given:

$\underline{F} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$, $\underline{e}_u = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} +2 \\ +1 \end{bmatrix}$, $\underline{e}_v = \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} +1 \\ +3 \end{bmatrix}$, see Figure 1

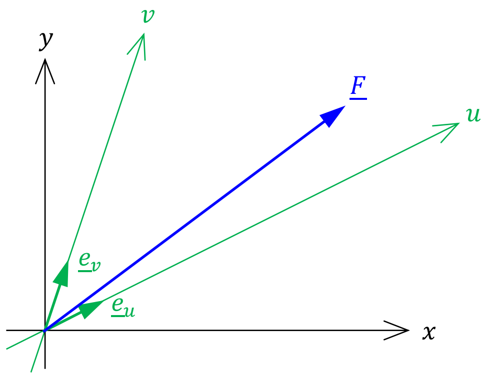

Figure 1: The given vector $\underline{F}$ and the given axes u and v.

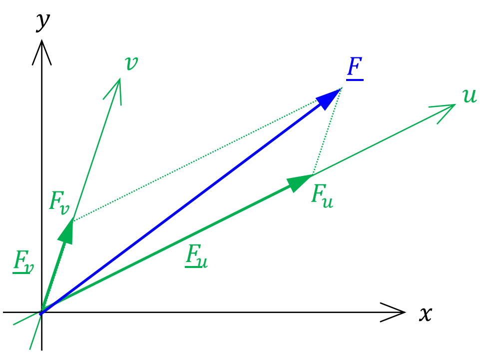

Figure 2: The coordinates F_u and F_v and the corresponding components $\underline{F}_u$ and $\underline{F}_v$.

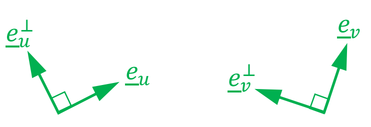

Figure 3: The unit vectors $\underline{e}_u^\perp$ and $\underline{e}_v^\perp$ constructed from the given unit vectors $\underline{e}_u$ and $\underline{e}_v$.

Applying the theorem

The coordinate F_u, see Figure 2:

$$\underline{e_v^{\perp}} := \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \cdot \underline{e_v} = \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} -3 \\ +1 \end{bmatrix}$$, see Figure 3

$$\underline{F}^T \cdot \underline{e_v^{\perp}} = \begin{bmatrix} 20 & 15 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{10}} = \frac{-45}{\sqrt{10}}$$

$$\underline{e_u}^T \cdot \underline{e_v^{\perp}} = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{10}} = \frac{-5}{\sqrt{5} \cdot \sqrt{10}}$$

$$F_u = \frac{\underline{F}^T \cdot \underline{e_v^{\perp}}}{\underline{e_u}^T \cdot \underline{e_v^{\perp}}} = \frac{\frac{-45}{\sqrt{10}}}{\frac{-5}{\sqrt{5} \cdot \sqrt{10}}} = \frac{-45}{-5} \cdot \sqrt{5} = 9\sqrt{5}$$

The component $\underline{F_u}$, see Figure 2:

$$\underline{F_u} = F_u \cdot \underline{e_u} = 9\sqrt{5} \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix}$$

The projection matrix P_u

$$(\underline{e_u}) \cdot (\underline{e_v^{\perp}})^T = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{5} \cdot \sqrt{10}} \cdot \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$$

$$P_u = \frac{(\underline{e_u}) \cdot (\underline{e_v^{\perp}})^T}{(\underline{e_u})^T \cdot (\underline{e_v^{\perp}})} = \frac{\frac{1}{\sqrt{5} \cdot \sqrt{10}} \cdot \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}}{\frac{-5}{\sqrt{5} \cdot \sqrt{10}}} = \frac{1}{5} \cdot \begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$$

Alternatively, the component $\underline{F_u}$ follows from

$$\underline{F_u} = P_u \cdot \underline{F} = \frac{1}{5} \cdot \begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix}$$

The coordinate F_v, see Figure 2:

$$\underline{e_u^{\perp}} := \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \cdot \underline{e_u} = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} -1 \\ +2 \end{bmatrix}$$, see Figure 3

$$\underline{F}^T \cdot \underline{e_u^{\perp}} = \begin{bmatrix} 20 & 15 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} = \frac{10}{\sqrt{5}}$$

$$\underline{e_v}^T \cdot \underline{e_u^{\perp}} = \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5} \cdot \sqrt{10}}$$

$$F_v = \frac{\underline{F}^T \cdot \underline{e_u^{\perp}}}{\underline{e_v}^T \cdot \underline{e_u^{\perp}}} = \frac{\frac{10}{\sqrt{5}}}{\frac{5}{\sqrt{5} \cdot \sqrt{10}}} = \frac{10}{5} \cdot \sqrt{10} = 2\sqrt{10}$$

The component $\underline{F_v}$, see Figure 2:

$$\underline{F_v} = F_v \cdot \underline{e_v} = 2\sqrt{10} \cdot \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Note that

$$\underline{F} = \begin{bmatrix} 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \underline{F_u} + \underline{F_v}$$

The projection matrix P_v

$$(\underline{e_v}) \cdot (\underline{e_u^{\perp}})^T = \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5} \cdot \sqrt{10}}$$

$$P_v = \frac{(\underline{e_v}) \cdot (\underline{e_u^{\perp}})^T}{(\underline{e_v})^T \cdot (\underline{e_u^{\perp}})} = \frac{\frac{1}{\sqrt{5} \cdot \sqrt{10}} \cdot \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}}{\frac{5}{\sqrt{5} \cdot \sqrt{10}}} = \frac{1}{5} \cdot \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}$$

Alternatively, the component $\underline{F_v}$ follows from

$$\underline{F_v} = P_v \cdot \underline{F} = \frac{1}{5} \cdot \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Note that

$$P_u + P_v = \frac{1}{5} \cdot \begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix} + \frac{1}{5} \cdot \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 Acknowledgement

The authors acknowledge the support of Charlotte Creusen and Ad Klein both affiliated with the Department of Engineering of Zuyd University of Applied Sciences.