Multiplical concept

by Dmitrii V. Guryanov

dmitriigur76@gmail.com

Abstract. The purpose of this article is to introduce and to describe a concept of math calculus "Multiplical". To my total surprise I have found that currently such a concept does not exist among set of math definitions in its direct and explicit form. Nevertheless there are number of areas of its practical use, where this concept would be suitable and potentially would be naturally used in its direct and explicit form, especially, in statistics, finance and economy researches and analysis and many other areas. Moreover from my perspective this concept perfectly fits into the coherent system of standard mathematical concepts and operators and should take its rightful place there. In this article also other topics are considered and some interesting conclusions are made.

Keywords: Multiplical, Multiplicand function, Factorial-multiplication, Factorial, Factor-anti-derivative, Factorization, Factor-derivative, Arithmetical function growth, Geometrical function growth, Accelent, Acceleration, Deceleration.

Following related articles:

- Singular properties
- Hyper-operator analysis

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Multiplical

The concept of math calculus "multiplical" has the same type of relation towards the product operator $\prod$ as the concept of math calculus "integral" has it towards the summation operator $\sum$ (as a continuous one has it towards a discrete one) and has the same type of relation towards concept math calculus "integral" as the product operator has it towards the summation operator (as a multiplicative one has it towards a summative one). The definition of the multiplical depends on and is conditioned by its position in the bottom right corner of the following table which could be a puzzle under other circumstances.

Table of concept/operator interrelations

	Discrete	Continuous
Summative	$\sum$	$\int$
Multiplicative	$\prod$	ʅ

Multiplical is an equivalent of product of infinite quantity of infinitively close to 1 (due to infinitively small power) factors which are equal to multiplicand function values raised to the power element of multiplication and is expressed as follows:

$$F^{\bullet}(\mathrm{x}) = \text{ʅ}\, f(\mathrm{x})^{\mathrm{dx}}, \quad (1.1)$$

$$F^{\bullet}(\mathrm{x}) = \bullet\!\int f(\mathrm{x})^{\mathrm{dx}}, \quad (1.2)$$

where ***f*** – multiplicand function; **dx** – element of multiplication; $\mathbf{F}^{\bullet}$ – indefinite multiplical of the ***f*** or factor-anti-derivative of the ***f***; ʅ - primary multiplical sign; **•ʃ** - alternative multiplical sign, used in circumstances of the proper symbol absence, the bullet differs it from the integral sign.

$\mathbf{F}^{\bullet}\mathbf{(x)}$ is called as "multiplical of **f(x)**" or "multiplical of **f(x)** over **x**".

An operation of searching for indefinite multiplical or factor-anti-derivative is called as "factorial-multiplication", a reverse operation of searching for factor-derivative is called as "factorization". The factorization (breaking down into a set of factors) of function is related to the factorial-multiplication of function the same way as the differentiation (breaking down into a set of differences in the sense of a set of increments or addends) of function is related to the integration of function. Those are mutually reverse operations of calculus.

The **function factorial f** represents the relative function change with respect to changes in the function argument or in the element of multiplication and has the following general definition:

$$\mathbf{f}f(\mathrm{x}) = \frac{f(\mathrm{x} + \mathbf{d}\mathrm{x})}{f(\mathrm{x})}. \qquad (2)$$

In accordance to how the whole integrand expression ***f*****(x)dx** is a differential of the anti-derivative **dF(x)**, the whole multiplicand expression $\boldsymbol{f}\mathbf{(x)}^{\mathbf{dx}}$ is a factorial of the factor-anti-derivative $\mathbf{fF}^{\bullet}\mathbf{(x)}$, which is by the way one of the infinite quantity of infinitely close to 1 factors that were mentioned in the multiplical definition, and that's is why the process is called no other way than **factorial-multiplication**:

$$F^{\bullet}(\mathrm{x}) = \bullet\!\int \mathbf{f}\boldsymbol{F}^{\bullet}(\mathbf{x}), \quad (3)$$

On my opinion the concept of factorial is way too great and fundamental to use the term for naming **x!**. Further in the context of this article and by the default the "factorial" term is not used with reference to **x!**.

As well as the integral the multiplical can be in definite and indefinite forms. In accordance to how the **dF(x)** changes its arithmetical sign to opposite when someone does an integration in an opposite to argument growth direction **dx<0** (the accumulating result of integration is not being added but instead subtracted by this differential in the case), **fF•(x)** is also changes to its multiplicative inverse when someone does a factorial-multiplication in an opposite to argument growth direction (the accumulating result of factorial-multiplication is not being multiplied by but instead divided by this factorial in the case):

$$\bullet\int_{x_0}^{x_1} f(x)^{dx} = 1/\bullet\int_{x_1}^{x_0} f(x)^{dx}, \quad (4)$$

where $\mathbf{x_0}$ is the begin of the factorial-multiplication segment; $\mathbf{x_1}$ is the end of the factorial-multiplication segment;

A solution of a definite multiplical can be got as ratio of indefinite multiplical at the ending point to indefinite multiplical at the beginning point of the segment of multiplication respectively:

$$\bullet\int_{x_0}^{x_1} f(x)^{dx} = \frac{F^{\bullet}(x_1)}{F^{\bullet}(x_0)}. \quad (5)$$

Just like an integral of a sum or difference equals to the sum or difference of the integrals, a multiplical of a product or ratio equals to the product or ratio of the multiplicals respectively:

$$\bullet\int (f_1(x) \times f_2(x))^{dx} = \bullet\int f_1(x)^{dx} \times \bullet\int f_2(x)^{dx}, \quad (6.1)$$

$$\bullet\int \left({f_1(x)}/{f_2(x)} \right)^{dx} = {\bullet\int f_1(x)^{dx}}/{\bullet\int f_2(x)^{dx}}, \quad (6.2)$$

Just like a definite integral of a segment equals to the sum of definite integrals of composite segments without gaps and under the condition of one direction of integration and its continuity, a definite multiplical of a segment equals to the product of definite multiplicals of composite segments without gaps and under the condition of one direction of multiplication and its continuity:

$$\bullet\int_{x_0}^{x_2} f(x)^{dx} = \bullet\int_{x_0}^{x_1} f(x)^{dx} \times \bullet\int_{x_1}^{x_2} f(x)^{dx}, \quad (7.1)$$

$$\bullet\int_{x_2}^{x_0} f(x)^{dx} = \bullet\int_{x_1}^{x_0} f(x)^{dx} \times \bullet\int_{x_2}^{x_1} f(x)^{dx}. \quad (7.2)$$

<u>It is forbidden for a multiplicand function to be negative inside of segment of factorial-multiplication</u> by two reasons: firstly, there is an uncertainty in the sign of **fF•(x)** with infinitely small and not necessarily rational exponent **dx**, and secondly, if the sign of **fF•(x)** is nevertheless defined as negative, then it still makes no sense to represent the multiplical as the product of an infinite number of negative multipliers, because then there inevitably arises an uncertainty

in the evenness or oddness of the quantity of these multipliers, and hence the uncertainty of the state of positivity or negativity of the factorial-multiplication result. Multiplicand function modulus has to be submitted for the purpose. For the same reason, there is no designation of the module of the multiplicand function in the record of the multiplicand itself, the entire responsibility for submitting the allowed type of function is on the analyst.

An integration of a constant gives us a linear function or an arithmetical progression; in return a factorial-multiplication of a constant gives us an exponential function or a geometrical progression. Indefinite multiplical of **f(x) = A** is expressed as follows:

$$F^{\bullet}(x) = B \cdot A^{x} \quad (8),$$

where **A** - constant, **B** – non-zero finite arbitrary constant (arbitrary multiplier) that shall be included as multiplier into an indefinite multiplical expression in the correspondence to how an arbitrary constant is included as an addend into an indefinite integral expression. <u>**B** can be a negative</u> which gives us an opportunity to have indefinite multiplical as a function that is below x-axis.

Ranges of arbitrary constants

Arbitrary constant	Unreachable small	Neutral	Unreachable large
Integral arbitrary constant addend **C**	$-\infty$	0	$+\infty$
Absolute value of multiplical arbitrary constant multiplier **B**	0	1	$+\infty$

Multiplical can be expressed via integral, however this expression is indirect and bulky by the definition, it requires some additional operations: raising **e** to power of integral of natural logarithm of multiplicand function:

$$F^{\bullet}(x) = e^{\int \ln(f(x))dx}. \quad (9)$$

To those who thinks the multiplical is a useless and redundant entity I offer equally and without any prejudice to re-consider the reason-ability of the product operator $\prod$ existence because obviously this operator can be expressed via sum operator $\sum$ exactly the same manure, and who knows, maybe it is also redundant according to them. The following expression could seem bulky but principally not bulkier that the indirect multiplical expression that is via the integral, but most importantly, it works:

$$\prod_{i=1}^{N} a_i = sign([a_1; a_2; ..a_N]) \cdot e^{\sum_{i=1}^{N} \ln|a_i|}, \quad (10)$$

where **sign** – a helper function that returns **–1** in case if number of negative multipliers in the passed as argument array is odd, otherwise it returns **+1**. In addition it returns **0** in case if there is at least one zero multiplier in the passed array.

Anyways as a compromise the multiplical can be considered as a shorter version of the above expression with usage of the integral. Personally I consider the shorter version as more intuitive, more primary by its nature, is something that directly reflects the mathematical essence of the conducted operation. On my opinion the multiplical has every right to take its rightful place in the coherent system of standard mathematical concepts and operators. The expression with usage of integral could be considered as an indirect expression that is used in circumstances of lack of the required math apparatus.

In fairness, it should be noted that the indirect multiplical expression gives us the opportunity to analytically describe formulas of indefinite multiplicals for a large number of analytically given functions using existing operators and existing functions.

A direct translation of a multiplical arbitrary constant multiplier **B** to an integral arbitrary constant addend **C** which is used in the indirect multiplical expression $\mathbf{e}^{\int \ln(f(x))dx}$ and the reverse translation of them both are possible via the following equations:

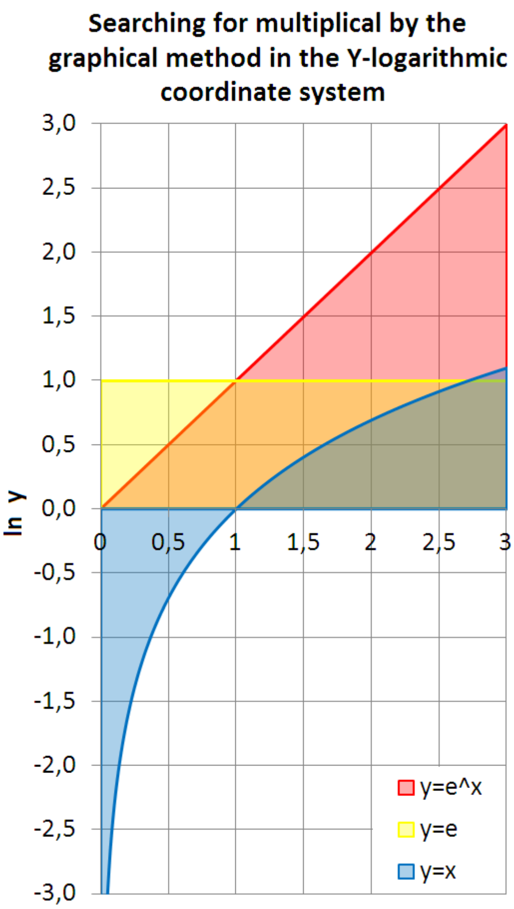
$$C = \ln|B|, \quad (11.1)$$

$$B = \pm e^{C}. \quad (11.2)$$

At $\mathbf{x_0} = \mathbf{x_1}$ a definite multiplical always returns one. This result corresponds to the following conclusion. As a result of the sum operator always represents a certain alteration of 0 the same way a result of the product operator represents a certain alteration of 1, therefore product of a zero quantity of multipliers gives us 1 as a result (without alteration of 1) and which is also being confirmed by equations 5 and 9:

$$\int_{x_0}^{x_1} \ln f(x)dx = 0 \text{ , at } x0 = x1 \quad (12.1)$$

$$F^{\bullet}(x) = e^{\int_{x_0}^{x_1} \ln f(x)dx} = e^0 = 1. \quad (12.2)$$



Just like the definite integral the definite multiplical can be solved graphically (see the diagram). If we build an analyzed function graph in a coordinate system where the Y-axis marked up in units of **ln y** (a natural logarithm of **y**) then if we measure an area of curvilinear trapezoid formed by the function graph in this coordinate system and limited by $\mathbf{x_0}$ and $\mathbf{x_1}$ at the left and at the right respectively, and then if we raise the **e** number to power of this area then we get a value of definite multiplical. In other words a natural logarithm of a function definite multiplical equals to an area of curvilinear trapezoid formed by the analyzed function and limited by $\mathbf{x_0}$ and $\mathbf{x_1}$ at the left and at the right respectively geometrically measured in a Y-axis natural logarithmic coordinate system. And since the multiplical neutral element is 1 (0 in **ln y** units) the measured below that Y coordinate curvilinear trapezoid area shall be counted as negative. The said is confirmed by the indirect multiplical expression with integral usage.

According to the multiplical definition a solution of the definite multiplical is also can be obtained via math limit of product operator:

$$\bullet\int_{x_0}^{x_1} f(x)^{dx} = \lim_{\Delta x \to 0}\Big(\prod_{i=1}^{N} f(x_i)^{\Delta x}\Big), \quad (13.1)$$

$$N = (x_1 - x_0) / \Delta x, \quad (13.2)$$

$$x_i = x_0 + \Delta x \cdot (i - ½). \quad (13.3)$$

For finite values of Δx (the length of an elementary segment) and for somewhat greater practical accuracy, it is proposed to take the values of the function in the middle of an elementary segment, as shown above. A more numerically accurate method is to use in the iterations a geometrical-average value as the multiplier which is received out of pair of multiplicand function values taken at the beginning and at the ending of an elementary segment respectively. Because of presence of two (an even number) close to each other function values as multipliers in this method, the latter provokes making a factorial-multiplication of negative function zones, which is forbidden.

The summation operator and the product operator can be given a general definition of a recursive incremental iterator of the first and second order respectively (according to the hyper-operator order used in the basis). The integral and the multiplical can be given a general definition of a recursive incremental iterator in limit of the first and second order respectively. Also the anti-derivative and the factor-anti-derivative, the derivative and the factor-derivative can be given a general definition of an anti-derivative of the first and second order, and a derivative of the first and second order respectively.

Examples of factor-anti-derivative for known functions

Function	Factor-anti-derivative
0	does not exists
1	B
a	$B \cdot a^x$
$a \cdot x^n$	$B \cdot e^{n \cdot x \cdot (\ln(\sqrt[n]{a} \cdot x) - 1)}$
$a^{n \cdot x}$	$B \cdot a^{(½ \cdot n \cdot x^2)}$
$e↗^x$	$B \cdot e↗^x$

where ↗ - a designation of the operator of power tower with left associative property.

Multiplical usage examples

$$I(t_0, t_1) = \bullet\int_{t_0}^{t_1} (1 + i(t))^{dt}, \quad (16.1)$$

where **t** – time, year; $\mathbf{t_0}$ – control period beginning timestamp, year; t_1 – control period ending timestamp, year; **i(t)** – time function of money inflation or economical growth or interest rate on year basis, u.f.; **I(t₀, t₁)** – factor function of money depreciation or economical growth or exponent debt growth over the control period, u.f.

$$S(t_1) = \bullet \int_0^{t_1} (1 - m(t))^{dt}, \quad (16.2)$$

where **m(t)** – function of year based mortality rate in an elementary group in dependence of the elementary group age, u.f.; t_1 – age, year; $S(t_1)$ – function of expected fraction of survivals out of all born in dependence of age t_1, u.f.

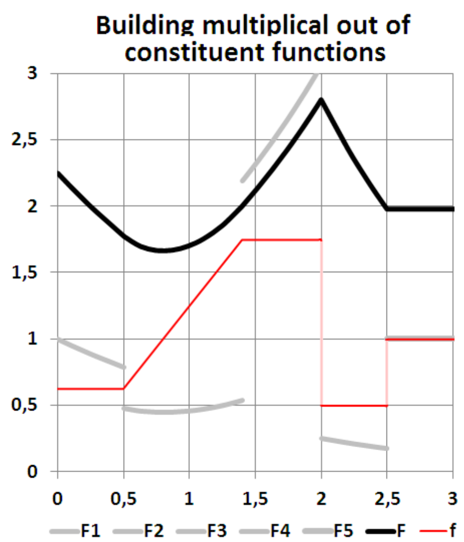
Arbitrary multipliers "B" coordination rule

If an analyzed function is defined via series of functions (further constituent functions) each applied for each argument intervals (function domains) located one just after another being adjacent, in other words if an analyzed functions is defined with interruptions then building its indefinite multiplical implies taking indefinite multiplicals for each of constituent functions in order to use those multiplicals as constituent indefinite multiplicals of the analyzed function indefinite multiplical for respective function domains. Further if the analysis implies building a continuous indefinite multiplical of the analyzed function then a mandatory operation of mutual coordination of arbitrary multipliers B must be conducted, of those arbitrary multipliers which belong to each of constituent indefinite multiplicals.# The coordination of all pairs of adjacent constituent indefinite multiplical arbitrary multipliers must meet the following equation:

$$B_1 \cdot F^{\bullet}{}_1(x) = B_2 \cdot F^{\bullet}{}_2(x), \quad (14)$$

где **0** и **1** - indexes of mutually adjacent the previous and the next constituent indefinite multiplicals and their arbitrary multipliers **B**; **x** - junction point of adjacent the previous and the next constituent indefinite multiplicals under indexes 0 and 1.

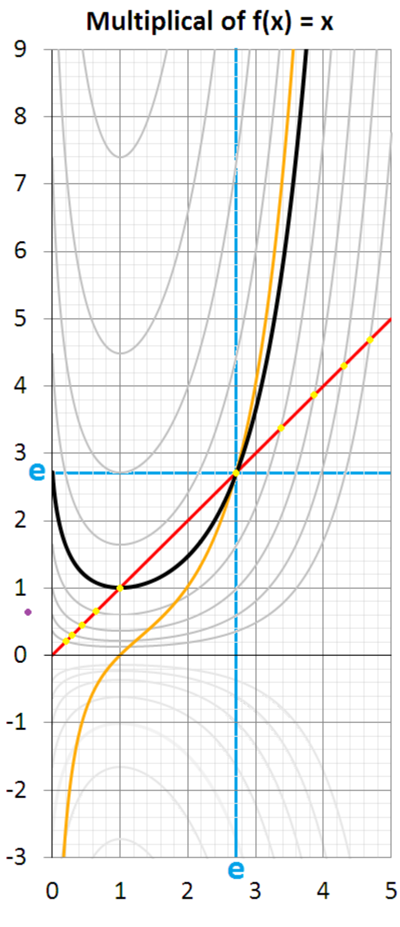
The solution of the above equations is carried out for each junction of the constituent indefinite multiplicals, and sequentially in the order of the values of the argument at the junction points in one of two directions: in the direction of their growth or in the direction of their decrease. Thus, the arbitrary multiplier **B** is determined for each next constituent indefinite multiplical by the already known value for each previous one. The value of the arbitrary multiplier **B** for the first constituent indefinite multiplical in the calculation sequence is set by the analyst.



On the diagram the is an example of arbitrary multipliers coordination. Here the multiplicand function (red) consists of five analytically defined linear functions and for each of them an indefinite multiplical is build (gray). Then a coordination of arbitrary multipliers **B** is carried out in the direction from left to right. So for the first (the leftmost) constituent indefinite multiplical the arbitrary multiplier is set to 2.25, for the second its calculated value is 3.709623, for the third it is 0.914782, for the fourth 11.20608 and for the fifth 1.980973. As the result of the conducted coordination a continuous function of the analyzed function indefinite multiplical is build (black) out of five constituent indefinite multiplicals. A similar procedure must be conducted for indefinite integral arbitrary addends **C** in similar cases.

As it is visible the multiplical has no interruptions of its derivative in points where multiplicand function has interruptions of its derivative (multiplicand function breaking points), because the there is no interruption of multiplicand function as multiplical function factor-derivative. In points, where multiplicand function has interruptions, its multiplical has interruptions of its derivative (multiplical breaking point).

So called "Continuous factorial"



On the diagram there is a series of graphs (in gray) from an infinite set of graphs of the indefinite multiplical of **f(x) = x** (indirectly $\mathbf{e^{x \cdot \ln(x) - x + C}}$) (in red) presented. Each of presented multiplicals differs from the others by its own value of an arbitrary constant multiplier **B**. And for one of them – the one that is lined through point (**x=1,y=1**) (in black) a derivative is drawn (in orange).

For the indefinite multiplical of **f(x) = x** the following remarkable ratio is valid:

$$\frac{F^{\bullet}(0) = F^{\bullet}(e) = F^{\bullet\prime}(e)}{F^{\bullet}(1) = F^{\bullet\prime\prime}(1)} = e, \qquad (17.1)$$

also a property of its derivative is:

$$F^{\bullet\prime}(1) = 0, \qquad (17.2)$$

where $\mathbf{F^{\bullet}(x)}$ – the indefinite multiplical of **f(x) = x**; $\mathbf{F^{\bullet\prime}(x)}$ – the derivative of $\mathbf{F^{\bullet}(x)}$; $\mathbf{F^{\bullet\prime\prime}(x)}$ – the second derivative of $\mathbf{F^{\bullet}(x)}$.

Regarding the indefinite multiplical of **f(x) = x**, a persistent thought does not leave me that this beautiful function may claim to play a role of so called "continuous factorial".

The **Gamma** function shifted one unit left (the **Pi** function) is a generalization for **x!** for real numbers and from my perspective is unsuitable for the role of the so called "continuous factorial". So, the first reference point of **x!** (points where **x=y**), is a point at **x=1**, and the second such point is at **x=2**, while for the indefinite multiplical of **f(x) = x**, in particular expressed by $\mathbf{e^{x \cdot \ln(x) - x + 1}}$, the second such point is at **x=e**, which testifies for an exclusivity of the latter curve comparing to the generalization for **x!**.

It would seem, what relation can **e** number have to building of so called ”continues factorial”. And as it turned out a very direct one, cause as it turned out the process of factorial-multiplication of function **f(x) = x** is one of methods to find the **e**. To confess, the shown on the diagram graphs are built not via the indirect multiplical expression $\pm e^{x \cdot \ln(x) - x + C}$ which I was not thinking of back then, therefore not via already known the **e** number, but in fact via the process of factorial-multiplication of **f(x) = x** using the numerical method staring from point (**x=1,y=1)** and iterating to both possible directions along the x-axis. And what an amazement I experienced when I found the **e** number as **x** tended to zero in the limit. But on the other hand, what is really to wonder here about, what other finite number could be found in the case as any other number would be a new notable math constant by its definition, and finding such a number would cause even greater excitement.

A building generalization for **x!** for real numbers returns us to the fact that initially **x!** is a discrete function and this fact reasonably raises two related to itself questions. The first one is why not to consider the set of multipliers starting not from 1 but from some other real number, for example from 0.5 making the set to look as follows: 0.5, 1.5, 2.5 and etc.? The second one is why the 1 and not any other positive real number is chosen as the set step size. What is so special about the 1 as the set initial point and as the set step size? This perspective makes the generalization for **x!** for real numbers to look as some special not a general function building.

Changing the set step size means changing quantity of multipliers that are effectively used for the function result calculation for a given argument value. In order to preserve sameness of the function result magnitude order for a given argument and since we have a deal with в set of multipliers, a potential set step size change have to be counterbalanced by raising each multiplier of the set to power of the set step size change (increase) multiplicity, in our case of a set step size change relative to 1 as the default set step size. In this context we can write new general definition of **x!** :

$$x! = \prod_{i=0}^{N(x)} (b + s \cdot i)^{S}, \quad (18.1)$$

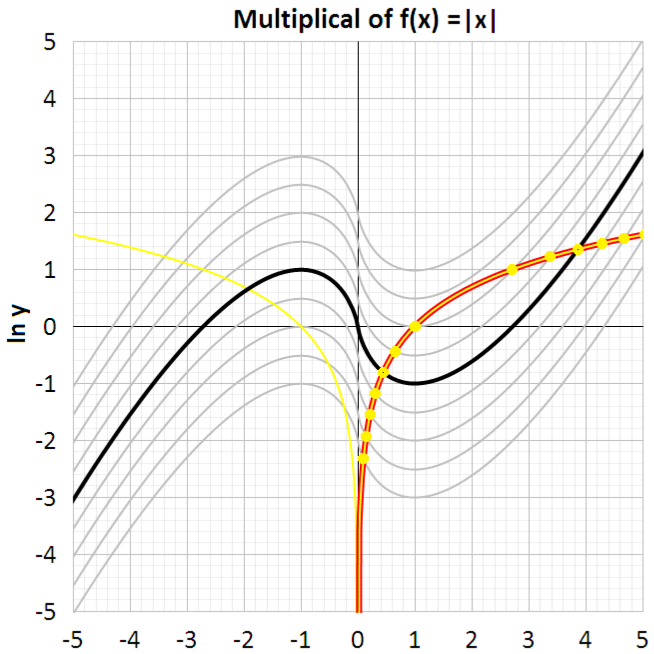
$$N(x) = \text{round}\left(\frac{(x - b)}{s}\right), \quad (18.2)$$

where **b** – the set/function initial point, the default value is 1; **s** – the set step size, the default value is 1; **N** – the product operator iterations quantity excluding the zero iteration.

Examples of the set: $0.5 \cdot 2.5^2 \cdot 4.5^2 \cdot 6.5^2 \cdot 8.5^2 \cdot 10.5^2$ and etc.; $1.5 \cdot 1.6^{0.1} \cdot 1.7^{0.1} \cdot 1.8^{0.1} \cdot 1.9^{0.1} \cdot 2.0^{0.1}$ and etc.; $1 \cdot 2^1 \cdot 3^1 \cdot 4^1 \cdot 5^1 \cdot 6^1$ and etc. From the described point of view the last set of multipliers – the one used in the initial version of **x!**, is the default one but the same time seems to be a specific building from a plenty of possible.

If we start to gradually reduce the set step size **s** then at the each next factorial-multiplication iteration the function value for a certain argument would be closer and closer to its value measured at the previous state of the set step size. Reducing the set step size to zero in the limit technically means replacing the defined above function with multiplical of **f(x) = x**. First of all this measure gives us the desired function continuity and also makes a function value for a certain argument to be indifferent to the set multipliers quantity or to the set step size in the limit, makes it to tend to its determined value in the limit. Therefore this defines locus of points for all its allowed arguments function values within a general determined position which depends only on the function initial point **b**. In fact each function initial point of its own infinite set defines one function locus of points of its own infinite set. Each of those function locus of points differs from the others in the set by its own indefinite multiplical arbitrary constant multiplier **B** and for all of them the described above remarkable equal to **e** ratio is valid. In reverse, a definite multiplical arbitrary constant multiplier **B** causes an existence of up to two possible function initial points **b**.

So as it is visible on the diagram it would be possible to draw a multiplical graph starting it from any point (except point **x=0**) of the **f(x) = x** graph. Yellow points represent samples of such initial points from which multiplical functions are drawn to the left direction. Also we can witness that the solution for initial points does not exist for all arbitrary multiplier constants **B** as some of multiplical graphs don't have intersection points with the **f(x) = x** graph.



In the second diagram in a naturally logarithmic along the y-axis coordinate system, a series of graphs of the indefinite multiplical (in gray in general and black at **B=0**) of the function **f(x) = x** (red) module (yellow) for various values of an arbitrary multiplier **B** is shown. Yellow shows the graph of the modulus of **f(x) = x**.

Geometrical function growth

A function derivative shows a function growth in a point. But as it turned out the function growth can be a different kind. Therefore it should be clarified that the described kind of growth is arithmetical as it shows how much the function will grow absolutely if the function argument will grow by one and if this growth will be constant within the argument growth. Graphically this is solved by drawing a tangent to the graph at the given point, more precisely, not just a tangent, not just a straight tangent, but a linear function graph that is tangent to the function graph at the given point and which is expressed by following general equation:

$$y = b \cdot x + c, \quad (37)$$

where **b** and **c** – constants of the tangent linear function, which determination gives it a tangency to the function graph at given point.

b numerically shows the absolute function growth with the grows of the argument by 1, therefore shows the function arithmetical growth at given point. The tangent of the slope reflects the arithmetical function growth.

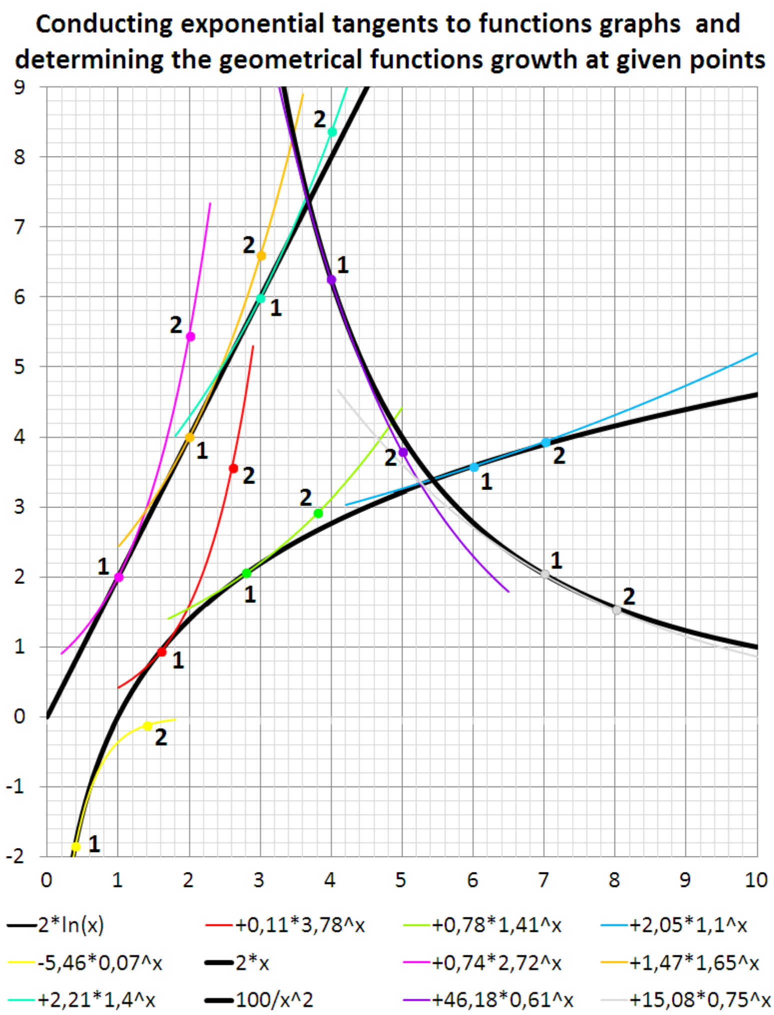
The mentioned above factor-derivative shows a function geometrical growth in a point as it shows how many times the function will grow if the argument will grow by one and if this growth will be constant within the argument growth, therefore it shows the relative function growth. Graphically this is solved by drawing an exponential function graph that is tangent to the function graph at the given point and which is expressed by following general equation:

$$y = b \cdot a^{x}, \quad (38.1)$$

$$a = e^{\left(\frac{f\prime(x)}{f(x)}\right)}, \quad (38.2)$$

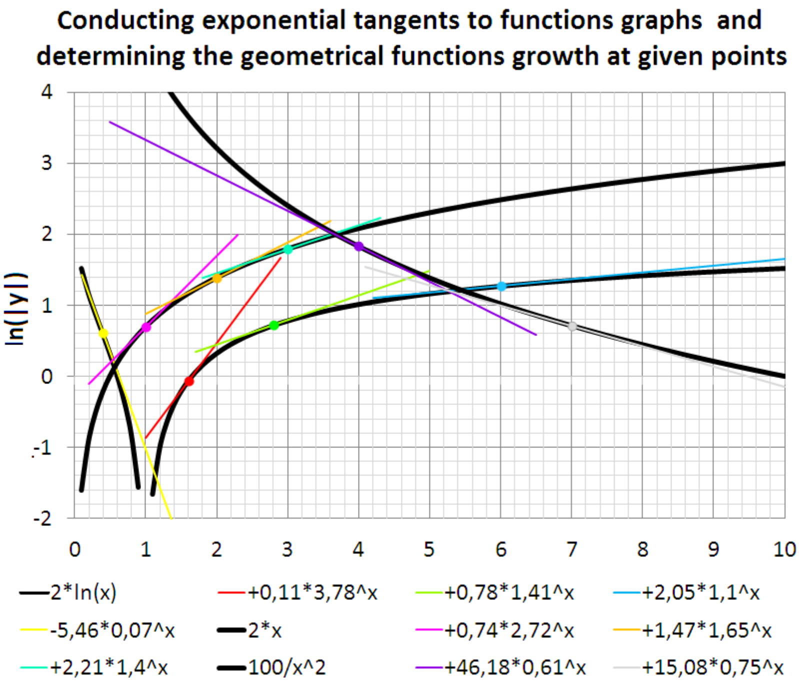
$$b = \frac{f(x)}{|f(x)|} \cdot a^{\left(\frac{\ln|f(x)|}{\ln a}\right) - x}, \quad (38.3)$$

where **a(x)** and **b(x)** – constants of the tangent exponential function, power base and multiplier respectively, which values provide it with a tangency to the analyzed function graph at given point; **f(x)** – the analyzed function; **f'(x)** – the analyzed function derivative.



a numerically shows the relative function growth with the grows of the argument by 1, therefore shows the function geometrical growth at the given point. <u>Factor-derivative can not be negative</u>.

On the diagram there is a number of tangent exponential functions graphs (in colors) drawn to 3 analyzed functions (in black): **f(x) = 2·ln(x), f(x) = 2·x and f(x) = 100/x²**. Here the absolute x-axis difference between the 2nd and the 1st points of the same color reflects a growth of function argument by 1. The same time the relative Y-axis difference between the 2nd and the 1st points of the respective pair or how many times point 2 is located farther from the x-axis than point 1 to indicates a geometrical growth value of the respective analyzed function at the point of contact (the 1st point). It is obvious that through one pair of points 1 and 2 one can draw one exponential graph of $y = b \cdot a^x$.



Building the all above functions and their respective tangent exponential functions in a coordinate system where the Y-axis marked up in units of ln |y| (a natural logarithm of absolute value of **y**) visually degrades all these graphs as follows: linear to logarithmic, exponential to linear, and respectively visually degrade the function geometrical growth to the function arithmetical growth. The **e** number raised to power of the tangent of the slope of the degraded to a line tangent exponential graph reflects the geometrical function growth at the point of contact. In other words a natural logarithm of an analyzed function geometrical growth at given point equals to the tangent of the slope of the tangent exponential function drawn through the given point of the analyzed function in a Y-axis natural logarithmic coordinate system.

In points where function crosses the x-axis its factor-derivative has interruptions. For example it is visible that the factor-derivative of **y=2·ln(x)** has an interruption at **x=1** tending to **0** ($e^{-\infty}$) and to + ∞ ($e^{+\infty}$) as **x** approaches to the point from the left and from the right respectively.

Like searching for a derivative implies omitting a common constant addend, searching for a factor-derivative implies omitting a common constant multiplier.

Examples of factor-derivative of know functions

Function	Factor-derivative
0	uncertainty
B	1
$b \cdot x^a$	$e^{a/x}$
$b \cdot a^x$	a
$b \cdot e\nearrow^x$	$e\nearrow^x$

where ↗ - a designation of the operator of power tower with left associative property.

Those who wish can practice in searching for factor-derivatives and factor-anti-derivative of known functions.

Through the analyzed function there is interdependence between the function derivative (arithmetical growth) and the function factor-derivative (geometrical growth):

$$f^{\bullet}(x) = e^{\frac{f'(x)}{f(x)}}, \quad (39.1)$$

$$f'(x) = f(x) \cdot \ln f^{\bullet}(x), \quad (39.2)$$

if $f(x) \neq 0$ and $f^{\bullet}(x) \neq 0$ and $f(x) \neq \infty$ and $f^{\bullet}(x) \neq \infty$ and $f'(x) \neq \infty$

where ***f*(x)** – the analyzed function; ***f*•(x)** – the analyzed function factor-derivative; ***f*′(x)** – the analyzed function derivative.

It is obvious that it is not possible to restore a function by its known derivative or factor-derivative solely, but it is possible to restore a function if both are known together and the factor-derivative is not equal to 0 or to 1 and has finite value:

$$f(x) = \frac{f'(x)}{\ln f^{\bullet}(x)}$$

if $f(x) \neq 0$ and $f^{\bullet}(x) \neq 0$ and $f^{\bullet}(x) \neq 1$ and $f(x) \neq \infty$ and $f^{\bullet}(x) \neq \infty$ and $f'(x) \neq \infty$. (39.3)

The same way as function derivative can be expressed via function differential: ***f*′(x) = d*f*(x) / dx**, function factor-derivative can be expressed via function factorial, using there not the division operator but rightfully the root extracting operator which is an operator of one hyper-operator order higher than the division operator:

$$f^{\bullet}(x) = \sqrt[dx]{\mathbf{f}f(x)}\,. \quad (40)$$

And therefore vice versa function factorial can be expressed via function factor-derivative, which is similar to how function differential can be expressed via function derivative: **d*f*(x) = *f*′(x) · dx**, but again here not using the multiplication operator but the exponentiation operator instead which is also an operator of one hyper-operator order higher than multiplication operator:

$$\mathbf{f}f(x) = f^{\bullet}(x)^{dx} \quad (41)$$

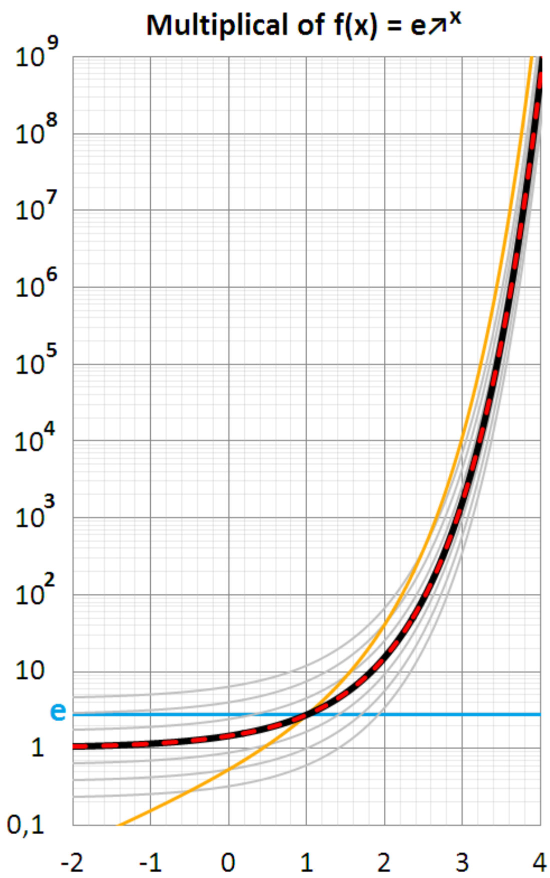
Accelent

I set a task to find and formulate a function whose growth or, in other words, whose factor-derivative, and automatically, therefore, the multiplical would be equal to the function itself for all values of argument. The first such function suggests itself, and this is the function **y = 1**, which exists in accordance with the existence of the function **y = 0** for the problem of finding a derivative and an integral equal to the function itself, where 1 and 0 are obviously the values of neutral arbitrary constants for the multiplical and integral respectively. Also, by analogy with the search for a derivative, the second function with a similar property should presumably be exponential and the same way as **y = e^{x}** function does it should have as its basis the **e** number, and as we already know the **e** number is really directly related to the factorial-multiplication and factorization of functions.

As you know, the integration and differentiation operations use the addition and subtraction operators, that is, binary operators of the first order, on the other hand the factorial-multiplication and factorization operations use the multiplication and division operations, that is, binary operators of the second order. Exponentiation function $\mathbf{y = e^x}$ is a single action with the power operator, that is, with a binary operator of the third order, and also has the **e** number as the first operand and the argument **x** as the second. You can notice that this function uses a binary operator standing two orders higher than the operators used in the integration and differentiation operations. Then we do a bold guess that the function we are looking for exactly copies the function $\mathbf{y = e^x}$ in part of operands and as $\mathbf{y = e^x}$ is a single action with a binary operator standing two orders higher than the operators used in the factorial-multiplication and factorization operations, that is an action with some binary operator of the 4^{th} order. Therefore this is a hyper-exponentiation function of the 4^{th} order with base **e**. We write this function and, in particular, the proposed operator as follows:

$$y = e{\nearrow}^{x}, \quad (43)$$

where ↗ - the designation of the operator of the 4^{th} order. The operator is called by its own name – Acceleration; super-scripted **x** – accelerator, **e** – accelerating; **y** – acceleration. The expression can be read as “the **e** number in acceleration of **x**”, “**e** number in **x**-th acceleration”, “**e** number accelerated **x** times”, ”accelerated **x** times **e** number”, ”acceleration of **e** number in **x**”. The curve $\boldsymbol{y = a{\nearrow}^{x}}$ is called as “Accelent”.



My guess is correct, as it turned out $\mathbf{y = e{\nearrow}^{x}}$ meets the requirements only if the Acceleration is a power tower but with left associative property. It should not be confused with the Tetration that is the power tower with right associative property.

On the diagram there are an accelent function $\mathbf{y=e{\nearrow}^{x}}$ (lined in red dotted line) and a set of its indefinite multiplicals (in gray). A graph of one indefinite multiplical – the one with arbitrary multiplier constant equal to 1 (in black) is the same as the graph of the accelent function, and that’s why I had to display the latter in dotted line. You can notice that a Y-logarithmic coordinate system visually degrades the accelent a one order down to visual exponent which is also shifted to the right by 1 on the x-axis.

Specially noted that the acceleration operator is obtained by the necessity of formulating an analog of $\mathbf{e^x}$ for the operations of factorial-multiplication and factorization, obtained not by the method of extrapolation but rather by method of shifting a series of 3 consecutive hyper-operators one order up in the intended direction of mathematical analysis, similar to how bridge spans are pushed during the construction of the latter. The conducted analysis ordained the inner logic of this operator of the 4th order and not vice a versa when we have an operator and then we try to apply it.

Illustration of the hyper-operator series shifting

	Addition	Multiplication	Raising to power	Acceleration
The initial state	Summation and Integration	Operation with the element of integration; a linear equation of the function arithmetical growth tangent	The function arithmetical growth is same as the function itself e^x	
Shifting one order up (rightward)		Product and Factorial-multiplication	Operation with the element of multiplication; an exponent equation of the function geometrical growth tangent	The function geometrical growth is the same as the function itself $e\nearrow^x$

The acceleration can be expressed compactly via operators of lower order:

$$a \nearrow^{\mathrm{n}} = a^{\left(a^{(\mathrm{n}-1)}\right)}. \quad (44)$$

A hyper-root of the 4th order is called as “Deceleration” and designated as follows:

$$a \searrow_n, \quad (45)$$

where ↘ - designation of the operator; subscribed **n** – decelerator; **a** – decelerating. The operation result is “deceleration”. The expression can be read as: “**a** in deceleration of **n**”, ”**a** in **n-th** deceleration”, ”**n-th** deceleration of **a**”, ”**a** decelerated **n** times”, ”decelerated **n** times **a**”, ”deceleration of **a** in **n**”.

The deceleration shows a number, which has to be raised to power of itself 1 times less than the decelerator value in order the decelerating to be obtained and applying the left associative property in the raising to power sequence for the purpose.

The deceleration is solved recurrently using root operator (a inverse binary operator one order lower) and also compactly:

$$a \searrow_n = \sqrt[\left((a\searrow_n)^{(n-1)}\right)]{a}, \quad (46)$$

There is nothing out of the ordinary, as extracting the root is also solved recurrently in the exactly same scheme, but what is distinctive and natural, using the division operator (a inverse binary operator one order lower):

$$\sqrt[n]{a} = \frac{a}{\left(\sqrt[n]{a}\right)^{(n-1)}}. \quad (47)$$

A consequence of a common solution scheme for the two inverse hyper-operators and similarity to how it is forbidden to submit a lower than 0 value to the radical expression, it is forbidden to submit a lower than 1 value as the decelerating. Recalling to how the complex numbers set was defined at the time, it is becoming to be interesting what set of numbers could be defined using decelerating less than 1.

The logarithm of the 4th order with own name “Accelerator extraction” and “Natural accelerator extraction” is defined as nested logarithm:

$$c \nwarrow_a = \log_a\left(\log_a c\right) + 1, \quad (48.1)$$

$$c \nwarrow = \ln\left(\ln c\right) + 1, \quad (48.2)$$

Factorial-multiplication for zero function values

Of factorial analytical interest is the process of factorial-multiplication of the modules of functions that intersect the x-axis in the vicinity of a given intersection (hereinafter, the zero point).

In a neighborhood of the zero point, the intersecting function can be approximately represented as a polynomial:

$$y = b_1 \cdot (x - c)^1 + b_2 \cdot \mathrm{sign}(x - c) \cdot (x - c)^2 + b_3 \cdot (x - c)^3 + \ldots + b_n \cdot \mathrm{sign}(x - c) \cdot |x - c|^n, \quad (15.1)$$

where $\mathbf{b_1}$,$\mathbf{b_2}$,$\mathbf{b_3}$, $\mathbf{b_n}$ are the multipliers of the polynomial; **c** – zero point X coordinate; **sign** - a function that returns **–1** if the passed argument is negative, **+1** otherwise.

In the infinitely close approximation to the zero point, the polynomial function can be reduced to a function of one term - the first from the left, which is with a non-zero factor b. Considering the exact position of the zero point on the x-axis as not important in the case in order to shorten the notation we will take C = 0 (the zero point is located at the point of origin). After simplification, the equation has the general look of a power function dependent on the arithmetic sign of the argument:

$$y = b_n \cdot \mathrm{sign}(x) \cdot |x|^n \text{ if } n > 0, \quad (15.2)$$

Since it is allowed to carry out factorial-multiplication of only positive functions domains, we transform the function up to its module, and we obtain a multiplicand function allowed for factorial-multiplication (hereinafter, the multiplicand). In addition, this transformation describes the case of not crossing the x-axis functions, but touching it. Thus, the present analysis covers all possible cases of contact between the graph of the function and the x-axis at one point:

$$y = |b_n| \cdot |x|^n. \quad (15.3)$$

Further, the multiplicand is divided into two domains: the one to the left and the one to the right of the zero point, represented by the following constituent functions applicable under appropriate conditions:

$$y = |x|^n \cdot b, \quad \text{at } b_n \cdot x \geq 0, \quad (15.4)$$

$$y = -|x|^n \cdot b, \quad \text{at } b_n \cdot x \le 0, \qquad (15.5)$$

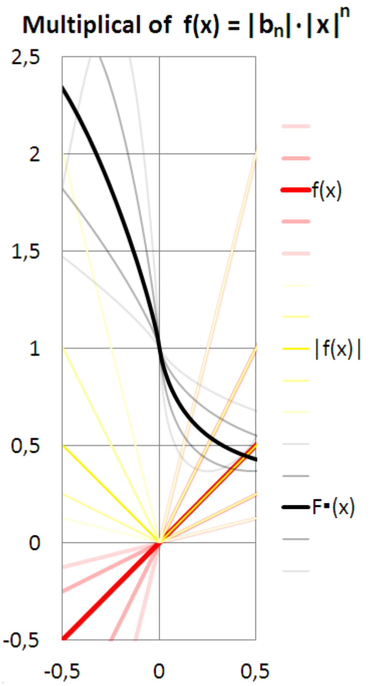
Indefinite multiplical for each of constituent functions:

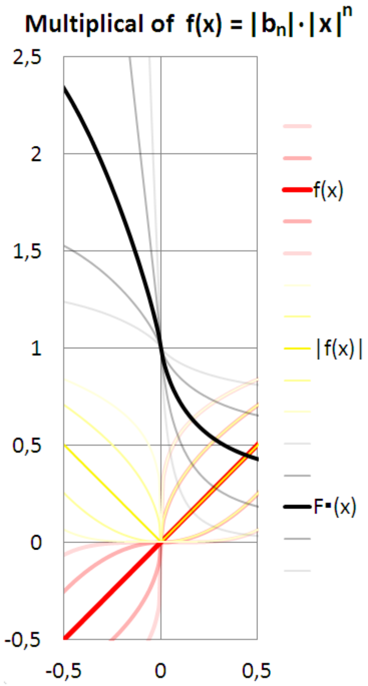
- $\int (|x|^n \cdot b)^{dx} = B \cdot e^{n \cdot x \cdot \left(\ln\left(x \cdot \text{sign}(b) \cdot \sqrt[n]{|b|}\right) - 1\right)}, \ \text{at } b_n \cdot x > 0, \qquad (15.6)$
- $\int (-|x|^n \cdot b)^{dx} = B \cdot e^{n \cdot x \cdot \left(\ln\left(-x \cdot \text{sign}(b) \cdot \sqrt[n]{|b|}\right) - 1\right)}, \ \text{at } b_n \cdot x < 0, \qquad (15.7)$

Since both of these indefinite multiplicals do not exist at the zero point, we determine their values when approaching this point infinitely close from the left and from the right separately using the previous equations:

$$\lim_{x \to 0} B \cdot e^{n \cdot x \cdot \left(\ln\left(x \cdot \text{sign}(b) \cdot \sqrt[n]{|b|}\right) - 1\right)} = B, \qquad \text{at } (x \cdot b) \ge 0, \qquad (15.8)$$

$$\lim_{x \to 0} B \cdot e^{n \cdot x \cdot \left(\ln\left(-x \cdot \text{sign}(b) \cdot \sqrt[n]{|b|}\right) - 1\right)} = B, \qquad \text{at } (x \cdot b) \le 0, \qquad (15.9)$$





On the first diagram there are presented: function **$y = b_n \cdot \text{sign}(x) \cdot |x|^n$** at **n=1** and different **b_n** (from 0.25 to 4 in shades of red and at **b_n=1** in bright red), function module: **$y = |b_n| \cdot |x|^n$** (in shades of yellow and at **b_n=1** in bright yellow), indefinite multiplical of function module at **B=1** и different **b_n** (in shades of gray and at **b_n=1** in black).

On the second diagram there are presented: function **$y = b_n \cdot \text{sign}(x) \cdot |x|^n$** at **$b_n$=1** and different **n** (from 0.25 to 4 in shades of red and at **n=1** in bright red), function module: **$y = |b_n| \cdot |x|^n$** (in shades of yellow and at **n=1** in bright yellow), indefinite multiplical of function module at **B=1** и different **n** (in shades of gray and at **n=1** in black).

It is easy to see that when approaching the zero point, the indefinite multiplical of the multiplicand (hereinafter the multiplical) tends to some finite value on both sides of this point, and moreover, on both sides it tends to the same value which depends only on the single for the two constituent multiplicals, the values of an arbitrary multiplier **B**, and, what is remarkable, does not depend in any way on **b_n** and on **n**, that is, on the values of all derivatives of the multiplicand. In the vicinity of the zero point, the constituent multiplicals of the two multiplicand domains are located at the junction with each other. At the same time, the two domains are still separated by the zero point, where there is an interruption of the constituent multiplicals, and possibly an interruption of the multiplical, but the factorial-multiplication operation bypassed this point, leaving the question open.

It would seem that in the process of factorial-multiplication, the transition through this point of the multiplicand, which has zero value, as through a factor for the intermediate result of factorial-multiplication (hereinafter referred to as the intermediate result) should nullify this result and automatically slam to zero the entire domain of the multiplical located to the right of the zero point, thereby completely making the factorial-multiplication process meaningless to the right of this point, assuming that we do operation in the direction of growth of the argument. This statement would be just if the zero would be a zero as a multiplier.

But as you know, when we carry out factorial-multiplication, we divide the linear segment of factorial-multiplication along the x-axis into infinitely small, but not zero length segments **dx**. From this position, the problem of a dimensionless point and at the same time the solution of the problem for us is the fact that the point, being a dimensionless quantity, leaves a projection of zero length on the x-axis.

One so-called geometric approach comes to the following simple conclusion. Since we are factorial-multiplicating along the linear continuum of the x-axis, influencing the state of the function by something that also has a dimension related to the linear dimensions of the x-axis leaves its mark there. And if at the same time something leaves a projection of zero length on the axis, then this something, no matter what it is, effectively leaves absolutely nothing, it simply does not exist for the operation being performed. And assuming that the function on both sides tends to the finite and to the same identical value, this point can be ignored, the function is "glued", and the continuity of the latter is stated.

Another so-called abstract approach does not ignore the state of the function at a point, and implies a transition from the analysis of the properties of functions in extremely small linear segments to the analysis of their properties at dimensionless points. This approach involves finding the “connecting” multiplier (hereinafter referred to as the multiplier) at a dimensionless point, when passing through which, in the process of factorial-multiplication, the intermediate result is multiplied by this multiplier or divided by it, depending on the direction of factorial-multiplication: in the direction of increasing or decreasing the argument, respectively. Thus, the "fate" of the factorial-multiplication being carried out depends on the state and value of this multiplier.

It is obvious that the size of the multiplier at the zero point is equal to the value of the multiplicand measured at the zero point, that is, zero which is raised to power of the element of multiplication, the size of which also has a zero value. Obviously, in this case we are not talking about the differential of the argument **dx**, since one with a zero length does not make sense, but no one has canceled the multiplication element that is necessarily applied as power to the value of the multiplicand, even if it has a zero value. Thus, the problem of determining the value of the multiplier is reduced to determining the result of $\mathbf{0^0}$.

The exponentiation operator is hyper-operator, namely, the product operator $\prod$ with the number of iterations corresponding to the exponent value. In our case, the result of the operator $\prod$ is a multiplier for the intermediate result. On the one hand, it can be assumed that the result of $\mathbf{0^0}$ is nothing, an uncertainty, since there are no factors at all, even if those factors are zeros. In this case, there is no operation of multiplication by zero at all, since the zeros themselves are absent. Given this, it is already clear that the multiplical will not turn into zero. But the multiplical can cease to exist as a result of multiplication by uncertainty. On the other hand, if $\prod$ has a zero number of multipliers, as in our case, then it is obviously inactive, does not iterate and does not increment anything, and if so, then it must leave unchanged the result of which it is a multiplier, pass through itself its value in transit, but does not destroy it.

It turns out that the expectation of the result of raising to a power depends on our idea of the operator's function and logic, on its formal definition, which in turn is determined by the context of its application. Above, the definition of the product operator was given as a recursive incremental iterator, which implies a certain initial state of the result of the operator, in relation to which an increment is made starting from the first iteration. The math analysis implies the transit without change in the case of operator inactivity, which is in full accordance with how the summation operator $\sum$ does not return uncertainty in the absence of addends, in the absence of iterations, in its inaction, but returns 0 as a neutral value, thereby does not change the result of the previous summation, but passes it through itself in transit.

So, the only multiplier that leaves the result of multiplication unchanged is 1. And this means that $\prod$ with a zero number of multipliers, regardless of their possible state (including the state of uncertainty) and value, must return 1 as a result of its inaction, as a neutral value. Thus, it can be said that the desired multiplier at the zero point is equal to 1. For your information, one of coming next related articles is about the hyper-operator analysis, where the topic of neutral elements (values) of hyper-operators is touched upon.

We single out the zero point itself and its infinitely small neighborhood on both sides of it into a separate third domain of the multiplicand, the one that is not included in the first two. Let's designate it as [0;0] from zero inclusive to zero inclusive. Next, it is necessary to make an assumption that the finite value of the multiplier at the singularity point, in our case at the zero point, can be extended to an infinitesimal neighborhood of this point towards that boundary on which the multiplical has finite and the same value as well as at the zero point. The statement is based on the assumption that within the considered infinitesimal interval the function is monotone under the given conditions, it simply has nowhere to go, there are no known reasons for a different behavior of the function. In our case, to the right of the zero point, to the left of it, the constituent multiplicals of the first and second domains tend to 1 (finite value) when approaching it from both sides, therefore, the value of the multiplier at the zero point extends to the entire infinitesimal neighborhood of the zero point, that is, over the entire previously defined third domain of the multiplicand.

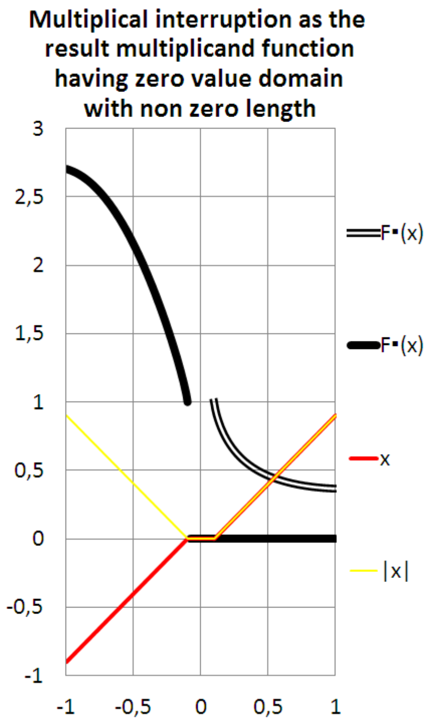
By extending the value of the multiplier to an infinitely small neighborhood, one should not understand the construction of a monotonic function with a constant value within this interval equal to the value of the multiplier, but it should be understood that during factorial-multiplication, the intermediate result at the input to this neighborhood is multiplied or divided by this multiplier, depending on the direction factorial-multiplication, then it is passed to the output from the given neighborhood.

If the extension of the value of the function is possible only in one of the two directions of the neighborhood, then it makes no sense to say that the multiplier must be divided into two multipliers, each of which is equal to square root of its original value, since the impossibility of spreading the value of the multiplier in both directions for one singularity point indicates the presence of an interruption there, which in turn makes senseless analytical work to find the continuity of the function at the singularity point. In this regard, for the sake of simplicity, speaking of the multiplier, we can omit the mention of the neighborhood of the point, and consider the multiplier as a property of the point, and the domain under study, as the domain of the point. From this perspective, the question of how to extend the value of the multiplier to the neighborhood of the point disappears, since the intermediate result will be multiplied or divided by the multiplier once when passing through the singularity point.

When building the multiplical of the function modulus, the result of factorial-multiplicating the third abstract domain of the multiplicand requires matching its arbitrary multiplier with arbitrary multipliers and two other constituent definite multiplicals. Next, we “glue” all the constituent multiplicals of all three domains of the multiplicand and obtain a continuous indefinite multiplical of the analyzed function.

Evidence of the identity of the multiplier to 1 at the zero point is also the identity of the results of factorial-multiplication obtained through two different approaches: through the so-called geometric approach, and through the so-called abstract approach with an analysis of the properties of functions at dimensionless points.

It can be argued that in the process of factorial-multiplication, the transition through the zero value of the multiplicand, due to the intersection of the function and x-axis or the contact of the x-axis at one point, does not lead to any changes in the intermediate result. Even a possible break in the function (Interruption of the first derivative) does not affect this result, since the value of the multiplical at the zero point does not depend on $\mathbf{b_n}$. But what is remarkable is that a function break at a singularity point, such as zero, leads to a break in the multiplical at this point, but obviously not to a break in its factor-derivative (multiplicand), and that is not observed when the function breaks outside singularity points, where only the second the derivative of the multiplical is interrupted. It can be concluded that the behavior of the derivative and factor-derivative is different at singularity points, where the interdependence between them is violated.



And now let's consider a slightly different case, specifically, one where the multiplicand has a zero value over an interval that is not zero in length. So, inside this interval, the intermediate result is multiplied by the multiplier $\mathbf{0^{dx}}$ equal to zero, since **dx** in this case is infinitely small, but not a zero value.

In this case, the multiplical "collapses" to zero to the right of the entry point of the multiplicand into the horizontal segment with zero value. Further, the multiplical is not restored, even despite the subsequent “dawn” of the multiplicand to non-zero values, because anything (the intermediate result) once multiplied by zero then gives only zero as a result (the solid black graph on the diagram). In this regard, we can say that the point where the indefinite multiplical touches the x-axis (we are not talking about an infinitely close approximation of the x-axis) means, in fact, the point of its interruption. If factorial-multiplication is carried out in the direction opposite to the direction of growth of the argument, then at the entry point on the left into the interval with a zero value of the multiplicand, the intermediate result will cease to exist as a result of an attempt to perform the operation of division by zero, which is also an interruption point for the multiplical, after which, to the left, it will not recover either (hollow black graph in the diagram). Thus, for the horizontal interval of the multiplicand with zero value, there are two points at which the multiplical is interrupted, these are the points of the beginning and end of this interval. The interval itself is the domain of uncertainty of the multiplical. It can also be argued that the multiplical of the function **y=0** is not **y=0**, but it simply does not exist, since the entire domain of the multiplicand from $-\infty$ to $+\infty$ is the domain of uncertainty of its multiplical.

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