A method for studying the Anderson transition in the orthogonal symmetry class employing a random walk expansion, the statistics of asymptotic walks and summation method

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May 28, 2022

Abstract

I propose a method to study the Anderson transition in the orthogonal symmetry class. This method employs a virtual lattice characterised by an arbitrary spectral dimension instead of a concrete lattice with a given integer or fractal dimension. This method makes it possible to simulate numerically infinite size system on a computer. Moreover, the computational complexity does not increase exponentially as the dimensionality increases. Thus, we can avoid the curse of dimensionality. Also, we can estimate the critical exponent numerically without resorting to the finite size scaling method often used in previous numerical studies of critical phenomena.

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1 Introduction

The Anderson transition is a disorder driven quantum phase transition exhibiting critical phenomena. The critical exponent is thought to depend only on fundamental properties of the system such as dimensionality and symmetry. In common with many studies of critical phenomena, numerical study of the Anderson transition often uses finite size scaling. Finite size scaling makes it possible to estimate the critical exponent from simulations of finite size system performed on a computer. However, such studies require exponential numerical complexity as the dimensionality of the system increases. Another problem is what kind of dimensionality is relevant for the Anderson transition such as fractal dimension or spectral dimension[1]. Also, to study the Anderson transition for non-integer dimensionality, it has been necessary to prepare a suitable fractal lattice. Besides it, there does not exist the theory of the critical phenomena of the Anderson transition without qualitative problem[2]. In this paper, I propose a method to study the Anderson transition that addresses these problems.

2 Anderson's model of localisation and random walks

In this section, I review some basic knowledge needed later in this paper. I describe Anderson's model of localisation, the characterisation of a lattice by the spectral dimension of a random walk, and the random walk expansion of the Green's function.

2.1 Anderson's model of localisation

The Hamiltonian of Anderson's model of localisation is

$$H = \sum_x U(x)|x\rangle\langle x| + \sum_{x,y} V(x,y)|x\rangle\langle y| \tag{1}$$

$$V(x,y) = \begin{cases} h & (x,y \text{ are nearest neighbours}) \\ 0 & (\text{otherwise}) \end{cases} \tag{2}$$

Here, x and y are sites of a given lattice or graph. The site energies U are random variables that are independently and identically distributed according to the box distribution,

$$p(U) = \begin{cases} 1/W & (|U| \leq W/2) \\ 0 & (\text{otherwise}) \end{cases} \tag{3}$$

The strength of disorder is controlled by the parameter W.

2.2 Asymptotic behaviour of random walk and spectral dimension

On a given general lattice, including fractal lattice, the probability of a random walk starting from site x and ending at site y after t steps is, for long times, given by

$$P_{xy}(t) \sim t^{-d/2} \tag{4}$$

Here, d is the spectral dimension of the lattice.

2.3 Random walk expansion of the Green's function of Anderson's model of localisation

The Green's function of Anderson's model of localisation is defined by

$$G(E) = (H - E)^{-1}. \tag{5}$$

For two arbitrary sites x and y, the Green's function between these sites can be expressed by a random walk expansion[3],

$$\begin{aligned} G(x, y; E) \quad &= \quad \sum_{\gamma: x \mapsto y} V(\gamma(0), \gamma(1)) V(\gamma(1), \gamma(2)) \cdots V(\gamma(|\gamma| - 1), \gamma(|\gamma|) \\ &(-1)^{|\gamma|} \quad \prod_{k=0}^{|\gamma|} \frac{1}{U(\gamma(k) - E)} \end{aligned} \tag{6}$$

Here γ is a finite path that starts at site x and ends at site y. The number of steps in the path is denoted by $|\gamma|$. The sum is over all paths connecting sites x and y. For Anderson's model of localisation, the hopping term is restricted to nearest neighbour sites. The expansion is then simplified

$$G(x, y; E) = \sum_{t=0}^{\infty} h^t \sum_{\gamma \in \Gamma_t(x,y)} \prod_{k=0}^{t} \frac{1}{U(\gamma(k)) - E} \tag{7}$$

Here, $\Gamma_t(x, y)$ is the set of paths which starts at site x and ends at site y with number of steps $|\gamma| = t$. We rewrite this equation in terms of the number of visits,

$$G(x, y; E) = \sum_{t=0}^{\infty} h^t \sum_{\gamma \in \Gamma_t(x,y)} \prod_{\alpha} [U(\alpha) - E]^{-\#(\gamma,\alpha)} \tag{8}$$

Here, $\#(\gamma, \alpha)$ is the number of visits to site α for path γ.

3 Statistical random walk summation and typical critical Green's function

In this section, I explain the central ideas of this paper, i.e., how to calculate the typical critical Green's function and conjugate product of wave function using a statistical random walk summation. First I recall the spectral expansion of the Green's function. Then I relate the spectral expansion with the random walk expansion using natural approximation by asymptotic behaviour for long time of random walk.

3.1 Spectral expansion of the Green's function and conjugate product of wavefunctions

The Green's function may also be expressed using the eigenvalues and eigenfunctions of the Hamiltonian

$$G(x, y; E) = \sum_{n} \frac{\psi_n^*(x)\psi_n(y)}{\epsilon_n - E} \tag{9}$$

Here, $\psi_n(x)$ is the value of the n-th eigenfunction at site x and ϵ_n is the n-th eigenvalue. To relate this expression with the random walk expansion, we rewrite the above equation,

$$\begin{aligned} G(x,y;E) &= \sum_n \frac{\psi_n^*(x)\psi_n(y)}{\epsilon_n}\frac{1}{1-(E/\epsilon_n)} \\ &= \sum_n \frac{\psi_n^*(x)\psi_n(y)}{\epsilon_n}\sum_{k=0}^{\infty}\left(\frac{E}{\epsilon_n}\right)^k \\ &= \sum_{k=0}^{\infty} E^k\left(\sum_n \psi_n^*(x)\psi_n(y)\frac{1}{\epsilon_n^{k+1}}\right) \end{aligned} \tag{10}$$

When $\epsilon_n = 0$, we replace ϵ_n with an arbitrary small quantity $\varepsilon > 0$. This replacement has meaning when taking a limit $\varepsilon \to 0$ after re-summation. The expanded form is just a formal series.

3.2 Asymptotic approximation for random walk expansion

Eq. (8) is a sum over all possible paths. This means that even when treating a three dimensional cubic lattice, the sum includes paths on two dimensional lattices and fractal lattices. However, the contribution of such irregular paths is expected to be irrelevant. Therefore, I propose to approximate by summing only relevant paths. Such relevant paths will be made of statistically typical and commonplace paths for the corresponding lattice. I assume that such typical paths have approximately the same statistical properties, i.e. the distribution of the number of visits are the same. This means the distribution of the number of visits is determined only by the number of steps of a path (and starting at site x and end at site y). Under this assumption, we can approximate Eq.(8) as

$$G(x,y;E) \simeq \sum_{t=0}^{\infty} A_{xy}(t)h^t \prod_{i_1=1}^{v_1(t)}\left(\frac{1}{u_{i_1}-E}\right)\prod_{i_2=1}^{v_2(t)}\left(\frac{1}{u_{i_2}-E}\right)^2\cdots\prod_{i_t=1}^{v_t(t)}\left(\frac{1}{u_{i_t}-E}\right)^t \tag{11}$$

Here, $A_{xy}(y)$ is the number of statistically typical path and $v_j(t)$ is the distribution of the number of visits by a statistically typical path when the number of steps of in the path is t. The u_{i_j} are independently and identically distributed random variables obeying Eq.(3). I note that the following equation folds

$$v_1(t) + 2v_2(t) + \cdots + tv_t(t) = t + 1 \tag{12}$$

3.3 High dimensional approximation of the Green's function

However, an exact calculation of $v_j(t)$ seems not to be easy. For this reason, we use an approximation which is expected to be valid in high dimensions. In high dimensions, the typical path will be well approximated by self avoiding random walk because the probability to return to a site which a path has already visited is expected to be very

low. In high dimension, we approximate

$$v_1(t) \simeq t+1 \tag{13}$$
$$v_2(t) \simeq \cdots = v_t(t) = 0 \tag{14}$$

This approximation should be valid for dimensions $d > 2$ i.e. when the return probability tends to zero for long times. The corresponding approximation for the Green's function is

$$G(x,y;E) \simeq \sum_{t=0}^{\infty} A_{xy}(t) h^t \prod_{i=1}^{t+1} \left(\frac{1}{u_i - E} \right) \tag{15}$$

To compare with the spectral expansion of the Green's function, we rewrite this equation as

$$\begin{aligned} G(x,y;E) &\simeq \sum_{t=0}^{\infty} A_{xy}(t) h^t \prod_{i=0}^{t} \left(\frac{1}{u_i - E} \right) \\ &= \sum_{t=0}^{\infty} A_{xy}(t) h^t \prod_{i=1}^{t+1} \frac{1}{u_i} \left(\frac{1}{1 - (E/u_i)} \right) \\ &= \sum_{t=0}^{\infty} A_{xy}(t) h^t \prod_{i=1}^{t+1} \frac{1}{u_i} \left(\sum_{n=0}^{\infty} (E/u_i)^n \right) \qquad (16) \\ &= \sum_{t=0}^{\infty} A_{xy}(t) h^t \prod_{i=1}^{t+1} \left(\sum_{n=1}^{\infty} (E^{n-1}/u_i^n) \right) \qquad (17) \end{aligned}$$

Here, we get a correspondence with the spectral expansion of Eq. (10).

3.4 Typical value and typical critical Green's function

We take the typical value of u_i^n.

$$u_i^{-n} = \left(\pm \frac{W}{2} \right)^{-n} \tag{18}$$

The absolute value is taken to avoid cancellation of terms. Then we have

$$\begin{aligned} G_{\text{typical}}(x,y;E) &\simeq \sum_{t=0}^{\infty} A_{xy}(t) h^t \prod_{i=1}^{t+1} \left[\sum_{n=1}^{\infty} \left(\frac{E^{n-1}}{(W/2)^n} \right) \right] \\ &= \sum_{t=0}^{\infty} A_{xy}(t) h^t \prod_{i=1}^{t+1} \left(\frac{1}{(W/2) - E} \right) \\ &= \sum_{t=0}^{\infty} A_{xy}(t) h^t \left(\frac{1}{(W/2) - E} \right)^{t+1} \end{aligned} \tag{19}$$

However, this expression is not suitable to calculate the conjugate product of wave functions because, if we multiply E and take a limit $E \to 0$, it always tends to zero for a finite cut-off of terms. So, we need to transform expression a little further

$$\begin{aligned}\frac{1}{(W/2)-E} &= -\frac{1}{E}\frac{1}{1-(\frac{W}{2E})} \\ &= -\frac{1}{E}\sum_{n=0}^{\infty}\left(\frac{W}{2E}\right)^n \\ &= -\sum_{n=0}^{\infty}\left(\frac{1}{E}\right)^{n+1}\left(\frac{W}{2}\right)^n \end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}G_{\text{typical}}(x,y;E) &\simeq \sum_{t=0}^{\infty}A_{xy}(t)h^t\left(-\sum_{n=0}^{\infty}\left(\frac{1}{E}\right)^{n+1}\left(\frac{W}{2}\right)^n\right)^{t+1} \\ &= \sum_{t=0}^{\infty}A_{xy}(t)h^t\left[\sum_{n=0}^{\infty}\left(\frac{1}{E}\right)^{n+1}\left(\frac{W}{2}\right)^n\right]^{t+1} \\ &= \sum_{t=0}^{\infty}A_{xy}(t)h^t\left(\frac{2}{W}\right)^{t+1}\left[\sum_{n=0}^{\infty}\left(\frac{W}{2E}\right)^{n+1}\right]^{t+1} \\ &= \sum_{t=0}^{\infty}A_{xy}(t)h^t\left(\frac{2}{W}\right)^{t+1}\left[\sum_{n=1}^{\infty}\left(\frac{W}{2E}\right)^{n}\right]^{t+1} \\ &= \sum_{t=0}^{\infty}A_{xy}(t)h^t\left(\frac{2}{W}\right)^{t+1}\sum_{n=1}^{\infty}\left(\frac{W}{2E}\right)^{n} \\ &\quad D(n,t+1) \\ &= \sum_{n=1}^{\infty}\left(\frac{W}{2E}\right)^{n}\sum_{t=0}^{\infty}A_{xy}(t)h^t\left(\frac{2}{W}\right)^{t+1} \\ &\quad D(n,t+1) \\ &= \sum_{n=1}^{\infty}\left(\frac{W}{2E}\right)^{n}\sum_{t=|x-y|}^{\infty}A_{xy}(t)h^t\left(\frac{2}{W}\right)^{t+1} \\ &\quad D(n,t+1) \end{aligned} \tag{21}$$

Here,

$$D(n,t) = \sum_{n_1+\cdots+n_t=n;n_j\geq 1} 1 \tag{22}$$

For $t > n$, we have

$$D(n,t) = 0 \tag{23}$$

So,

$$n \geq t \tag{24}$$

For $t = n$, we have

$$D(n, n) = 1 \tag{25}$$

Thus, we have

$$\begin{aligned} G_{\text{typical}}(x, y; E) &\simeq \sum_{n=1}^{\infty} \left(\frac{W}{2E}\right)^n \sum_{t=|x-y|}^{n-1} A_{xy}(t) h^t \left(\frac{2}{W}\right)^{t+1} \\ & D(n, t+1) \\ &= \sum_{n=|x-y|+1}^{\infty} \left(\frac{W}{2E}\right)^n \sum_{t=|x-y|}^{n-1} A_{xy}(t) h^t \left(\frac{2}{W}\right)^{t+1} \\ & D(n, t+1) \end{aligned} \tag{26}$$

3.5 Padé analysis for typical critical Green's function

For simplicity of calculation, we define z,

$$z = \frac{W}{2E} \tag{27}$$

We approximate by truncating the series at $2(N+|x-y|)+1$ (using some positive integer N) and then replacing it with a Padé approximant. The Padé approximant allows us to extract $1/E$ behaviour of the typical critical Green's function.

$$\begin{aligned} G_{\text{typical}}(x, y; E) &\simeq \sum_{n=|x-y|+1}^{\infty} z^n \sum_{t=|x-y|}^{n-1} A_{xy}(t) h^t \left(\frac{2}{W}\right)^{t+1} \\ & D(n, t+1) \\ &\simeq \sum_{n=|x-y|+1}^{2(N+|x-y|)+1} z^n \sum_{t=|x-y|}^{n-1} A_{xy}(t) h^t \left(\frac{2}{W}\right)^{t+1} \\ & D(n, t+1) \\ &= z^{|x-y|+1} \sum_{n=0}^{2N+|x-y|} z^n \sum_{t=|x-y|}^{n+|x-y|} A_{xy}(t) h^t \\ & \left(\frac{2}{W}\right)^{t+1} D(n + |x-y| + 1, t+1) \\ &\simeq z^{|x-y|+1} [N/(N + |x-y|)] \\ &= z^{|x-y|+1} \frac{a_N z^N + O(z^{N-1})}{b_{N+|x-y|} z^{N+|x-y|} + O(z^{N+|x-y|-1})} \\ &\propto z \frac{a_N}{b_{N+|x-y|}} \; for \; z \to \infty \\ &\propto \frac{a_N W}{2 b_{N+|x-y|} E} \; for \; E \to 0 \end{aligned} \tag{28}$$

3.6 Estimation of $A_{xy}(t)$

$A_{xy}(t)$ is the number of typical paths of length t which start at site x and end at site y. In the special case of $t = |x - y|$,

$$A_{xy}(t) = 1 \tag{29}$$

In the case of a d-dimensional cubic lattice, we have to consider parity. For example, if $|x - y|$ is even,

$$A_{xy}(t) = 0 \text{ for odd } t \tag{30}$$

When t increases the typical path will be composed of connected pieces where each piece is a random walk. Let's divide the straight path starting from site x and ending at site y into M pieces. Each piece has t/M steps. Let D^* be the diffusion constant of the lattice. The diffusion constant D^*, which depends on details of the lattice, is expected to be an irrelevant variable. However, estimation of $A_{xy}(t)$ and $A_{xx}(t)$ depends on strength of disorder W. To take the effect of disorder on a typical path into consideration, we propose the following approximation. For a given arbitrary diffusion constant D^*, we define an effective diffusion constant D

$$D = \frac{D^*}{\sqrt{W/h + 1}} \tag{31}$$

In limit of $W \to 0$, D becomes D^*, i.e., natural diffusion. In limit of $W \to \infty$, D becomes 0, i.e., ballistic motion. As disorder W increases, the effective diffusion constant D becomes smaller because the possible paths are restricted by the higher potential. From the coincidence with the diameter $|x-y|/M$ of a piece and the diameter $\sqrt{4Dt/M}$ of a random walk with t/M steps, we have

$$\frac{|x - y|}{M} = \sqrt{4D\frac{t}{M}} \tag{32}$$

$$\Rightarrow |x - y| = \sqrt{4DtM} \tag{33}$$

$$\Rightarrow 4DtM = |x - y|^2 \tag{34}$$

$$\Rightarrow M = \frac{|x - y|^2}{4Dt} \tag{35}$$

$$\Rightarrow M = \frac{l}{t} \tag{36}$$

Here,

$$l \equiv \frac{|x - y|^2}{4D} \tag{37}$$

When the strength of disorder W increases, the number of pieces M increases. Since l is expected to be larger than $|x - y|$, we have

$$D \leq \frac{|x - y|}{4} \tag{38}$$

$A_{xy}(t)$ is equal to the product of the number typical paths for each piece. If we consider large t limit this number is determined by the spectral dimension d and

$$\begin{aligned} A_{xy}(t) &= k^t \left[\left(\frac{t}{M} \right)^{-d/2} \right]^M \\ &= k^t \left(\frac{t^2}{l} \right)^{-dl/2t} \end{aligned} \tag{39}$$

Here, k is the mean number of edges connected to a site (in the case of a d-dimensional cubic lattice $k = 2d$). The approximation above may fail for small t but should work well for large (but not too large) t. If the calculated probability $(t/M)^{-d/2}$ is greater than 1, we replace it with unity.

For $t \gg l$, effect of the difference between x and y should become negligible and in this case we expect $A_{xy}(t) \approx A_{xx}(t)$ where $A_{xx}(t)$ is estimated in next section. For this reason, we use following approximation. When $t \geq l$ we set

$$A_{xy}(t) = A_{xx}(t) \tag{40}$$

We take l as the boundary value because, as will be clear after $A_{xx}(t)$ is calculated in the next section,

$$A_{xy}(l) = A_{xx}(l) \tag{41}$$

For $t < l$, we use Eq.(39) for the estimation of $A_{xy}(t)$.

3.7 Estimation of $A_{xx}(t)$

$A_{xx}(t)$ is the number of typical loops of length t. In case of a d-dimensional cubic lattice, we have to consider parity. For example, if $|x - y|$ is even,

$$A_{xx}(t) = 0 \text{ for odd } t \tag{42}$$

Using asymptotic formula for large t, we obtain

$$A_{xx}(t) \simeq k^t t^{-d/2} \tag{43}$$

For $t = 0$, we use

$$A_{xx}(t = 0) = 1 \tag{44}$$

3.8 Calculation of $D(n, t)$

$D(n, t)$ is equal to number of ways of partitioning a positive integer n into the sum of t positive integers. This is the same as the number of ways of placing $t - 1$ partitions between n balls, therefore,

$$D(n, t) = \binom{n-1}{t-1} \tag{45}$$

4 Numerical implementation and calculation of the localisation length

In this section, I discuss the numerical implementation of the method.

4.1 Scaling before Padé approximation

We need to calculate the Padé approximant of following series with cut-off $2N+|x-y|$, Eq. (28).

$$G_{\text{typical}}(x,y;E) \simeq z^{|x-y|+1} \sum_{n=0}^{2N+|x-y|} c_n z^n \tag{46}$$

where

$$c_n = \sum_{t=|x-y|}^{n+|x-y|} A_{xy}(t) \frac{1}{h} \left(\frac{2h}{W}\right)^{t+1} \binom{n+|x-y|}{t} \tag{47}$$

The number of terms should be enough amount such that $2N + |x - y| \geq 2l$. The coefficients of z^n grow rapidly with n, leading to numerical difficulties. This can be handled by a suitable scaling before performing the Padé approximation. I define a variable ζ and small positive real numbers p by

$$\zeta \equiv pz \tag{48}$$

Then we have

$$G_{\text{typical}}(x,y;E) \simeq z^{|x-y|+1} \frac{1}{h} \sum_{n=0}^{2N+|x-y|} \zeta^n \frac{c_n}{p^n} \tag{49}$$

We choose p as

$$p \equiv \left(\frac{c_{2N+|x-y|}}{c_1}\right)^{1/(2N+|x-y|)} \tag{50}$$

After this scaling, the Padé approximation is performed,

$$\begin{aligned} G_{\text{typical}}(x,y;E) &\simeq z^{|x-y|+1} \frac{1}{h} [N/(N+|x-y|)] \\ &= \frac{1}{h} \frac{a_N \zeta^N + O(\zeta^{N-1})}{b_{N+|x-y|}\zeta^{N+|x-y|} + O(\zeta^{N+|x-y|-1})} \end{aligned} \tag{51}$$

Finally, we get back Eq. (28) and the typical critical Green's function.

$$\begin{aligned} G_{\text{typical}}(x,y;E) &\simeq z^{|x-y|+1} \frac{1}{h} \frac{a_N \zeta^N + O(\zeta^{N-1})}{b_{N+|x-y|}\zeta^{N+|x-y|} + O(\zeta^{N+|x-y|-1})} \\ &\propto z^{|x-y|+1} \zeta^{-|x-y|} \frac{a_N}{h b_{N+|x-y|}} \; for\; \zeta \to \infty \\ &\propto z \frac{a_N}{p^{|x-y|} h b_{N+|x-y|}} \; for\; z \to \infty \\ &\propto \frac{W a_N}{2 p^{|x-y|} h b_{N+|x-y|}} \frac{1}{E} \; for\; E \to 0 \end{aligned} \tag{52}$$

4.2 Conjugate product of wavefunctions for $E = 0$

The spectral expansion of the Green's function is

$$G_{\text{typical}}(x, y; E) = \frac{A}{E} + \sum_{\epsilon_n \neq 0} \frac{A'}{E - \epsilon_n} \tag{53}$$

Here, A and A' are, respectively, the sums of product of eigenfunctions with corresponding energies $\epsilon_n = 0$ and $\epsilon_n \neq 0$. We extract A by taking the following limit

$$|\lim_{E \to 0} EG_{\text{typical}}(x, y; E)| = A = \frac{Wa_N}{2hb_N} \tag{54}$$

While A is a sum of the product of eigenfunctions, if we take large $|x - y|$, the most extended state remains. If $A \geq 1$, we replace it with $1/A$. We expect to obtain the absolute value of the conjugate product of wave functions because we have taken the absolute value when calculating the typical value

$$\lim_{E \to 0} EG_{\text{typical}}(x, y; E) = |\psi^*(x)\psi(y)| = \frac{Wa_N}{2p^{|x-y|}hb_N} \tag{55}$$

Here, ψ is most extended state for $E = 0$.

4.3 Inverse localisation length and the critical exponent

Calculated product of wave function is related with distance of t. So, it is identified with $t = |x - y|^2/4D$. For sufficiently large $|x - y|$, the inverse localisation length λ may be estimated by fitting $\ln|\psi^*(x)\psi(y)|$ by linear function for various values of $|x - y|$ as its slope. If a negative localization length is obtained after fitting, we replace it with its absolute value. Near the critical disorder of the Anderson transition we expect the leading order behaviour to be

$$\lambda(W) = Aw^{\nu} \tag{56}$$

with

$$w \equiv (W - W_c)/W_c \tag{57}$$

By fitting numerical data with the above equations, i.e., determining A, W_c and ν, we obtain estimates of the critical disorder W_c and the critical exponent ν.

5 Results

I report numerical results for various dimensions d. For the results reported below the Padé approximant was calculated using the Python module mpmath with 300 to 500 digits.

5.1 $d = 3$

I calculated the inverse of the localisation length for the following of parameters: $2N + |x - y| = 200$, $h = 1$,$|x - y| = 40, 42, \cdots, 50$ $W = 0.8, 1.2, \cdots, 3.2$, $D^* = 10$. Digits for this calculation is 1000. The result is plotted in Fig.1. The estimates of the critical exponent and critical disorder are

$$\begin{aligned} \nu &= 1.20 \pm 0.34 \\ W_c &= -0.07 \pm 0.54 \\ A &= 0.49 \pm 0.28 \end{aligned} \tag{58}$$

$\pm$ here and appearing later is derived from the covariance matrix of fitting and not statistical error. The value for the exponent is consistent with previous study[4] with 2σ.

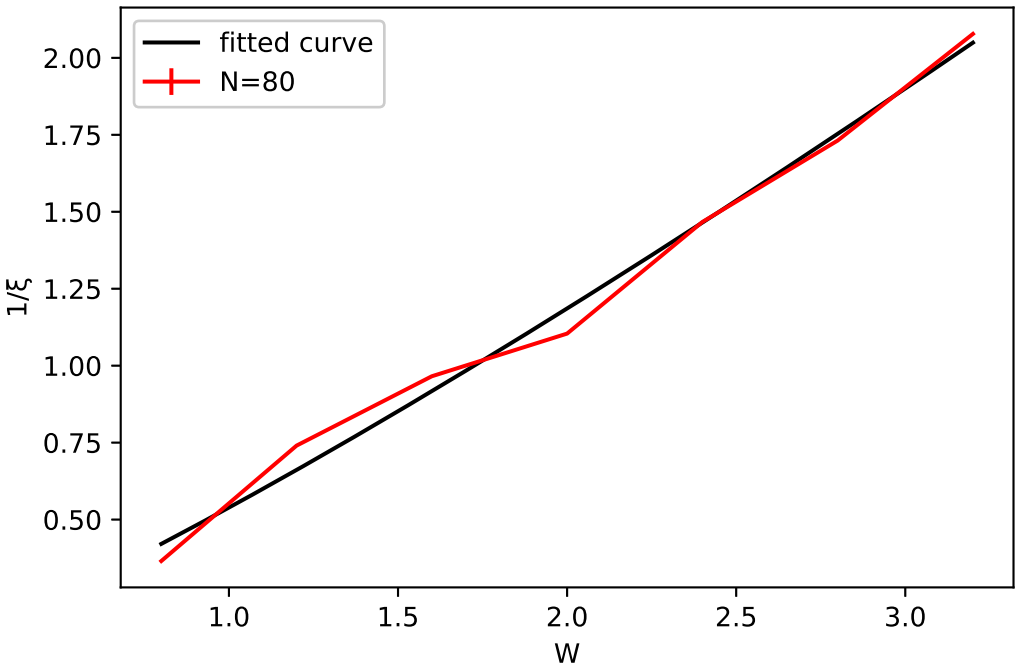

Figure 1: The inverse localisation length $\lambda = 1/\xi$ for $d = 3$.

Some data is far from fitting curve. So, I fitted the data with $W = 1.2, 1.6$ removed. The result is plotted in Fig.2. The estimates of the critical exponent and critical disorder

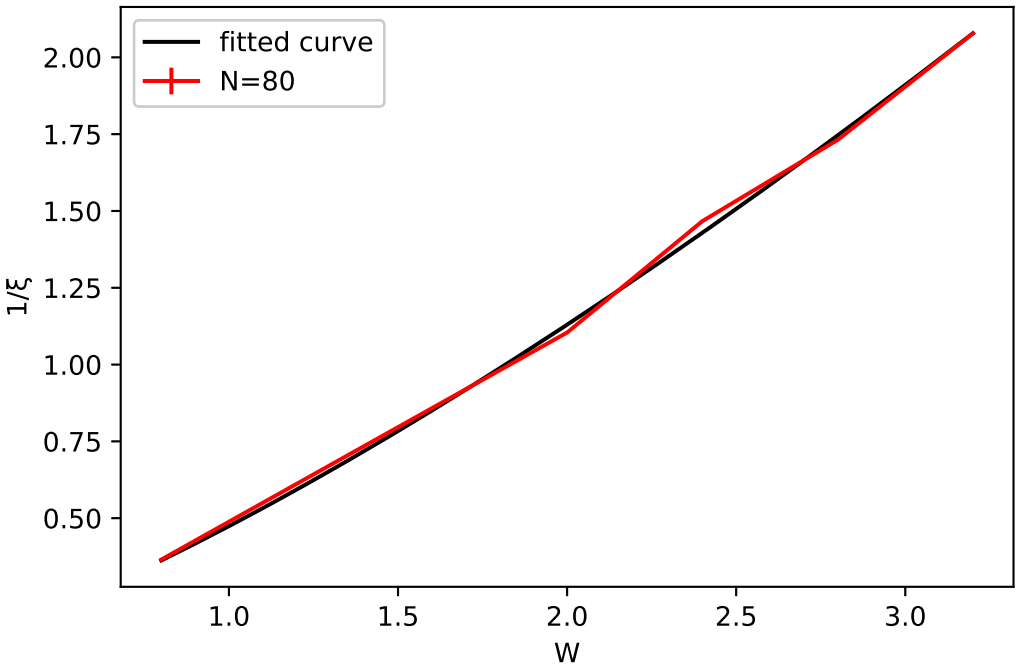

Figure 2: The inverse localisation length $\lambda = 1/\xi$ for $d = 3$ with data removed.

are

$$\begin{aligned} \nu &= 1.36 \pm 0.19 \\ W_c &= -0.11 \pm 0.28 \\ A &= 0.41 \pm 0.14 \end{aligned} \tag{59}$$

The estimated value is smaller than previous studies. This is thought to be because of high dimensional approximation. This problem will be solved by further improvement of high dimensional approximation in Appendix.A.

5.2 $d = 6$

I calculated the inverse of the localisation length for the following of parameters: $2N + |x - y| = 200$, $h = 1$,$|x - y| = 40, 42, \cdots, 50$ $W = 0.8, 1.0, \cdots, 2.8$, $D^* = 10$. Digits for this calculation is 1000. The result is plotted in Fig.3. The estimates of the critical

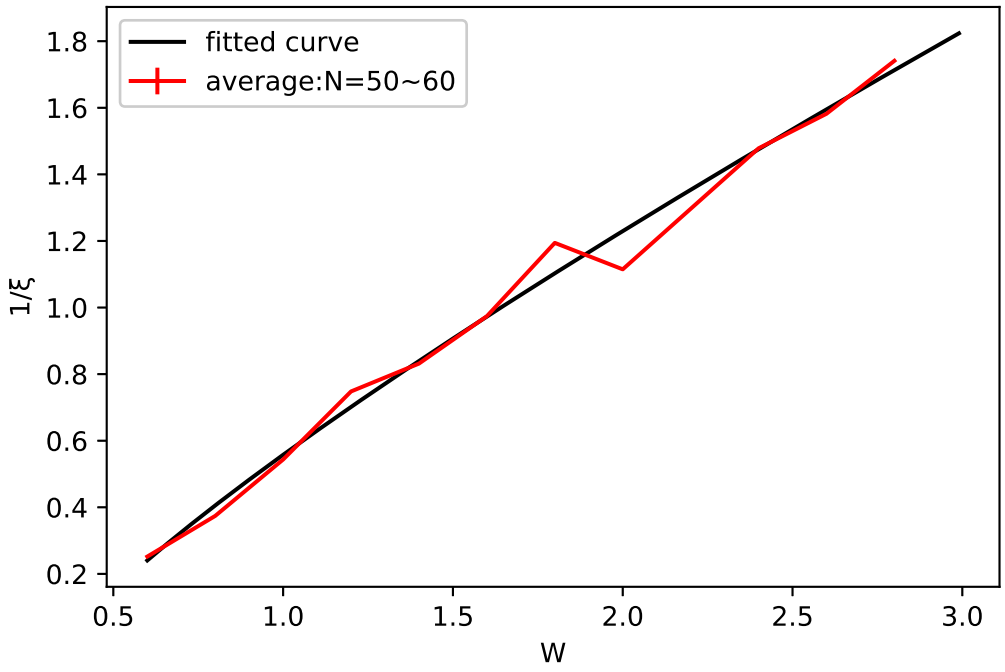

Figure 3: The inverse localisation length $\lambda = 1/\xi$ for $d = 6$.

exponent and critical disorder are

$$\begin{aligned} \nu &= 0.83 \pm 0.10 \\ W_c &= 0.37 \pm 0.12 \\ A &= 0.82 \pm 0.10 \end{aligned} \tag{60}$$

Estimated value of the critical exponent is consistent with previous study[5] with 2σ.

5.3 $d = 7$

I calculated the inverse of the localisation length for the following of parameters: $2N + |x - y| = 200$, $h = 1$,$|x - y| = 40, 42, \cdots, 50$ $W = 0.8, 1.0, \cdots, 2.6$, $D^* = 10$. Digits for this calculation is 1000. The result is plotted in Fig.4. The estimates of the critical exponent and critical disorder are

$$\begin{aligned} \nu &= 0.79 \pm 0.011 \\ W_c &= 0.39 \pm 0.11 \\ A &= 0.84 \pm 0.09 \end{aligned} \tag{61}$$

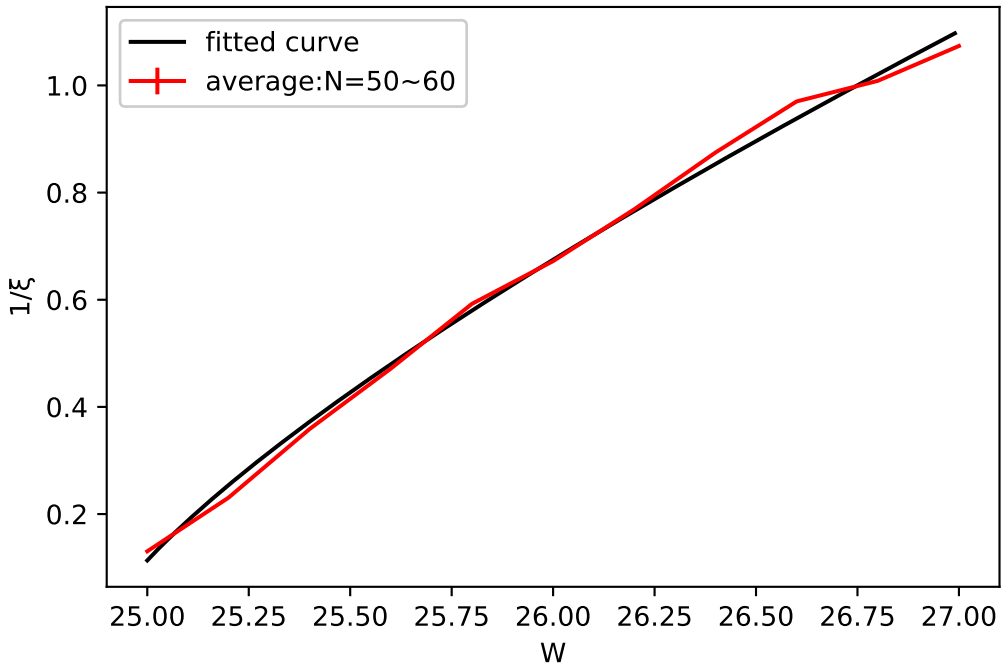

Figure 4: The inverse localisation length $\lambda = 1/\xi$ for $d = 7$.

5.4 $d = 2.54$

I calculated the inverse of the localisation length for the following of parameters: $2N + |x - y| = 210$, $h = 1$,$|x - y| = 50, 52, \cdots, 60$ $W = 0.05, 0.15, \cdots, 0.95$, $D^* = 10$. Digits for this calculation is 1000. The result is plotted in Fig.5. The estimates of the critical exponent and critical disorder are

$$\begin{aligned} \nu &= 1.15 \pm 0.50 \\ W_c &= -0.54 \pm 0.44 \\ A &= 2.03 \pm 01.07 \end{aligned} \tag{62}$$

The value for the exponent is smaller than previous studies. The reason of this inconsistency is high dimensional approximation.

Some data is far from fitting curve. So, I fitted the data with $W = 0.25, 0.35$ removed. The result is plotted in Fig.6. The estimates of the critical exponent and

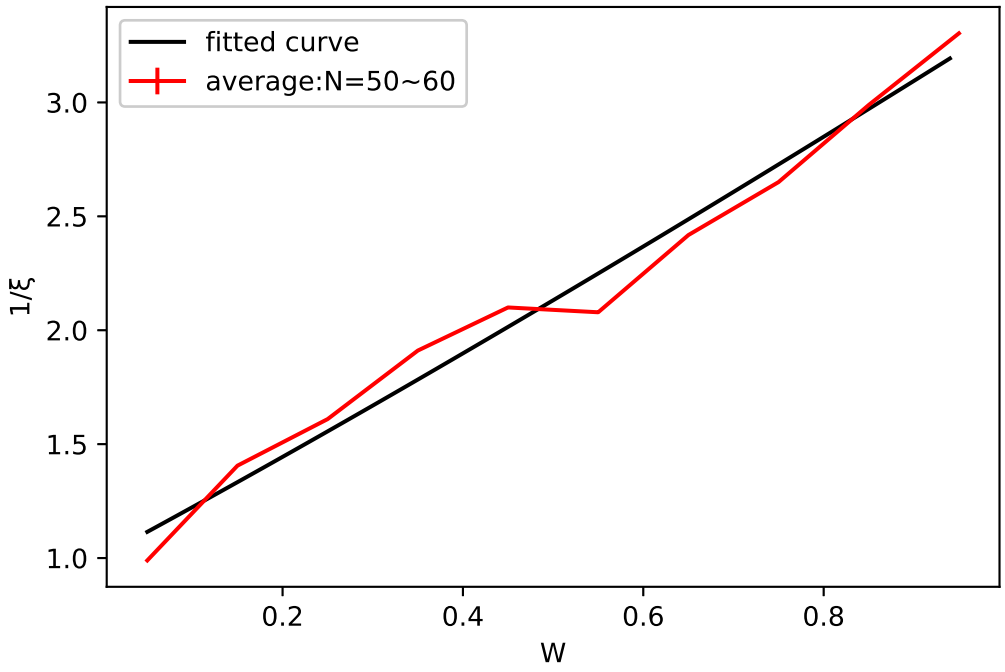

Figure 5: The inverse localisation length $\lambda = 1/\xi$ for $d = 2.54$.

critical disorder are

$$\begin{aligned} \nu &= 1.55 \pm 0.89 \\ W_c &= -0.83 \pm 0.75 \\ A &= 1.33 \pm 1.53 \end{aligned} \tag{63}$$

The estimated value is consistent with previous studies[6].

6 Conclusion

I proposed a new method to study the Anderson transition in the orthogonal symmetry class. The relevant dimension for the critical phenomena of the Anderson transition is the spectral dimension. This method employs a virtual lattice characterised by an arbitrary spectral dimension instead of a concrete lattice with a given integer or spectral dimension. The next step is to build an analytical theory of Anderson transition based on this new theoretical framework. Since the spectral dimension is just a parameter, it is possible to study non-integer dimension more easily than before. In contrast with other

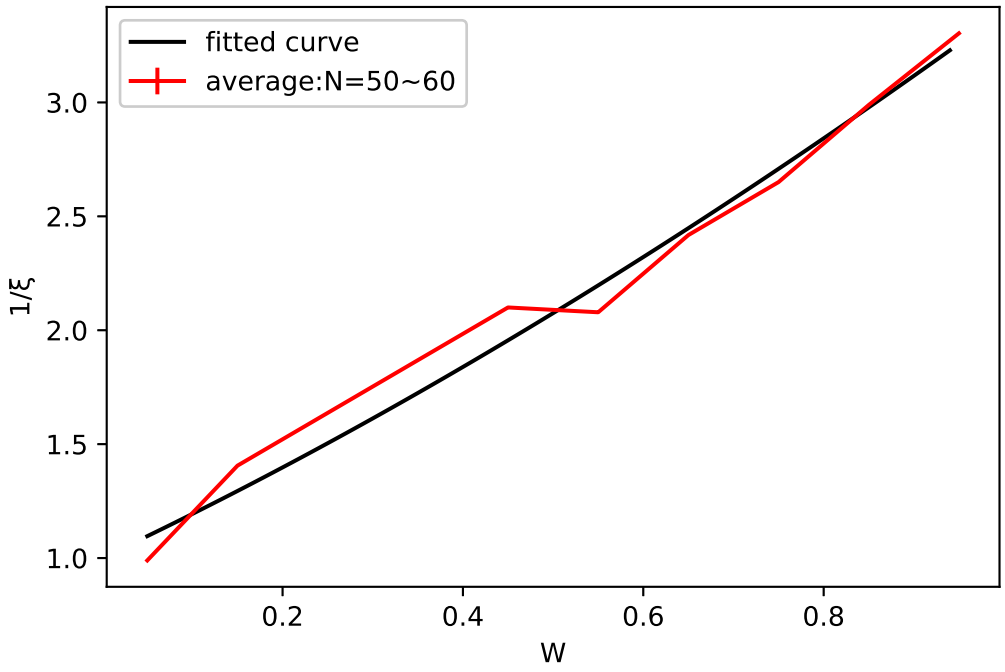

Figure 6: The inverse localisation length $\lambda = 1/\xi$ for $d = 3$ with data removed.

methods, the numerical complexity does not increase exponentially with dimension and finite size scaling is not required. In terms of SRWS, I can say that Hamiltonian was dead and lattice melted.

I thank Keith Slevin for suggesting to calculate the typical Green's function.

A Simple approximation for $v_2(t)$

In this appendix, I give further approximation for high dimensional approximation in Sec.3.3. Detailed of this study will be done in the future paper. If we focus on short steps, we may give approximation for $v_2(t)$. Similar approximation is possible but complicated. So, we see approximation for $v_2(t)$ only. On graph which have k-nodes on average, walk goes back previous site with probability $1/k$. Therefore, we may approximate $v_2(t)$ using following equation.

$$2v_2(t) \simeq \frac{1}{k} v_1(t) \tag{64}$$

And, if we ignore higher order,

$$v_3(t) \simeq \cdots \simeq v_t(t) \simeq 0 \tag{65}$$

From Eq(12), we have

$$\begin{aligned} & v_1(t) + 2v_2(t) \simeq t + 1 \\ \Rightarrow \quad & v_1(t) + \frac{1}{k} v_1(t) \simeq t + 1 \\ \therefore \quad & v_1(t) \simeq \lfloor \frac{t+1}{1+k^{-1}} \rfloor \qquad (66) \\ \therefore \quad & v_2(t) \simeq t + 1 - v_1(t) \qquad (67) \end{aligned}$$

Because $v_1(t)$ must be integer, I took floor function as an approximation. Formulation of typical critical Green function is similar to case of $v_2(t) = 0$.

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