For GA Newcomers: Demonstrating the Equivalence of Different Expressions for Vector Rotations

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Abstract

As an example for newcomers to GA who may have difficulty applying its identities to real problems, we use those identities to prove the equivalence of two expressions for rotations of a vector. Rather than simply present the proof, we first review the relevant GA identities, then formulate and explore reasonable conjectures that lead, promptly, to a solution.

1 Introduction

A particularly useful feature of GA is its ability to express rotations conveniently. For example (Fig. 1), the vector $\mathbf{v}'$ that results from the rotation of vector $\mathbf{v}$ through the angle θ about an axis perpendicular to the bivector $\hat{\mathbf{B}}$, and in the sense of the rotation of $\hat{\mathbf{B}}$ itself, is

$$\mathbf{v}' = \left[e^{-\hat{\mathbf{B}}\theta/2}\right]\mathbf{v}\left[e^{\hat{\mathbf{B}}\theta/2}\right]. \tag{1.1}$$

Macdonald ([1], p. 89) begins the derivation of that formula by expressing $\mathbf{v}$ as the sum of its components parallel and perpendicular to $\hat{\mathbf{B}}$ ($\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$, respectively). Then, Macdonald notes that while the vertical component is unaffected by the rotation, the parallel component becomes $\mathbf{v}_\parallel e^{\hat{\mathbf{B}}\theta}$. Thus, $\mathbf{v}'$ is also

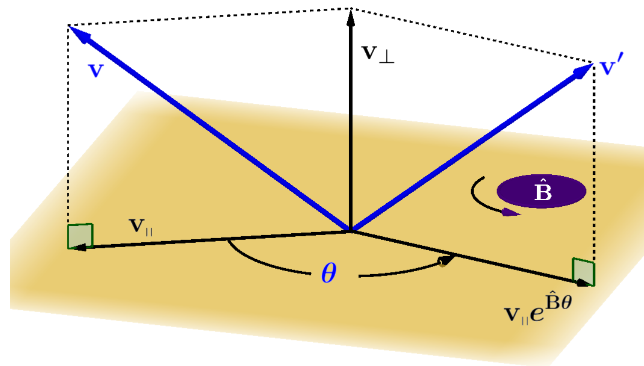

Figure 1: Relations between vector $\mathbf{v}$; its components perpendicular and parallel to $\hat{\mathbf{B}}$; and the rotated vector $\mathbf{v}'$.

$$\begin{aligned}\mathbf{v}' &= \mathbf{v}_\perp + \mathbf{v}_\parallel \underbrace{\left[\cos\theta + \hat{\mathbf{B}}\sin\theta\right]}_{=e^{\hat{\mathbf{B}}\theta}} \\ &= \mathbf{v}_\perp + \mathbf{v}_\parallel \cos\theta + \mathbf{v}_\parallel \hat{\mathbf{B}}\sin\theta. \end{aligned} \tag{1.2}$$

How might we demonstrate that Eqs. (1.1) and (1.2) are equivalent? We begin by expanding Eq. 1.1 :

$$\begin{aligned}\mathbf{v}' &= \left[\cos\frac{\theta}{2} - \hat{\mathbf{B}}\sin\frac{\theta}{2}\right]\mathbf{v}\left[\cos\frac{\theta}{2} + \hat{\mathbf{B}}\sin\frac{\theta}{2}\right] \\ &= \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\hat{\mathbf{B}}\cos\frac{\theta}{2}\sin\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\cos\frac{\theta}{2}\sin\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2}. \end{aligned} \tag{1.3}$$

To make further progress, we need to review a bit.

2 From 3D Euclidean GA: some identities that we will use . . .

For any vector $\mathbf{v}$ and any unit bivector $\hat{\mathbf{B}}$,

1. The multiplicative inverse of $\hat{\mathbf{B}}$: $\hat{\mathbf{B}}^{-1} = -\hat{\mathbf{B}}$
2. $\hat{\mathbf{B}} \cdot \mathbf{v} = -\mathbf{v} \cdot \hat{\mathbf{B}}$
3. $\hat{\mathbf{B}} \wedge \mathbf{v} = \mathbf{v} \wedge \hat{\mathbf{B}}$
4. $\mathbf{v}\hat{\mathbf{B}} = \mathbf{v} \cdot \hat{\mathbf{B}} + \mathbf{v} \wedge \hat{\mathbf{B}}$
5. $\hat{\mathbf{B}}\mathbf{v} = \hat{\mathbf{B}} \cdot \mathbf{v} + \hat{\mathbf{B}} \wedge \mathbf{v} = -\mathbf{v} \cdot \hat{\mathbf{B}} + \mathbf{v} \wedge \hat{\mathbf{B}}$

6. The components of $\mathbf{v}$ parallel to and perpendicular to $\hat{\mathbf{B}}$ are:
   (a) $\mathbf{v}_{\shortparallel} = \left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1} = \left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\left({}^{-}\hat{\mathbf{B}}\right)$
   (b) $\mathbf{v}_{\perp} = \left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1} = \left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\left({}^{-}\hat{\mathbf{B}}\right)$
7. From 3, 4, and 5 (above),
   (a) $\hat{\mathbf{B}}\mathbf{v} = {}^{-}\mathbf{v}\hat{\mathbf{B}} + 2\mathbf{v}\wedge\hat{\mathbf{B}}$
   (b) $\hat{\mathbf{B}}\mathbf{v} = \mathbf{v}\hat{\mathbf{B}} - 2\mathbf{v}\cdot\hat{\mathbf{B}}$
8. From trigonometry:
   (a) $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \sin\alpha$
   (b) $\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha$

3 Demonstration of the Equivalence of Our Two Expressions for $\mathbf{v}'$

After reviewing the identities in Section 2 , several possible routes might suggest themselves. For example, we can combine the two $\cos\frac{\theta}{2}\sin\frac{\theta}{2}$ terms in Eq. (1.2) to obtain

$$\mathbf{v}' = \mathbf{v}\cos^2\frac{\theta}{2} + \left(\mathbf{v}\hat{\mathbf{B}} - \hat{\mathbf{B}}\mathbf{v}\right)\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2}.$$

Now, from point 7b in Section 2 , we see that $\mathbf{v}\hat{\mathbf{B}} - \hat{\mathbf{B}}\mathbf{v} = 2\mathbf{v}\cdot\hat{\mathbf{B}}$. Therefore,

$$\begin{aligned}
\mathbf{v}' &= \mathbf{v}\cos^2\frac{\theta}{2} + 2\mathbf{v}\cdot\hat{\mathbf{B}}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2} \\
&= \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right] - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2} \\
&= \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2}\ . \qquad (3.1)
\end{aligned}$$

We now have a $\sin\theta$ term in this expression for $\mathbf{v}'$, just as we do in Eq. (1.2). We can demonstrate the equality of those terms (i.e., that $\mathbf{v}_{\|}\hat{\mathbf{B}} = \mathbf{v}\cdot\hat{\mathbf{B}}$) by noting that $\mathbf{v}_{\|} = \left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\left(\hat{\mathbf{B}}^{-1}\right)$, so that $\mathbf{v}_{\|}\hat{\mathbf{B}} = \left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\left(\hat{\mathbf{B}}^{-1}\right)\hat{\mathbf{B}} = \mathbf{v}\cdot\hat{\mathbf{B}}\left(\hat{\mathbf{B}}^{-1}\hat{\mathbf{B}}\right) = \mathbf{v}\cdot\hat{\mathbf{B}}$.

What to do with the factor $\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}$ in Eq. (1.3) may not be clear. One idea is to "reverse" the product $\hat{\mathbf{B}}\mathbf{v}$ to obtain $\mathbf{v}\hat{\mathbf{B}}$, so that the $\hat{\mathbf{B}}$ in that part will cancel with the second $\hat{\mathbf{B}}$. We can do this in either of two ways, using items 7a and 7b :

$$\begin{aligned}
\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= \left[\mathbf{v}\hat{\mathbf{B}} + 2\mathbf{v}\wedge\hat{\mathbf{B}}\right]\hat{\mathbf{B}} \\
&= \mathbf{v}\hat{\mathbf{B}}\hat{\mathbf{B}} + 2\left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}} \\
&= {}^{-}\mathbf{v} + 2\left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}},
\end{aligned}$$

and

$$
\begin{aligned}
\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= \left[\mathbf{v}\hat{\mathbf{B}} - 2\mathbf{v}\cdot\hat{\mathbf{B}}\right]\hat{\mathbf{B}} \\
&= \mathbf{v}\hat{\mathbf{B}}\hat{\mathbf{B}} - 2\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}} \\
&= {}^{-}\mathbf{v} - 2\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}}.
\end{aligned}
$$

These approaches will work, but—at least when I attempted them —they turned out to be tedious, and not at all insightful. So, let's look for a different idea. First, let's note that we're trying to demonstrate the equivalence between (1) a relation that's expressed in terms of the two vectors $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$ (i.e., Eq. (1.2)), and (2) a relation that's expressed in terms of products of $\mathbf{v}$ and $\hat{\mathbf{B}}$ (i.e., Eq. (3.1)). If we recall the derivations of items 6a and 6b , ([1], p. 119) we can see that the product $\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}$ is indeed a sum or difference of $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$. Let's find out what that specific sum/difference is:

$$
\begin{aligned}
\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= \left[\hat{\mathbf{B}}\cdot\mathbf{v} + \hat{\mathbf{B}}\wedge\mathbf{v}\right] \\
&= \left[{}^{-}\mathbf{v}\cdot\hat{\mathbf{B}} + \mathbf{v}\wedge\hat{\mathbf{B}}\right]\hat{\mathbf{B}} \\
&= {}^{-}\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}} + \left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}} \\
&= \left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\left({}^{-}\hat{\mathbf{B}}\right) - \left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\left({}^{-}\hat{\mathbf{B}}\right) \\
&= \mathbf{v}_{\parallel} - \mathbf{v}_{\perp}.
\end{aligned}
$$

Substituting this result into Eq. (3.1),

$$
\mathbf{v}' = \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta - \left(\mathbf{v}_{\parallel} - \mathbf{v}_{\perp}\right)\sin^2\frac{\theta}{2}\ .
$$

Now we can see that the terms $\cos^2\frac{\theta}{2}$ and $\sin^2\frac{\theta}{2}$ might be combined per the double-angle formulas (items 8a and 8b) if we write $\mathbf{v}$ as $\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ in the $\cos^2$ term:

$$
\mathbf{v}' = \left(\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}\right)\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta - \left(\mathbf{v}_{\parallel} - \mathbf{v}_{\perp}\right)\sin^2\frac{\theta}{2}\ .
$$

The rest is simple:

$$
\begin{aligned}
\mathbf{v}' &= \mathbf{v}_{\perp}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) + \mathbf{v}_{\parallel}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta \\
&= \mathbf{v}_{\perp} + \mathbf{v}_{\parallel}\cos\theta + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta\ . \qquad (3.2)
\end{aligned}
$$

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).